

# Using Collocation Method for Solving Differential Equations with Time Lag

\*Fadhel S. Fadhel

\*\* Nadia K. Marie

Date of acceptance 4/2/2003

## Abstract :-

In this paper, the collocation method will be used to solve ordinary differential equations of retarded arguments. Also, some examples are presented in order to illustrate this approach.

## Introduction:

Real life problems of differential equations with retarded arguments are of great importance. One reason for the importance is that they describe processes with “after-effects”, that is, time lag (if the independent variable denotes time). For this reason, the theory of delay differential equations had been very intensively worked out in the last decade, by now many fundamental parts of this theory have been developed almost as completely as the corresponding parts of the theory of ordinary equations without time lag [6].

The general form of the n-th order DDE is given by: -

$$F(t, x(t), x(t-\tau_1), x(t-\tau_2), x(t-\tau_3), \dots, x(t-\tau_k), x(t-\tau_{k+1}), \dots, x(t-\tau_n)) = 0 \dots \dots \dots (1)$$

Where F is a given function and  $\tau_1, \tau_2, \dots, \tau_k$  are given fixed positive number called the “time delays” [1,2].

In some of Literatures, equation (1) is called a differential equation with deviating argument [3,4], or an equation with time lag [5], or a differential difference equation [2,7], or a functional differential equation [2].

Other Literatures writes equation (1) in the following form:

$$F(t, x(t), x(k_1 t), x(k_2 t), \dots, x(k_n t), x(t), x(k_1 t), x(k_2 t), \dots, x(k_n t), x^{(m)}(t), x^{(m)}(k_1 t), x^{(m)}(k_2 t), \dots, x^{(m)}(k_n t)) = 0 \dots \dots \dots (2)$$

Where  $k_1, k_2, \dots, k_n$  are constants not equals to one.

Some times, DDE’s are so complicated to be solved analytically or when the solution  $\phi_k(t)$  in the successive steps becomes so complicated such that the next step solution  $\phi_{k+1}(t)$  could not be evaluated. Therefore, in this article a new approach will be used to solve delay differential equations, which has its basis on the collocation method.

## Collocation Method:

This method is based on the eigenfunction expansion of the exact solution, which will be transformed into a linear system of algebraic equations. The most powerful and advantageous of this method is to evaluate the solution for any point in the domain of solution. So the first step, we choose the collocation points  $t_0, t_1, \dots, t_k$  in the closed interval  $[a, b]$ , such that :

$$t_j = a + j \left( \frac{b-a}{k} \right) \quad , \quad j = 0, 1, \dots, k.$$

\* Department of Mathematics and Computer Applications, College of Science, Saddam University, Baghdad, Iraq

\*\* Department of Mathematics, Science College for Women, Baghdad University, Baghdad, Iraq

and we force the approximate solution  $x(t) = w(t) + \sum_{i=0}^N a_i w_i(t)$  to satisfy DDE's at this set of points, when  $w(t)$  satisfies the non-homogeneous initial and boundary conditions, while  $w_i(t)$  satisfies the homogeneous conditions.

**Solution of DDE Using Collocation Method:**

We will discuss briefly how to solve linear and non-linear DDE's of the first and second order using collocation method, which has novalty of accuracy.

The basic idea of this method is to suppose that the solution of DDE is given by

$$x(t) = w(t) + \sum_{i=0}^N a_i w_i(t)$$

Where  $w(t)$  satisfy the non-homogeneous conditions (initial function) and  $w_i(t)$  satisfies the homogeneous condition, and the problem is to evaluate  $a_i$ 's by direct substitution in the DDE, and we choose the collocation points  $t_0, t_1, K, t_k$  in the closed interval  $[a,b]$ , which gives a linear system of  $a_0, a_1, K, a_n$ . Then Gauss elimination technique could be used to solve the resulting system of equations.

The following examples illustrations this approach.

**Example (1):**

Consider the linear DDE of the first order  $x'(t) = x(t - 1)$ , with the initial condition

$$x(t) = \phi_0(t) = t, \text{ for } -1 \leq t \leq 0,$$

and we want to find the solution in the first step interval  $[0,1]$ . Suppose the solution is given by:

$$x(t) = a_0 t + a_1 t^2 + a_2 t^3$$

which could be differentiated to be as follows:

$$x'(t) = a_0 + 2a_1 t + 3a_2 t^2$$

Upon substituting in the original DDE we have

$$a_0 + 2a_1 t + 3a_2 t^2 = t - 1$$

Evaluating this equation for different time steps.

If  $t=0$ , we get  $a_0 = -1$

if  $t = \frac{1}{2}$ , we get  $a_0 + a_1 + \frac{3}{4} a_2 = -\frac{1}{2}$

if  $t=1$ , we get  $a_0 + 2a_1 + 3a_2 = 0$

So, we have the system of equations.

$$a_0 = -1$$

$$a_0 + a_1 + \frac{3}{4} a_2 = -\frac{1}{2}$$

$$a_0 + 2a_1 + 3a_2 = 0$$

Solving this linear system, gives

$$a_0 = -1, a_1 = \frac{1}{2} \text{ and } a_2 = 0$$

So, the solution is of the form

$$x(t) = -t + \frac{t^2}{2}, \text{ for } t \in [0,1].$$

In order to find the solution in the second time step interval  $[1,2]$ , with the initial condition

$$x_1(t) = \frac{t^2}{2} - t, \text{ for } 0 \leq t \leq 1$$

Suppose the solution of the second interval is of the form

$$x(t) = \frac{\frac{t}{2} - \frac{1}{2}}{a_0 + a_1 + a_2} - \frac{1}{2} \frac{a_0 t + a_1 t^2 + a_2 t^3}{a_0 + a_1 + a_2}$$

With derivative

$$x'(t) = \frac{1}{2(a_0 + a_1 + a_2)} - \frac{1}{2} \frac{a_0 + 2a_1t + 3a_2t^2}{a_0 + a_1 + a_2}$$

Substituting in the original DDE , we get :-

$$\frac{1}{2(a_0 + a_1 + a_2)} - \frac{a_0 + 2a_1t + 3a_2t^2}{2(a_0 + a_1 + a_2)} = \frac{(t-1)^2}{2} - (t-1)$$

Evaluating this equation for different time steps  $t = 1, \frac{3}{2}, 2$ , we have the system of equations: -

$$\begin{aligned} a_0 + 2a_1 + 3a_2 &= 1 \\ a_0 + 9a_1 + 24a_2 &= 4 \\ 3a_1 + 11a_2 &= 1 \end{aligned}$$

and by solving this linear system, we get:

$$a_0 = -\frac{2}{7}, a_1 = \frac{6}{7} \text{ and } a_2 = -\frac{1}{7}$$

Hence, the solution in the second step interval [1,2] is given by:

$$x(t) = -\frac{7}{6} + \frac{t^3}{6} - t^2 + \frac{3}{2}t, \text{ for } t \in [-1,2]$$

and so on , we proceed to the next intervals.

**Example (2):**

Consider the linear DDE of the second order

$$x''(t) = x'(t-1) + 1, \text{ with the initial condition}$$

$$x(t) = \phi_0(t) = 1, \text{ for } -1 \leq t \leq 0.$$

To find the solution in the first interval [0,1], we suppose that:

$$x(t) = 1 + a_0t^2 + a_1t^3 + a_2t^4$$

Upon derivative twice both sides and put in the original equation, we get: -

$$2a_0 + 6a_1t + 12a_2t^2 = 1$$

Evaluating this equation for different time steps, we have the system of equations

$$\begin{aligned} a_0 &= \frac{1}{2} \\ 2a_0 + 3a_1 + 3a_2 &= 1 \\ 2a_0 - 6a_2 &= 1 \end{aligned}$$

Upon substituting and simplification, we get:

$$x(t) = \frac{t^2}{2}, \text{ for } t \in [0,1].$$

In order to find the solution in the second step interval, suppose that:

$$x(t) = \frac{\frac{1}{2}t^2 - \frac{1}{2}}{a_0 + a_1 + a_2} + \frac{1}{2} \frac{a_0t^2 + a_1t^3 + a_2t^4}{a_0 + a_1 + a_2}.$$

Similarly, we get the solution

$$x(t) = \frac{t^3}{6} + \frac{1}{3}, \text{ for } t \in [1,2]$$

and so on for  $t \geq 2$ .

We can also solve non-linear delay differential equation of the form

$$x'(t) = x(t)[ax''(t-\tau) + bx'''(t-\tau)]$$

Where a,b are given constants and n is any positive number . The collocation method could be used also and given raise to a linear system of equations with  $a_0, a_1, K, a_n$  as an unknown.

Using the same steps above, we get: -

### Example (3): -

Consider the non-linear DDE of the first order  $x'(t) = x'^2(t-1) + t$ , with the initial condition  $x(t) = \phi_0(t) = 1$ , for  $-1 \leq t \leq 0$ , and the problem is to find the solution in the first step interval  $[0,1]$ , suppose that: -

$$x(t) = 1 + a_0 t + a_1 t^2 + a_2 t^3$$

which has the following derivative:

$$x'(t) = a_0 + 2a_1 t + 3a_2 t^2$$

Upon substituting in the original DDE and taking  $t = 0, \frac{1}{2}, 1$ , we get the following linear system of equations:

$$a_0 = 0$$

$$a_0 + a_1 + \frac{3}{4}a_2 = \frac{1}{2}$$

$$a_0 + 2a_1 + 3a_2 = 1$$

and by solving this system, we get the solution:

$$x(t) = \frac{t^2}{2}, \text{ for } t \in [0,1].$$

Which is the desired solution in the first interval. In order to find the solution in the second interval, suppose that:

$$x(t) = \frac{\frac{1}{2}t - \frac{1}{2}}{a_0 + a_1 + a_2} + \frac{1}{2} \frac{a_0 t + a_1 t^2 + a_2 t^3}{a_0 + a_1 + a_2}$$

$$x(t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{1}{3}, \text{ for } t \in [1,2]$$

and so on for  $t \geq 2$ .

### References:

1. AL-Saady, A.S., 2000. "Cubic Spline Technique in Solving Delay Differential Equations", M.Sc. Thesis, Department of Mathematics, College of Science, Saddam University,
2. Bellman, R. and Cooke, K.L., 1963 "Differential Difference Equations", Academic Press Inc., New York, .
3. Driver, R.D., 1971. "Ordinary and Delay Differential Equations", Springer Verlag, New York, Inc. .
4. El'sgol'c, L.E., "Qualitative Methods in Mathematical Analysis", Trans. Math. Mono., Vol. 12, American Mathematical Society, 1964.
5. Halanay, A. 1966. "Differential Equations, Stability, Oscillations, Time lags" New York, Academic,
6. Marie, N.K., "Variational Formulation of Delay Differential Equations", M.Sc. Thesis, 2001 Department of Mathematics, College of Education \ Ibn AL-Haitham, University of Baghdad.
7. Pinney, E., 1959 "Ordinary Difference Differential Equations", University of California.

## استخدام طريقة الرصف لحل معادلات تفاضلية تباطؤية

نادية خزعل مرعي\*\*

فاضل صبحي فاضل\*

\* جامعة النهرين  
\*\* مدرس مساعد / قسم الرياضيات / كلية العلوم للبنات / جامعة بغداد

### الخلاصة:-

في هذا البحث ، استخدمت طريقة الرصف ( collocation method ) لحل معادلات تفاضلية تباطؤية . كما وقدمت بعض الأمثلة لتوضيح نتائج هذه الطريقة .