The Periodic Coincidence Points of continuous maps and the Lindemann's independence Theorem For Exponentials

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Date of acceptance 26 / 6 / 2005

Abstract

This paper give a proof of known conditions for the existence of periodic coincidence points of continuous maps ,using Lindemann theorem on transcendental numbers.

Introduction

Let M be a compact connected oriented manifold of dimension l. Let $f, g : M \to M$ be two continuous maps (for more details of compact connected oriented manifold see [3]) . A point $x \in M$ is called a coincidence point if $f(x) = g(x)$ and called a periodic coincidence point if there exists a positive integer k such that $f^k(x) = g^k(x)$, following [7] . A continuous self map f of M . induces homomorphism of rational homology of M, $f^{*q} : H^q(M) \to H^q(M)$

In [4] , H. T. Ku and L. N. Mann define $g_*^q : H^q(M) \to H^q(M)$ by $< g_*^q \alpha \cup \beta, [M] >= < \alpha \cup g^{*(n-q)} \beta, [M] >$ $, \alpha \in H^q(M), \beta \in H^{n-q}(M)$.

Lemma(1) :-

If $\deg g \neq 0$ *then* $g_*^q = (\deg g)(g^{*q})^{-1}$.

Proof :-

See [4].

Lemma(2) :-

If $\deg g \neq 0$ ***and*** $f^{*q} \circ g^{*q} = g^{*q} \circ f^{*q}$ ***,all q, then*** $(f^k)^{*q} \circ (g_*^k)^q = (f^{*q} \circ g_*^q)^k$***,all k and all q .***

Proof :-

See [4].

n [4] the authors define the Lefschetz number to the pair of continuous maps (f , g) by $L_{f,g} = \sum_{q=0}^{n} (-1)^q Tr(f^{*q} \circ g_*^q)$, were *Tr* is the *Trace* of the matrix which represent the linear transformation $f^{*q} \circ g_*^q$. Also they define the Euler characteristic of f and g by

$$\chi_{f,g}(M) = \sum_{q=0}^{n} (-1)^q \dim \operatorname{Im}(f^{*q} \circ g_*^q)$$

and they prove the following Fuller coincidence theorem :

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Theorem (3) :-

Suppose f , g are continuous self – maps of a compact connected oriented manifold of dimension l . If $\chi_{f,g}(M) \neq 0$ ***,deg g ≠0 and*** $f^{*q} \circ g^{*q} = g^{*q} \circ f^{*q}$***, all q ,then there exists x ∈ M such that*** $f^k(x) = g^k(x)$ ***for some k ,***

$1 \leq k \leq \max[\sum_{q odd} \dim \mathrm{Im}(f^{*q} \circ g_*^q),$

$\sum_{q even} \dim \mathrm{Im}(f^{*q} \circ g_*^q)].$

M. O. Damen [1] , M. B. Milinovich [5] and Ivan Niven [6,p .117] gave the Lindemann theorem which state that " Given any distinct algebraic numbers $\alpha_1, \alpha_2, ..., \alpha_m$, the values $e^{\alpha_1}, e^{\alpha_2}, ..., e^{\alpha_m}$ are linearly independent over the field of algebraic numbers i.e. $\sum_{j=1}^{m} a_j e^{\alpha_j} = 0$ then $a_1 = a_2 = ... = a_m = 0$. " .

The main result

In this section we shall give a proof of a generalization of theorem (3) . Set $\lambda_0 = 0$, and let $\lambda_1, ..., \lambda_n$ be the distinct nonzero eigenvalues of $f^{*q} \circ g_*^q$, where $n = \dim \mathrm{Im}(f^{*q} \circ g_*^q)$. Let $m_q(\lambda_j)$ be the multiplicity of λ_j in $f^{*q} \circ g_*^q$, (j = 0, ... , n) .We then have :

Theorem (4) :-

If f , g are two continuous self– F-Salama***maps of a compact connected oriented manifold M , deg g ≠ 0 ,*** $f^{*q} \circ g^{*q} = g^{*q} \circ f^{*q}$ ***and f , g have no periodic coincidence point x of period k ,*** $1 \leq k \leq \max[\sum_{q odd} \dim \mathrm{Im}(f^{*q} \circ g_*^q), \sum_{q even} \dim \mathrm{Im}(f^{*q} \circ g_*^q)]$. ***Then*** $\sum_q (-1)^q m_q(\lambda_0) = \chi_{f,g}(M)$ ***and*** $\sum_q (-1)^q m_q(\lambda_j) = 0$, ***(j =1 ,..., n) .***

Proof :-

Let m be the maximum dimension for which $H^q(M) \neq 0$ .For $q = 0, \dots ,m$, let A_q be a matrix representing $f^{*q} \circ g_*^q$ with respect to some fixed basis for $H^q(M)$. Define

$B_k = diag[A_0^k, -A_1^k, ..., (-1)^m A_m^k]$,

for $k \geq 0$, and set

$$E = \sum_{k=0}^{\infty} \tfrac{1}{k!} B_k . \qquad (1)$$

$Tr(B_0) = Tr(diag[A_0^0, A_1^0, ..., (-1)^m A_m^0] = \sum_{q=0}^{m} (-1)^q \dim \mathrm{Im}(f^{*q} \circ g_*^q) = \chi_{f,g}(M)$, for $k \geq 1, Tr(B_k) = \sum_{q=0}^{m} (-1)^q Tr(f^{*q} \circ g_*^q)^k = \sum_{q=0}^{m} (-1)^q Tr(f^k)^{*q} \circ (g_*^k)^q = L_{f^k, g^k}$, the Lefchetz coincidence number of f^k and g^k, since by hypothesis $\deg g \neq o$ and $f^{*q} \circ g^{*q} = g^{*q} \circ f^{*q}$. If f and g have no periodic coincidence points x ,of period k , $k = 1$ . ... . max [

$\sum_{qodd} \dim \operatorname{Im}(f^{*q} o g_*^q), \sum_{qeven} \dim \operatorname{Im}(f^{*q} o g_*^q)]$ then by Lefchetz coincidence point theorem $Tr(B_k)=0$. We then have equating traces in (1) :

$$\sum_{j=0}^{n}\sum_{q=0}^{m}(-1)^q m_q(\lambda_j)e^{\lambda_j} \qquad (2)$$

$$=\sum_{j=0}^{n}[(-1)^0 m_0(\lambda_j)e^{\lambda_j}+(-1)^1 m_1(\lambda_j)e^{\lambda_j}+(-1)^2 m_2(\lambda_j)e^{\lambda_j}+\dots+(-1)^m m_m(\lambda_j)e^{\lambda_j}]$$

$$=m_0(\lambda_0)e^{\lambda_0}-m_1(\lambda_0)e^{\lambda_0}+m_2(\lambda_0)e^{\lambda_0}+\dots+(-1)^m m_m(\lambda_0)e^{\lambda_0}+m_0(\lambda_1)e^{\lambda_1}-m_1(\lambda_1)e^{\lambda_1}+m_2(\lambda_1)e^{\lambda_1}+\dots+(-1)^m m_m(\lambda_1)e^{\lambda_1}+\dots+m_0(\lambda_n)e^{\lambda_n}-m_1(\lambda_n)e^{\lambda_n}+m_2(\lambda_n)e^{\lambda_n}+\dots+(-1)^m m_m(\lambda_n)e^{\lambda_n}.$$

But the A_q are the rational matrices , so the eigenvalues $\lambda_0,\dots,\lambda_n$ are the distinct algebraic numbers , and we can apply the theorem of Lindemann to conclude that each of the coefficients in equation (2)vanishes , i. e. $\sum_q(-1)^q m_q(\lambda_0)=\chi_{f,g}(M)$ and $\sum_q(-1)^q m_q(\lambda_j)=0$, ($j=1$, ... , n) . So $\sum_{j=0}^{n}\sum_{q=0}^{m}(-1)^q m_q(\lambda_j)e^{\lambda_j}=m_0(\lambda_0)e^{\lambda_0}-m_1(\lambda_0)e^{\lambda_0}+\dots+(-1)^m m_m(\lambda_0)e^{\lambda_0}=\chi_{f,g}(M)e^{\lambda_0}$. □

The following corollary is the special case of theorem (4) , when g homotopic to the identity map which proved by F. B. Fuller [2] : -

Corollary (5) (Fuller)

Suppose f, g are continuous self – maps of a compact connected oriented manifold of dimension l . If $\chi_{f,g}(M)\neq 0$ and g homotopic to the identity map then f has a periodic fixed point.

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الخلاصة

ليكن M مطوي متراص مترابط ولتكن f و g دوال مستمره معرفه على M . يقال ان x نقطه تطابق اذا كان $f(x) = g(x)$ ويقال انها نقطه دوريه متطابقه اذا وجد $k \in N$ بحيث ان $f^k(x) = g^k(x)$.

هذا البحث يعطي برهان الشروط المعروفه لوجود النقاط الدوريه المتطابقه لدوال مستمره بأستخدام مبرهنة ليندمان على الأعداد الغير جبريه .