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Coincidence of Fixed Points with Mixed Monotone Property

Ali A. Kazem   *Amal M. Hashim**

Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq
Corresponding author: amalmhashim@yahoo.com
E-mail address: eali.nasir.sci@uobasrah.edu.iq

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Abstract:

The purpose of this paper is to introduce and prove some coupled coincidence fixed point theorems for self mappings satisfying $(\phi - \psi)$-contractive condition with rational expressions on complete partially ordered metric spaces involving altering distance functions with mixed monotone property of the mapping. Our results improve and unify a multitude of coupled fixed point theorems and generalize some recent results in partially ordered metric space. An example is given to show the validity of our main result.

Keywords: Common fixed point, Coupled fixed point, Metric space, Mixed monotone property, Ordered set, Partially ordered set, Rational contractive.

Introduction:

Banach[1] contraction theorem is one of the most essential methods in the study of nonlinear problems in analysis. Therefore, various generalizations of the Banach contraction theorem are obtained either by expanding the structure of the ambient space[2,3]or by weakening the contractive maps.

Fixed point theorems are being developed for use in partially ordered metric space (P.O.M.S) by Ran and Reurings[4], obtained analogue of Banach's theorem on (P.O.M.S). After this paper, numerous paper has been published on (P.O.M.S)[5-7].Bhaskar and Lakshmikantham8work on the development of coupled (F, P) results "for mixed monotone operators" in (P.O.M.S). This same work also includes discussions of (F, P) theorems with rational expressions in(P.O.M.S)[8-12].

The main aim of this paper is to demonstrate that some coupled coincidence fixed point results satisfy a contractive condition of rational type.

Preliminaries

Let $(\mathcal{M},\leq)$ be a partially ordered set, $\mathcal{F}:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$and $T:\mathcal{M}\to\mathcal{M}$ be maps.
The set of coincidence fixed points (coupled fixed point) of $\mathcal{F}$ and T is denoted by the Couple $\mathrm{C}(\mathcal{F},T)$and couple Fix$(\mathcal{F},T)$ respectively" N *and* $(\mathfrak{R}^+)$ stand for natural numbers and (the positive real numbers), respectively.

Definition 1[4]: Let$(\mathcal{M},\leq)$is a (P.O.S) together with$(\mathcal{M},d)$ as a metric space. Then$(\mathcal{M},d,\leq)$ is said to be a (P.O.M.S).
$\mathcal{M}$ is called a partially (P.O.C.M.S) whenever $(\mathcal{M},d)$ is a complete metric space.

Definition 2[4]: Let M be a (P.O.S), then $m,n\in\mathcal{M}$ are said to be comparable if $m\leq n$ or $m\geq n$.

Definition 3[13]: The Two maps $\mathcal{F}$ and Thave monotone property (MP) if $\mathcal{F}$ is monotone T-increasing in n and is monotone$T$-decreasing in m, that is, for any $m,n\in\mathcal{M}$.

$$n_1,n_2\in\mathcal{M}, Tn_1\leq Tn_2 \Rightarrow \mathcal{F}(n_1,m)\leq \mathcal{F}(n_2,m).$$

and

$$m_1,m_2\in\mathcal{M}, Tm_1\leq Tm_2 \Rightarrow \mathcal{F}(n,m_1)\geq \mathcal{F}(n,m_2).$$

If $T=I_d${identity map} this yields definition[8]

Definition 4[13]: An element$(n,m)\in\mathcal{M}\times\mathcal{M}$is called a coupled coincidence fixed point of the maps$\mathcal{F}:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$and $T:\mathcal{M}\to\mathcal{M}$ if

$$\mathcal{F}(n,m)=Tn,\ \ \mathcal{F}(m,n)=Tm.$$

If T=I_d{identity map}this yields definition[8]

Definition 5[13]: Let $\mathcal{M}$ be a nonempty set $\mathcal{F}$ and $T$are commutative if for all $m, n \in \mathcal{M}$,

$$\mathcal{F}(Tn, Tm) = T(\mathcal{F}(n, m)).$$

Definition 6[14]: The function $\psi: [0, \infty) \to [0, \infty)$ is called an altering distance function if satisfying the following conditions:

i- ψ is a continuous
ii- ii -ψnondecreasing function
iii-$\psi(t) = 0 \leftrightarrow t = 0$, and

$\phi: [0, \infty) \to [0, \infty)$ satisfying the following conditions:

i- ϕ is lower semi-continuous,
ii- $\phi(t) = 0 \leftrightarrow t = 0$.

Main result

Lemma1: Suppose that $(\mathcal{M}, d)$be a (P.O.M.S) and let $T: \mathcal{M} \to \mathcal{M}$ be a mapping and $\{Tq_n\}$ is a sequence in $\mathcal{M} \ni \lim_{n\to\infty} d(Tq_n, Tq_{n-1}) = 0$. If$\{Tq_n\}$is not a Cauchy sequence that is $\exists\, \varepsilon > 0$ for each τ in N and the sequences and$\{n(\tau)\}$with$n(\tau) > m(\tau) > \tau \ni$
$d(Tq_{m(\tau)}, Tq_{n(\tau)}) \geq \varepsilon$,
then$d(Tq_{m(\tau)}, Tq_{n(\tau)-1}) < \varepsilon$ and

(i) $\lim_{\tau\to\infty} d(Tq_{m(\tau)}, Tq_{n(\tau)}) = \varepsilon$
(ii) $\lim_{\tau\to\infty} d(Tq_{m(\tau)-1}, Tq_{n(\tau)-1}) = \varepsilon$
(iii) $\lim_{\tau\to\infty} d(Tq_{m(\tau)-1}, Tq_{n(\tau)}) = \varepsilon$
(iv) $\lim_{\tau\to\infty} d(Tq_{m(\tau)}, Tq_{n(\tau)-1}) = \varepsilon$

Proof: To show that the sequence$\{Tq_n\}$ is a Cauchy in$(\mathcal{M}, d)$. Let us assume otherwise then $\exists\varepsilon > 0$for each τ in $N \exists\, n(\tau) > m(\tau) > \tau$

$$d(Tq_{m(\tau)}, Tq_{n(\tau)}) \geq \varepsilon. \quad 1$$

Farther, It is possible to find $n(\tau)$in such a way that it is the smallest integer with $n(\tau) > m(\tau)$and satisfying

$$d(Tq_{m(\tau)}, Tq_{n(\tau)-1}) < \varepsilon. \quad 2$$

Now, from the above inequalities (1) and (2), it follows that,

$$\varepsilon \leq d(Tq_{m(\tau)}, Tq_{n(\tau)}) \leq d(Tq_{m(\tau)}, Tq_{n(\tau)-1}) + d(Tq_{n(\tau)-1}, Tq_{n(\tau)})$$

$$\varepsilon \leq d(Tq_{m(\tau)}, Tq_{n(\tau)}) \leq \varepsilon + d(Tq_{n(\tau)-1}, Tq_{n(\ell\tau)})$$

Letting $\tau \to \infty$ in the above inequality implies

$$\lim_{\tau\to\infty} d(Tq_{m(\tau)}, Tq_{n(\tau)}) = \varepsilon \quad 3$$

Again,

$$d(Tq_{m(\tau)-1}, Tq_{n(\tau)-1}) \leq d(Tq_{m(\tau)-1}, Tq_{m(\tau)}) + d(Tq_{m(\tau)}, Tq_{n(\tau)}) + d(Tq_{n(\tau)}, Tq_{n(\tau)-1})$$

and

$$d(Tq_{m(\tau)}, Tq_{n(\tau)}) \leq d(Tq_{m(\tau)}, Tq_{m(\tau)-1}) + d(Tq_{m(\tau)-1}, Tq_{n(\tau)-1}) + d(Tq_{n(\tau)-1}, Tq_{n(\tau)})$$

Letting $\tau \to \infty$ in the previous inequality and using (3), implies

$$\lim_{\tau\to\infty} d(Tq_{m(\tau)-1}, Tq_{n(\tau)-1}) = \varepsilon \quad 4$$

Again,

$$d(Tq_{m(\tau)-1}, Tq_{n(\tau)}) \leq d(Tq_{m(\tau)-1}, Tq_{n(\tau)-1}) + d(Tq_{n(\tau)-1}, Tq_{n(\tau)})$$

And

$$d(Tq_{m(\tau)-1}, Tq_{n(\tau)-1}) \leq d(Tq_{m(\tau)-1}, Tq_{n(\tau)}) + d(Tq_{n(\tau)}, Tq_{n(\tau)-1})$$

Letting $\tau \to \infty$ in the above inequality and using (4) it follows that

$$\lim_{\tau\to\infty} d(Tq_{m(\tau)-1}, Tq_{n(\tau)}) = \varepsilon \quad 5$$

Again,

$$d(Tq_{m(\tau)}, Tq_{n(\tau)-1}) \leq d(Tq_{m(\tau)}, Tq_{n(\tau)}) + d(Tq_{n(\tau)}, Tq_{n(\tau)-1})$$

And

$$d(Tq_{m(\tau)}, Tq_{n(\tau)}) \leq d(Tq_{m(\tau)}, Tq_{n(\tau)-1}) + d(Tq_{n(\tau)-1}, Tq_{n(\tau)})$$

Making $\tau \to \infty$ in the preceding relations and using (3), implies

$$\lim_{\tau\to\infty} d(Tq_{m(\tau)}, Tq_{n(\tau)-1}) = \varepsilon. \quad 6$$

Theorem 1: Let$(\mathcal{M}, d, \leq)$ be a (P.O.C.M.S) and suppose that $\mathcal{F}: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, $T: \mathcal{M} \to \mathcal{M}$.Satisfying the following condition

$$\psi(d(\mathcal{F}(m, n), \mathcal{F}(t, s))) \leq \psi(M_d(m, n, t, s)) - \phi(M_d(m, n, t, s)) + LN_d(m, n, t, s) \quad 7$$

For all$m, n, t, s \in M$with$Tm \geq Tt$and$Tn \leq Ts$ where$\phi \in \Phi$, $\psi \in \Psi$, $L \geq 0$, and

$$M_d(m, n, t, s) = Max\left\{\frac{d(Tm, \mathcal{F}(m, n))d(Tt, \mathcal{F}(t, s))}{d(Tm, Tt) + d(Tt, \mathcal{F}(m, n)) + d(Tm, \mathcal{F}(t, s))}, d(Tm, Tt)\right\}$$

$$N_d(m,n,t,s) = Min\begin{Bmatrix} d(Tm,\mathcal{F}(m,n)), d(Tt,\mathcal{F}(t,s)), \\ d(Tt,\mathcal{F}(m,n)), d(Tm,\mathcal{F}(t,s)) \end{Bmatrix}$$

So, assume $\mathcal{F}$ and T are continuous maps and satisfied (MP) property. T commutes with $\mathcal{F}$ and $\mathcal{F}(\mathcal{M}\times\mathcal{M}) \subseteq T(\mathcal{M})$. If $\exists$ a point $(q_0,p_0) \in \mathcal{M}\times\mathcal{M} \ni Tq_0 \le \mathcal{F}(q_0,p_0)$ and $Tp_0 \le \mathcal{F}(q_0,p_0)$ then $COC(\mathcal{F},T) \ne \emptyset$.

Proof: From the given hypothesis $\exists\ (q_0,p_0)\mathcal{M}\times\mathcal{M}$ such that $Tq_0 \le \mathcal{F}(q_0,p_0)$ and $Tp_0 \le \mathcal{F}(q_0,p_0)$. Since $F(\mathcal{M}\times\mathcal{M}) \subseteq T((\mathcal{M})\exists(q_1,p_1) \in \mathcal{M}\times\mathcal{M}$ such that $Tq_0 \le \mathcal{F}((q_0,p_0)$ and $Tp_0 \le \mathcal{F}(q_0,p_0)$, then $Tq_0 \le \mathcal{F}(q_0,p_0) = Tq_1$ and $Tp_0 \le \mathcal{F}(q_0,p_0) = Tp_1$. Similarly, there exists $(q_2,p_2) \in \mathcal{M}\times\mathcal{M}$ such that $Tq_2 = \mathcal{F}(q_1,p_1)$ and $Tp_2 = \mathcal{F}(p_1,q_1)$. Since $\mathcal{F}$ and T-satisfied (MP) property.

$Tq_1 \le \mathcal{F}(q_0,p_0) \le \mathcal{F}(q_0,p_1) \le \mathcal{F}(q_1,p_1) = Tq_2$ and $Tp_2 \le \mathcal{F}(q_1,p_1) \le \mathcal{F}(p_0,q_1) \le \mathcal{F}(p_0,q_0) = Tp_1$.

Continuing that way leads to the sequences $\{q_n\}$ and $\{p_n\}$ in $\mathcal{M}$ such that for all $n \in \mathbb{N}$,

$Tq_{n+1} = \mathcal{F}(q_n,p_n)$ and $Tp_{n+1} = \mathcal{F}(p_n,q_n)$, 8

for which

$$Tq_0 \le Tq_1 \le Tq_2 \le Tq_3 \dots\dots \le Tq_n \le Tq_{n+1} \dots\dots$$
$$Tp_0 \ge Tp_1 \ge Tp_2 \ge Tp_3 \dots\dots \ge Tp_n \ge Tp_{n+1} \dots\dots$$

If there exists $n_0 \in N$ such that $Tq_{n_0+1} = Tq_{n_0}$ and $Tp_{n_0+1} = Tp_{n_0}$, then $Tq_{n_0} = \mathcal{F}(q_{n_0},p_{n_0})$ and $Tp_{n_0} = F(p_{n_0},q_{n_0})$. Hence $(q_{n_0},p_{n_0}) \in$ Couple$C(\mathcal{F},T)$, let $Tq_{n+1} \ne Tq_\tau$ and $Tp_{n+1} \ne Tn$ for all $n \in N$.

From condition (7) and equality (8), it follows that,

$$\psi(d(Tq_n,Tq_{n+1}) = \psi\left(d\big(\mathcal{F}(q_{n-1},p_{n-1}),\mathcal{F}(q_n,p_n)\big)\right) \le \psi\big(M_d(q_n,p_n,q_{n-1},p_{n-1})\big) - \phi\big(M_d(q_n,p_n,q_{n-1},p_{n-1})\big) + LN_d(q_n,p_n,q_{n-1},p_{n-1})$$

Where

$$M_d(q_n,p_n,q_{n-1},p_{n-1}) = Max\left\{\frac{d(Tq_n,\mathcal{F}(q_n,p_n))d(Tq_{n-1},\mathcal{F}(q_{n-1},p_{n-1}))}{d(Tq_n,Tq_{n-1})+d(Tq_{n-1},\mathcal{F}(q_n,p_n))+d(Tq_n,\mathcal{F}(q_{n-1},p_{n-1}))}, d(Tq_n,Tq_{n-1})\right\}$$

$$= Max\left\{\frac{d(Tq_n,Tq_{n+1})d(Tq_{n-1},Tq_n)}{d(Tq_n,Tq_{n-1})+d(Tq_{n+1},Tq_{n-1})+d(Tq_n,Tq_n)}, d(Tq_n,Tq_{n-1})\right\}$$

Since $d(Tq_n,Tq_{n+1}) \le d(Tq_n,Tq_{n-1}) + d(Tq_{n-1},Tq_{n+1})$, it leads to the conclusion,

$$M_d(q_n,p_n,q_{n-1},p_{n-1}) = Max\{d(Tq_n,Tq_{n-1}), d(Tq_n,Tq_{n-1})\} = d(Tq_n,Tq_{n-1})$$

And

$$N_d = Min\begin{Bmatrix} d(Tq_n,\mathcal{F}(q_n,p_n))d(Tq_{n-1},\mathcal{F}(q_{n-1},p_{n-1})), \\ d(Tq_{n-1},\mathcal{F}(q_n,p_n)) + d(Tq_n,\mathcal{F}(q_{n-1},p_{n-1})) \end{Bmatrix} = Min\begin{Bmatrix} d(Tq_n,Tq_{n+1})d(Tq_{n-1},Tq_n), \\ d(Tq_{n-1},Tq_{n+1}) + d(Tq_n,Tq_n) \end{Bmatrix}$$

So,

$\psi(d(Tq_n,Tq_{n+1})) \le \psi d(Tq_n,Tq_{n-1})) \le \phi d(Tq_n,Tq_{n-1}) < d(Tq_n,Tq_{n-1})$ 9

Similarly, it yields,

$\psi(d(Tp_n,Tp_{n+1})) \le \psi\big(d(Tp_n,Tp_{n-1})\big) \le \phi\big(d(Tp_n,Tp_{n-1})\big) < d(Tp_n,Tp_{n-1})$ 10

Therefore, the sequences $\{(d(Tq_n,Tq_{n+1})\}$ and $\{(Tp_n,Tp_{n+1})\}$ are decreasing sequences in $\mathfrak{R}^+$, which is bounded below. So, there exists $\gamma_1,\gamma_2 > 0$. such that

$$\lim_{\tau\to\infty} d(Tq_n,Tq_{n+1}) = \gamma_1 \quad ,$$

$$\lim_{\tau\to\infty}(Tp_n,Tp_{n+1}) = \gamma_2. \qquad 11$$

Now to show $\gamma_1,\gamma_2 = 0$. Suppose that $\gamma_1,\gamma_2 > 0$. By taking the limit of the supremum in the inequalities (9) and (10) as $n\to\infty$, this implies

$$\psi(\gamma_1) \le \psi(\gamma_1) - \phi(\gamma_1) < \psi(\gamma_1) \text{ and } \psi(\gamma_2) \le \psi(\gamma_2) - \phi(\gamma_2) < \psi(\gamma_2).$$

Which is a contradiction. To our assumption $\gamma_1,\gamma_2 > 0$ on Hence, it is concluded that, so

$$\lim_{n\to\infty} d(Tq_n,Tq_{n+1}) = 0 \qquad 12$$

It is possible to demonstrate that in a similar way

$$\lim_{n\to\infty}(Tp_n,Tp_{n+1}) = 0 \qquad 13$$

Now, to show that $\{Tq_n\}$ is a Cauchy sequence in $(\mathcal{M},d)$. Let us assume otherwise. Then there exist an $\delta > 0$ and subsequences $\{Tq_{m(\tau)}\}$ and $\{Tq_{n(\tau)}\}$ of $\{Tq_n\}$, $n(\tau) > m(\tau) > \tau$ such that $d\big(Tq_{m(\tau)},Tq_{n(\tau)}\big) \ge \delta$, where is $n(\tau)$ a smallest positive integer satisfies that

$$d\big(Tq_{m(\tau)},Tq_{n(\tau)-1}\big) < \delta$$

Now from condition (7) Let $= q_{m(\tau)-1}$, $n = p_{m(\tau)-1}$, $t = p_{n(\tau)-1}$, $s = p_{n(\tau)-1}$. Then

$$M_d(q_{m(\tau)-1}, p_{m(\tau)-1}, q_{n(\tau)-1}, p_{n(\tau)-1})$$
$$= max\left\{d(Tq_{m(\tau)-1}, Tq_{n(\tau)-1}), \frac{d\left(Tq_{m(\tau)-1}, \mathcal{F}(q_{m(\tau)-1}, p_{m(\tau)-1})\right) d\left(Tq_{n(\tau)-1}, \mathcal{F}(q_{n(\tau)-1}, p_{n(\tau)-1})\right)}{d(Tq_{m(\tau)-1}, Tq_{n(\tau)-1}) + d\left(Tq_{n(\tau)-1}, F(q_{m(\tau)-1}, p_{m(\tau)-1})\right) + d\left(Tq_{m(\tau)-1}, F(q_{n(\tau)-1}, p_{n(\tau)-1})\right)}\right\}$$

$$= Max\left\{\frac{d(Tq_{m(\tau)-1}, Tq_{m(\tau)})d(Tq_{n(\tau)-1}, Tq_{n(\tau)})}{d(Tq_{m(\tau)-1}, Tq_{n(\tau)-1}) + d(Tq_{n(\tau)-1}, Tq_{m(\tau)}) + d(Tq_{m(\tau)-1}, Tq_{n(\tau)})}, d(Tq_{m(\tau)-1}, Tq_{n(\tau)-1})\right\} \quad 14$$

And

$$N_d(q_{m(\tau)-1}, p_{m(\tau)-1}, q_{n(\tau)-1}, p_{n(\tau)-1}) = Min\left\{\begin{array}{c} d\left(Tq_{m(\tau)-1}, \mathcal{F}(q_{m(\tau)-1}, p_{m(\tau)-1})\right), \\ d\left(Tq_{n(\tau)-1}, \mathcal{F}(q_{n(\tau)-1}, p_{n(\tau)-1})\right), \\ d\left(Tq_{n(\tau)-1}, \mathcal{F}(q_{m(\tau)-1}, p_{m(\tau)-1})\right) + \\ d\left(Tq_{m(\tau)-1}, \mathcal{F}(q_{n(\tau)-1}, p_{n(\tau)-1})\right) \end{array}\right\}$$
$$= Min\left\{\begin{array}{c} d(Tq_{m(\tau)-1}, Tq_{m(\tau)})d(Tq_{n(\tau)-1}, Tq_{n(\tau)}), \\ d(Tq_{n(\tau)-1}, Tq_{m(\tau)}) + d(Tq_{m(\tau)-1}, Tq_{n(\tau)}) \end{array}\right\} \quad 15$$

Letting $\tau \to \infty$in (14), (15), and by using lemma 1 and by (12). It can be concluded that,

$$M_d = Max\{0, \delta\} = \delta \quad 16$$
$$N_d = Min\{0,0, \delta, \delta\} = 0 \quad 17$$

Since$n(\tau) > m(\tau)$and$\{Tq_{m(\tau)}\} < \{Tq_{n(\tau)}\}$ applying condition (7) and using (14), (15), it follows that

$$\psi\left(d(Tq_{m(\tau)}, Tp_{n(\tau)})\right) = \psi\left(d\left(\mathcal{F}(q_{m(\tau)-1}, p_{m(\tau)-1}), \mathcal{F}(q_{n(\tau)-1}, p_{n(\tau)-1})\right)\right)$$
$$\leq \psi\left(M_d(q_{m(\tau)-1}, p_{m(\tau)-1}, q_{n(\tau)-1}, p_{n(\tau)-1})\right) - \phi\left(M_d(q_{m(\tau)-1}, p_{m(\tau)-1}, q_{n(\tau)-1}, p_{n(\tau)-1})\right) + LN_d(q_{m(\tau)-1}, p_{m(\tau)-1}, q_{n(\tau)-1}, p_{n(\tau)-1})$$

Letting limit supremum in both sides of the previous inequality, lemma 1, (16), (17) and the property of$\psi$and the continuity ofϕ, lead to the conclusion

$$\psi(\delta) \leq \psi(\delta) - \phi(\delta) < \psi(\delta).$$

Contradiction to our assumption$\delta > 0$. The sequence$\{Tq_n\}$is a Cauchy sequence in$\mathcal{M}$. Similarly, It is possible to demonstrate that the sequence$\{Tq_n\}$ is a Cauchy sequence in $\mathcal{M}$. So, by the completeness of $\mathcal{M}$,∃ points$u, v \in \mathcal{M}$such that$Tq_n \to u$,$Tp_n \to v$ as $n \to \infty$. Now from the commutativity of $T$and$\mathcal{F}$, it implies

$$T(Tq_{n+1}) = T(\mathcal{F}(q_n, p_n)) = \mathcal{F}(Tq_n, Tp_n)$$
$$T(Tp_{n+1}) = T(\mathcal{F}(p_n, q_n)) = \mathcal{F}(Tp_n, Tq_n) \quad 18$$

Letting $n \to \infty$ in (18) and from the continuity of $T$and $\mathcal{F}$, it follows

$$Tu = \lim_{n\to\infty} T(Tq_{n+1}) = \lim_{n\to\infty} \mathcal{F}(Tq_n, Tp_n) = \mathcal{F}\left(\lim_{n\to\infty} Tq_n, \lim_{n\to\infty} Tp_n\right) = \mathcal{F}(u, v)$$
$$Tv = \lim_{n\to\infty} T(Tp_{n+1}) = \lim_{n\to\infty} \mathcal{F}(Tp_n, Tq_n) = \mathcal{F}\left(\lim_{n\to\infty} Tp_n, \lim_{n\to\infty} Tq_n\right) = \mathcal{F}(v, u)$$

Therefore, $\mathcal{F}(u, v) = Tu$ and $\mathcal{F}(v, u) = Tv$, i.e., then $Couple\,C(\mathcal{F}, T) \neq \emptyset$.

Theorem 2: Let$(M, d, \leq)$be a (P.O.C.M.S) and suppose that $\mathcal{F}: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, $T: \mathcal{M} \to \mathcal{M}$. satisfying the following condition

$$\psi(d(\mathcal{F}(m, n), \mathcal{F}(t, s))) \leq \psi(M_d(m, n, t, s)) - \phi(M_d(m, n, t, s)) + LN_d(m, n, t, s) \quad 19$$

For all$m, n, t, s \in M$ with$Tm \geq Tt$,$Tn \leq Ts$and $Tm \neq Tt$ where $\phi \in \Phi$, $\psi \in \Psi, L \geq 0$,

$$M_d(m, n, t, s) = Max\left\{\begin{array}{c} \frac{d(Tm, \mathcal{F}(m,n))d(Tt, \mathcal{F}(t,s))}{d(Tm,Tt)}, \\ \frac{d(Tm,\mathcal{F}(m,n))[1+d(Tt,\mathcal{F}(t,s)]}{1+d(Tm,Tt)}, \\ d(Tm, Tt) \end{array}\right\}$$
$$N_d(m, n, t, s) = Min\left\{\begin{array}{c} d(Tm, \mathcal{F}(m, n)), d(Tt, \mathcal{F}(t, s)), \\ d(Tt, \mathcal{F}(m, n)), d(Tm, \mathcal{F}(t, s)) \end{array}\right\}$$

So, assume $\mathcal{F}$and$T$are continuous maps and satisfied (MP) property. Tcommutes with$\mathcal{F}$and$\mathcal{F}(\mathcal{M} \times \mathcal{M}) \subseteq T(\mathcal{M})$. If there exists a point $(q_0, p_0) \in \mathcal{M} \times \mathcal{M}$such that$Tq_0 \leq \mathcal{F}(q_0, p_0)$ and $Tp_0 \leq \mathcal{F}(p_0, q_0)$, then $Couple$ $C(\mathcal{F}, T) \neq \emptyset$

Proof.: From Th1andthe given hypothesis, it is obtained two sequences $\{q_n\}$and$\{p_n\}$in Msuch that for all $n \in N$,$Tq_{n+1} = F(q_n, p_n)$, $Tp_{n+1} =$

$F(p_n, q_n)$
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let $Tq_{n+1} \neq Tq_n$ and $Tp_{n+1} \neq Tp_n$ for all $n \in N$.
From condition (19) and equality (20), it follows that,

$\psi(d(Tq_n, Tq_{n+1}))$
$= \psi\left(d\left(\mathcal{F}(q_{n-1}, p_{n-1}), \mathcal{F}(q_n, p_n)\right)\right)$
$\leq \psi\left(M_d(q_n, p_n, q_{n-1}, p_{n-1})\right) - \phi\left(M_d(q_n, p_n, q_{n-1}, p_{n-1})\right) + LN_d(q_n, p_n, q_{n-1}, p_{n-1})$, Where

$$M_d(q_n, p_n, q_{n-1}, p_{n-1}) = Max\left\{\begin{array}{c}\frac{d\left(Tq_n, \mathcal{F}(q_n, p_n)\right) d\left(Tq_{n-1}, \mathcal{F}(q_{n-1}, p_{n-1})\right)}{d(Tq_n, Tq_{n-1})}, d(Tq_n, Tq_{n-1}), \\ \frac{d\left(Tq_n, \mathcal{F}(q_n, p_n)\right)\left[1 + d\left(Tq_{n-1}, \mathcal{F}(q_{n-1}, p_{n-1})\right)\right]}{1 + d(Tq_n, Tq_{n-1})}\end{array}\right\}$$

$$= Max\left\{\begin{array}{c}\frac{d(Tq_n, Tq_{n+1}) d(Tq_{n-1}, Tq_n)}{d(Tq_n, Tq_{n-1})}, \\ \frac{d(Tq_n, Tq_{n+1})[1 + d(Tq_{n-1}, Tq_n)]}{1 + d(Tq_n, Tq_{n-1})}, \\ d(Tq_n, Tq_{n-1})\end{array}\right\} = Max\{d(Tq_n, Tq_{n+1}), d(Tq_n, Tq_{n-1})\}$$

And

$$N_d = Min\left\{\begin{array}{l}d\left(Tq_n, \mathcal{F}(q_n, p_n)\right) d\left(Tq_{n-1}, \mathcal{F}(q_{n-1}, p_{n-1})\right), \\ d\left(Tq_{n-1}, \mathcal{F}(q_n, p_n)\right), d\left(Tq_n, \mathcal{F}(q_{n-1}, p_{n-1})\right)\end{array}\right\}$$
$$= Min\left\{\begin{array}{l}d(Tq_n, Tq_{n+1}), d(Tq_{n-1}, Tq_n), \\ d(Tq_{n-1}, Tq_{n+1}), d(Tq_n, Tq_n)\end{array}\right\} = 0$$

Therefore
$\psi(d(Tq_n, Tq_{n+1}))$
$\leq \psi(Max\{d(Tq_n, Tq_{n+1}), d(Tq_n, Tq_{n-1})\})$
$-\phi(Max\{d(Tq_n, Tq_{n+1}), d(Tq_n, Tq_{n-1})\})$
21
There are two cases, If $Max\{d(Tq_n, Tq_{n+1}), d(Tq_n, Tq_{n-1})\} = d(Tq_n, Tq_{n+1})$, then $\psi\left(d(Tq_n, Tq_{n+1})\right) \leq \psi d(Tq_n, Tq_{n+1}) - \phi\left(d(Tq_n, Tq_{n+1})\right)$, which is a contradiction since $\phi(t) < \psi(t)$.
Hence $Max\{d(Tq_n, Tq_{n+1}), d(Tq_n, Tq_{n-1})\} = d(Tq_n, Tq_{n-1})$. Therefore, the sequences $\{d(Tq_n, Tq_{n+1})\}$ is a decreasing sequence in $\mathfrak{R}^+$ and has a lower bound. So, $\exists\ \lambda \geq 0$, $\ni$

$\lim_{n\to\infty} d(Tq_n, Tq_{n+1}) = \lambda$
22

Now to show that $\lambda = 0$. Assume that $\lambda > 0$. By Letting the limit of the sup in the relation (21) as $n \to \infty$, it follows that

$$\psi(\lambda) \leq \psi(\lambda) - \phi(\lambda) < \psi(\lambda).$$

Which is a contradiction. Hence, $\lambda = 0$, that is,
$\lim_{n\to\infty} d(Tq_n, Tq_{n+1}) = 0$
23
Similarly, it can be concluded that,
$\lim_{n\to\infty} d(Tp_n, Tp_{n+1}) = 0$
24
Now, to show that $\{Tq_n\}$ is Cauchy sequence in $(\mathcal{M}, d)$. Let us assume otherwise. Then by lemma1 $\exists\ \mathcal{E} > 0$, such that for each $\tau \in N$ and subsequences $\{Tq_{m(\tau)}\}$ and $\{Tq_{n(\tau)}\}$ of $\{Tq_n\}$, $n(\tau) > m(\tau) > \tau$ such that $d\left(Tq_{m(\tau)}, Tq_\tau\right) \geq \mathcal{E}$, where is $n(\tau)$ a smallest positive integer satisfies that

$$d\left(Tq_{m(\tau)}, Tq_{n(\tau)-1}\right) < E$$

Now from condition (19) Let $= q_{m(\tau)-1}$, $n = p_{m(\tau)-1}$, $t = q_{n(\tau)-1}$, $s = p_{n(\tau)-1}$.
It follows that,
$\psi\left(d\left(Tq_{m(\tau)}, Tp_{n(\tau)}\right)\right)$
$= \psi\left(d\left(\mathcal{F}\left(q_{m(\tau)-1}, p_{m(\tau)-1}\right), \mathcal{F}\left(q_{n(\tau)-1}, p_{n(\tau)-1}\right)\right)\right)$
$\leq \psi\left(M_d\left(q_{m(\tau)-1}, p_{m(\tau)-1}, q_{n(\tau)-1}, p_{n(\tau)-1}\right)\right)$
$-\phi\left(M_d\left(q_{m(\tau)-1}, p_{m(\tau)-1}, q_{n(\tau)-1}, p_{n(\tau)-1}\right)\right)$

$+LN_d\left(q_{m(\tau)-1}, p_{m(\tau)-1}, q_{n(\tau)-1}, p_{n(\tau)-1}\right)$
25
Where

$$M_d\left(q_{m(\tau)-1}, p_{m(\tau)-1}, q_{n(\tau)-1}, p_{n(\tau)-1}\right) = Max\left\{\begin{array}{c}\frac{d\left(Tq_{m(\tau)-1}, \mathcal{F}\left(q_{m(\tau)-1}, p_{m(\tau)-1}\right)\right)\left[1 + d\left(Tq_{n(\tau)-1}, \mathcal{F}\left(q_{n(\tau)-1}, p_{n(\tau)-1}\right)\right)\right]}{1 + d\left(Tq_{m(\tau)-1}, Tq_{n(\tau)-1}\right)}, \\ \frac{d\left(Tq_{m(\tau)-1}, \mathcal{F}\left(q_{m(\tau)-1}, q_{m(\tau)-1}\right)\right) d\left(Tq_{n(\tau)-1}, \mathcal{F}\left(q_{n(\tau)-1}, p_{n(\tau)-1}\right)\right)}{d\left(Tq_{m(\tau)-1}, Tq_{n(\tau)-1}\right)}, d\left(Tq_{m(\tau)-1}, Tq_{n(\tau)-1}\right)\end{array}\right\}$$

$$= Max\left\{\begin{array}{c}\dfrac{d(Tq_{m(\tau)-1},Tq_{m(\tau)})[1+d(Tq_{n(\tau)-1},Tq_{n(\tau)})]}{[1+d(Tq_{m(\tau)-1},Tq_{n(\tau)-1})]},\\ \dfrac{d(Tq_{m(\tau)-1},Tq_{m(\tau)})d(Tq_{n(\tau)-1},Tq_{n(\tau)})}{d(Tq_{m(\tau)-1},Tq_{n(\tau)-1})}, d(Tq_{m(\tau)-1},Tq_{n(\tau)-1})\end{array}\right\}$$

And

$$N_d(q_{m(\tau)-1},p_{m(\tau)-1},q_{n(\tau)-1},p_{n(\tau)-1}) = Min\left\{\begin{array}{c} d\left(Tq_{m(\tau)-1},\mathcal{F}(q_{m(\tau)-1},q_{m(\tau)-1})\right),\\ d\left(Tq_{n(\tau)-1},\mathcal{F}(q_{n(\tau)-1},p_{n(\tau)-1})\right),\\ d\left(Tq_{n(\tau)-1},\mathcal{F}(q_{m(\tau)-1},p_{m(\tau)-1})\right),\\ d\left(Tq_{m(\tau)-1},\mathcal{F}(q_{n(\tau)-1},p_{n(\tau)-1})\right)\end{array}\right\}$$

$$= Min\left\{\begin{array}{c} d(Tq_{m(\tau)-1},Tq_{m(\tau)}), d(Tq_{n(\tau)-1},Tq_{n(\tau)}),\\ d(Tq_{n(\tau)-1},Tq_{m(\tau)}), d(Tq_{m(\tau)-1},Tq_{n(\tau)})\end{array}\right\}$$

Making $\tau \to \infty$ in previous equalities it follows that

$$M_d = Max\{0,0,\varepsilon\} = \varepsilon \qquad 26$$
$$N_d = Min\{0,0,\varepsilon,\varepsilon\} = 0 \qquad 27$$

Since $n(\tau) > m(\tau)$ and $\{Tq_{m(\tau)}\} < \{Tq_{n(\tau)}\}$ applying (25), (26) و (27), lemma 1and Taking limit sup in both sides of the above inequality, and the property of ψ and the continuity of ϕ, it yields

$$\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon).$$

A contradiction. Hence $\{Tq_n\}$ is a Cauchy sequence in $\mathcal{M}$. Similarly, It can conclude that the sequence $\{Tq_n\}$ is a Cauchy in $\mathcal{M}$. So by the completeness of $\mathcal{M}$ $\exists$ a point $(u,v) \in \mathcal{M}\times\mathcal{M}$ $\ni$ $Tq_n \to u$, $Tp_n \to v$ as $n \to \infty$. Now from the commutativity of T and $\mathcal{F}$, implies

$$T(Tq_{n+1}) = T(\mathcal{F}(q_n,p_n)) = \mathcal{F}(Tq_n,Tp_n)$$
$$T(Tp_{n+1}) = T(\mathcal{F}(p_n,q_n)) = \mathcal{F}(Tp_n,Tq_n)$$
28

Letting $n \to \infty$ in (18) and from the continuity of T and $\mathcal{F}$. It can be concluded that,

$$\begin{aligned} Tu &= \lim_{n\to\infty} T(Tq_{n+1}) \\ &= \lim_{n\to\infty} \mathcal{F}(Tq_n,Tp_n) \\ &= \mathcal{F}\left(\lim_{n\to\infty} Tq_n, \lim_{n\to\infty} Tp_n\right) \\ &= \mathcal{F}(u,v) \end{aligned}$$

$$\begin{aligned} Tv &= \lim_{n\to\infty} T(Tp_{n+1}) \\ &= \lim_{n\to\infty} \mathcal{F}(Tp_n,Tq_n) \\ &= \mathcal{F}\left(\lim_{n\to\infty} Tp_n, \lim_{n\to\infty} Tq_n\right) \\ &= \mathcal{F}(v,u) \end{aligned}$$

Where

Therefore, $\mathcal{F}(u,v) = Tu$ and $\mathcal{F}(v,u) = Tv$ then $COC(\mathcal{F},T) \neq \emptyset$.

Theorem 3: In addition to the hypotheses of Theorem 1 suppose that for every $\exists, (q^{\propto},p^{\propto}) \in M\times M$ there exists $(u,w) \in M\times M$ such that $(F(u,w),F(w,u))$ is comparable to $(F(q,p),F(p,q))$ and $(F(q^{\propto},p^{\propto}),F(p^{\propto},q^{\propto}))$. Then $CoC(\mathcal{F},T)$ has at most one element.

Proof. From Theorem 1 that there exists at least a coupled coincidence point. Suppose that (q,p) and $(q^{\propto},p^{\propto})$ are belong to the set $COC(\mathcal{F},T)$, It is necessary to demonstrate $Tq = Tq^{\propto}$ and $Tp = Tp^{\propto}$.
since there exists $(u,w) \in M\times M$ such that $(F(u,w),F(w,u))$ is comparable to $(F(q,p),F(p,q))$ and $(F(q^{\propto},p^{\propto}),F(p^{\propto},q^{\propto}))$.
Put $u_0 = u$ and $w_0 = w$ from (8) it can be constructed as sequences $\{u_n\}$ and $\{w_n\}$ such that $Tu_{n+1} = F(u_n,w_n)$ and $Tw_{n+1} = F(w_n,u_n)$.
Further, set $q = q_0$, $p = p_0$, $q^{\propto} = q_0^{\propto}$, $p^{\propto} = p_0^{\propto}$ and, in the same way, define the sequence $\{Tq_n\}$, $\{Tp_n\}$, $\{Tq_n^{\propto}\}$ and $\{Tp_n^{\propto}\}$. Then it is easy to show that,

$$Tq_n \to F(q,p),\ Tp_n \to F(p,q),\ Tq_n^{\propto} \to F(q^{\propto},p^{\propto}),\ Tp_n^{\propto} \to F(p^{\propto},q^{\propto}) \text{ as } n\to\infty \text{ for all } n \ge 0.$$

Since $(F(q,p),F(p,q)) = (Tq_1,Tp_1) = (Tq,Tp)$ and $(F(u,w),F(w,u)) = (Tu_1,Tw_1)$ are comparable, then It is easy to show that $(Tq,Tp) \le (Tu_n,Tw_n)$, that is, $Tq \le Tu_n$ and $Tp \ge Tw_n$ for all $n \ge 0$.
Thus from condition (7), it follows

$$\begin{aligned}\psi(d(Tq,Tu_{n+1})) &= \psi(d(F(q,p),F(u_n,w_n))) \\ &\le \psi(M_d(q,p,u_n,w_n)) - \phi(M_d(q,p,u_n,w_n)) \\ &\quad + LN_d(q,p,u_n,w_n)\end{aligned}$$

$$M_d(q,p,u_n,w_n) = Max\left\{\frac{d(Tq,F(q,p))d(Tu_n,F(u_n,w_n)))}{d(Tq,Tu_n)+d(Tu_n,F(q,p,))+d(Tq,F(u_n,w_n))}, d(TqTu_n)\right\}$$
$$= Max\left\{\frac{d(Tq,Tq)d(Tu_n,Tu_{n-1})}{d(Tq,Tu_n)+d(Tu_n,Tq)+d(Tq,Tu_{n-1})}, d(Tq,Tu_n)\right\}$$
$$= Max\{d(Tq,Tu_n)\} = d(Tq,Tu_n)$$

And
$N_d(q,p,u_n,w_n)$
$$= Min\begin{Bmatrix} d(Tq,F(q,p)), d(Tu_n,F(u_n,w_n)), \\ d(Tu_n,F(q,p,)), d(Tq,F(u_n,w_n)) \end{Bmatrix}$$
$$= Min\begin{Bmatrix} d(Tq,Tq), d(Tu_n,Tu_{n-1}), \\ d(Tu_n,Tq), d(Tq,Tu_{n-1}) \end{Bmatrix} = 0$$

Hence
$$\psi(d(Tq,Tu_{n+1})) \le \psi(d(Tq,Tu_n)) - \phi(d(Tq,Tu_n)) < \psi(d(Tq,Tu_n)) \quad 29$$

Therefore $\{d(Tq,Tu_n)\}$is a decreasing sequence. Hence, there exists$\lambda \ge 0 \ \ni \lim_{n\to\infty} d(Tq,Tu_n) = \lambda$

Passing the upper limit in (29) as $n \to \infty$, it is obtained that $\psi(\lambda) \le \psi(\lambda) - \phi(\lambda) < \psi(\lambda)$.

A contradiction. Then, $\lambda = 0$. This implies
$$\lim_{n\to\infty} d(Tq,Tu_n) = 0 \quad 30$$

In a similar, way, it leads to the conclusion that
$$\lim_{n\to\infty} d(Tp,Tw_n) = 0 \quad 31$$

Following the same lines as the previous
$$\lim_{n\to\infty} d(Tq^{\propto},Tu_n) = \lim_{n\to\infty} d(Tp^{\propto},Tw_n) = 0 \quad 32$$

From the triangle inequality, (30), (31), and (32), lead to the conclusion
$$d(Tq,Tq^{\propto}) \le d(Tq,Tu_{n+1}) + d(Tu_{n+1},Tq^{\propto})$$
$$d(Tp,Tp^{\propto}) \le d(Tp,Tw_{n+1}) + d(Tw_{n+1},Tp^{\propto})$$

Taking $n \to \infty$in the above inequality, to obtain $d(Tq,Tq^*) = 0$ and $d(Tp,Tp^{\propto}) = 0$.
Thus, $Tq = Tq^{\propto}$and$Tp = Tp^{\propto}$.

Theorem 4: If the hypotheses of Theorem 2 and Theorem 3 hold then $Couple$C$(\mathcal{F},T)$has at most one element.

Proof. From the condition, 19it implies that
$$\psi(d(Tq,Tu_{n+1})) = \psi\left(d(F(q,p),F(u_n,w_n))\right)$$
$$\le \psi(M_d(q,p,u_n,w_n)) - \phi(M_d(q,p,u_n,w_n)) + LN_d(q,p,u_n,w_n)$$

Where
$M_d(q,p,u_n,w_n) =$
$$Max\begin{Bmatrix} \frac{d(Tq,F(q,p))d(Tu_n,F(u_n,w_n))}{d(Tq,Tu_n)}, d(Tq,Tu_n) \\ \frac{d(Tq,F(q,p))[1+d(Tu_n,F(u_n,w_n))]}{1+d(Tq,Tu_n)} \end{Bmatrix} =$$
$$Max\left\{\frac{d(Tq,Tq)d(Tu_n,Tu_{n-1})}{d(Tq,Tu_n)}, \frac{d(Tq,Tq)[1+d(Tu_n,Tu_{n-1})]}{1+d(Tq,Tu_n)}, d(Tq,Tu_n)\right\}$$
$$= Max\{d(Tq,Tu_n)\} = d(Tq,Tu_n)$$

And
$N_d(q,p,u_n,w_n)$
$$= Min\begin{Bmatrix} d(Tq,F(q,p)), d(Tu_n,F(u_n,w_n)), \\ d(Tu_n,F(q,p)), d(Tq,F(u_n,w_n)) \end{Bmatrix}$$
$$= Min\begin{Bmatrix} d(Tq,Tq), d(Tu_n,Tu_{n-1}), \\ d(Tu_n,Tq), d(Tq,Tu_{n-1}) \end{Bmatrix} = 0$$

Then,
$$\psi(d(Tq,Tu_n)) \le \psi(d(Tq,Tu_n)) - \phi(d(Tq,Tu_n)) < \psi(d(Tq,Tu_n))$$

Which is a contradiction.
Following the same lines as Theorem 3, it can show $Couple$ C$(\mathcal{F},T)$has at most one element.

Corollary 1: Let$(\mathcal{M},d,\le)$be a (P.O.C.M.S) and suppose that $\mathcal{F}: \mathcal{M}\times\mathcal{M} \to \mathcal{M}$, Satisfying the following condition
$$\psi(d(\mathcal{F}(m,n),\mathcal{F}(t,s))) \le \psi(M_d(m,n,t,s)) - \phi(M_d(m,n,t,s)) + LN_d(m,n,t,s)$$
For all$m,n,t,s \in M$ $with m \ge t$and$n \le s$ where$\phi \in \Phi,\ \psi \in \Psi, L \ge 0$,
$$M_d(m,n,t,s) = Max\left\{\frac{d(m,\mathcal{F}(m,n))d(t,\mathcal{F}(t,s))}{d(m,t)+d(t,\mathcal{F}(m,n))+d(m,\mathcal{F}(t,s))}, d(m,t)\right\}$$
$$N_d(m,n,t,s) = Min\begin{Bmatrix} d(m,\mathcal{F}(m,n)), d(t,\mathcal{F}(t,s)), \\ d(t,\mathcal{F}(m,n)), d(m,\mathcal{F}(t,s)) \end{Bmatrix}$$with

So, assume $\mathcal{F}$is continuous mapping and satisfied (MP) property. If there exists a point $(q_0,p_0) \in \mathcal{M}\times\mathcal{M}$such that$q_0 \le \mathcal{F}(q_0,p_0)$ and$p_0 \le \mathcal{F}(p_0,q_0)$, then$Couple$ Fix$(F) \ne \emptyset$.

Proof. Take $Tm = I_d$ (the identity map) in Theorem 1, and Theorem 3.

Corollary2: Let$(\mathcal{M},d,\le)$ be a (P.O.C.M.S) and suppose that $\mathcal{F}: \mathcal{M}\times\mathcal{M} \to \mathcal{M}$, $T: \mathcal{M} \to \mathcal{M}$. Satisfying the following condition
$$d(\mathcal{F}(m,n),\mathcal{F}(t,s)) \le kM_d(m,n,t,s)$$
For all$m,n,t,s \in M$ with$Tm \ge Tt$and$Tn \le Ts$
$$M_d(m,n,t,s) = Max\left\{\frac{d(Tm,\mathcal{F}(m,n))d(Tt,\mathcal{F}(t,s))}{d(Tm,Tt)+d(Tt,\mathcal{F}(m,n))+d(Tm,\mathcal{F}(t,s))}, d(Tm,Tt)\right\}$$

$N_d(m,n,t,s) =$
$Min\begin{Bmatrix} d(Tm,\mathcal{F}(m,n)), d(Tt,\mathcal{F}(t,s)), \\ d(Tt,\mathcal{F}(m,n)), d(Tm,\mathcal{F}(t,s)) \end{Bmatrix}$
So, assume $\mathcal{F}$ and T are continuous mappings and satisfied (MP) property. T commutes with $\mathcal{F}$ and $\mathcal{F}(\mathcal{M}\times\mathcal{M}) \subseteq T(\mathcal{M})$. If $\exists$ a point $(q_0,p_0) \in \mathcal{M}\times\mathcal{M}$ such that $Tq_0 \le \mathcal{F}(q_0,p_0)$ and $Tp_0 \le \mathcal{F}(p_0,q_0)$, then $COC(\mathcal{F},T) \neq \emptyset$.

Proof. Take $L = 0$, $\psi(t) = t$ and $\phi(t) = (1-k)t$ and where $k \in [0,1)$ in Theorem1 and Theorem3.

Corollary3: Let $(\mathcal{M},d,\le)$ be a (P.O.C.M.S) and suppose that $\mathcal{F}:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$, $T:\mathcal{M}\to\mathcal{M}$. Satisfying the following condition

$d(\mathcal{F}(m,n),\mathcal{F}(t,s)) \le$
$\alpha\frac{d(Tm,\mathcal{F}(m,n))d(Tt,\mathcal{F}(t,s))}{d(Tm,Tt)+d(Tt,\mathcal{F}(m,n))+d(Tm,\mathcal{F}(t,s))} + \beta d(Tm,Tt)$
For all $m,n,t,s \in M$ with $Tm \ge Tt$ and $Tn \le Ts$.
So, assume $\mathcal{F}$ and T are continuous mappings and satisfied (MP) property. T commutes with $\mathcal{F}$ and $\mathcal{F}(\mathcal{M}\times\mathcal{M}) \subseteq T(\mathcal{M})$. If $\exists$ a point $(q_0,p_0) \in \mathcal{M}\times\mathcal{M}$ such that $Tq_0 \le \mathcal{F}(q_0,p_0)$ and $Tp_0 \le \mathcal{F}(p_0,q_0)$, then $Couple\ C(\mathcal{F},T) \neq \emptyset$.

Proof. Let $\alpha,\beta \ge 0, \alpha+\beta \in [0,1)$ for all $m,n,t,s \in M$ with $Tm \ge Tt$ and $Tn \le Ts$. Then
$d(\mathcal{F}(m,n),\mathcal{F}(t,s)) \le$
$\alpha\frac{d(Tm,\mathcal{F}(m,n))d(Tt,\mathcal{F}(t,s))}{d(Tm,Tt)+d(Tt,\mathcal{F}(m,n))+d(Tm,\mathcal{F}(t,s))} + \beta d(Tm,Tt)$

$$\le (\alpha+\beta)max\left\{\frac{d(Tm,\mathcal{F}(m,n))d(Tt,\mathcal{F}(t,s))}{d(Tm,Tt)+d(Tt,\mathcal{F}(m,n))+d(Tm,\mathcal{F}(t,s))}, d(Tm,Tt)\right\}$$
$$\le kM_d(m,n,t,s)$$

Where $k = \alpha+\beta \in [0,1)$. Therefore, it yields Corollary 2.

Corollary 4: Let $(M,d,\le)$ be a (P.O.C.M.S) and suppose that $\mathcal{F}:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$. Satisfying the following condition

$$\psi(d(\mathcal{F}(m,n),\mathcal{F}(t,s))) \le \psi(M_d(m,n,t,s)) - \phi(M_d(m,n,t,s)) + LN_d(m,n,t,s)$$
For all $m,n,t,s \in M$ with $m \ge t, n \le s$ and $m \neq t$ where $\phi \in \Phi$, $\psi \in \Psi, L \ge 0$,

$$M_d(m,n,t,s) = Max\left\{\frac{d(m,\mathcal{F}(m,n))d(t,\mathcal{F}(t,s))}{d(m,t)}, \frac{d(m,\mathcal{F}(m,n))[1+d(t,\mathcal{F}(t,s)]}{1+d(m,t)}, d(m,Tt)\right\}$$

$N_d(m,n,t,s) =$
$Min\begin{Bmatrix} d(Tm,\mathcal{F}(m,n)), d(Tt,\mathcal{F}(t,s)), \\ d(Tt,\mathcal{F}(m,n)), d(Tm,\mathcal{F}(t,s)) \end{Bmatrix}$
So, assume $\mathcal{F}$ is a continuous mapping and satisfied (MP) property. If there exists a point $(q_0,p_0) \in \mathcal{M}\times\mathcal{M}$ such that $q_0 \le \mathcal{F}(q_0,p_0)$ and $p_0 \le \mathcal{F}(p_0,q_0)$, then $Couple$ Fix$(\mathcal{F}) \neq \emptyset$

Corollary5: Let $(M,d,\le)$ be a (P.O.C.M.S) and suppose that $\mathcal{F}:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$, $T:\mathcal{M}\to\mathcal{M}$. Satisfying the following condition
$$\psi(d(\mathcal{F}(m,n),\mathcal{F}(t,s))) \le kM_d(m,n,t,s)$$
For all $m,n,t,s \in M$ with $Tm \ge Tt$, $Tn \le Ts$ and $Tm \neq Tt$ where $\phi \in \Phi$, $\psi \in \Psi, L \ge 0$,
$M_d(m,n,t,s) =$
$Max\begin{Bmatrix} \frac{d(Tm,\mathcal{F}(m,n))d(Tt,\mathcal{F}(t,s))}{d(Tm,Tt)}, \\ \frac{d(Tm,\mathcal{F}(m,n))[1+d(Tt,\mathcal{F}(t,s)]}{1+d(Tm,Tt)}, d(Tm,Tt) \end{Bmatrix}$
So, assume $\mathcal{F}$ and T are continuous mappings and satisfied (MP) property. T commutes with $\mathcal{F}$ and $\mathcal{F}(\mathcal{M}\times\mathcal{M}) \subseteq T(\mathcal{M})$. If there exists a point $(q_0,p_0) \in \mathcal{M}\times\mathcal{M}$ such that $Tq_0 \le \mathcal{F}(q_0,p_0)$ and $Tp_0 \le \mathcal{F}(p_0,q_0)$,
then $Couple\ (\mathcal{F},T) \neq \emptyset$
Proof. Take $L = 0$, $\psi(\varrho) = \varrho$ and $\phi(\varrho) = (1-k)\varrho$ and where $k \in [0,1)$ in Theorem 1 and Theorem 3.

Corollary6: Let $(M,d,\le)$ be a (P.O.C.M.S) and suppose that $\mathcal{F}:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$, $T:\mathcal{M}\to\mathcal{M}$. Satisfying the following condition
$(d(\mathcal{F}(m,n),\mathcal{F}(t,s))) \le$
$\alpha\frac{d(Tm,\mathcal{F}(m,n))d(Tt,\mathcal{F}(t,s))}{d(Tm,Tt)} +$
$\beta\frac{d(Tm,\mathcal{F}(m,n))[1+d(Tt,\mathcal{F}(t,s)]}{1+d(Tm,Tt)} + \delta d(Tm,Tt)$
$\forall m,n,t,s \in M$ with $Tm \ge Tt$, $Tn \le Ts$ and $Tm \neq Tt$.
So, assume $\mathcal{F}$ and T are continuous mappings and satisfied (MP) property. T commutes with $\mathcal{F}$ and $\mathcal{F}(\mathcal{M}\times\mathcal{M}) \subseteq T(\mathcal{M})$. If $\exists(q_0,p_0) \in \mathcal{M}\times\mathcal{M}$ such that $Tq_0 \le \mathcal{F}(q_0,p_0)$ and $Tp_0 \le \mathcal{F}(p_0,q_0)$, then $Couple(\mathcal{F},T) \neq \emptyset$

Proof. Let $\alpha, \beta, \delta \geq 0, \alpha + \beta + \delta \in [0,1)$ $\forall\, m, n, t, s \in M$ with, $Tn \leq Ts$ and $Tm \neq Tt$. Then

$$d(\mathcal{F}(m,n), \mathcal{F}(t,s)) \leq \alpha \frac{d(Tm, \mathcal{F}(m,n))d(Tt, \mathcal{F}(t,s))}{d(Tm,Tt) + d(Tt, \mathcal{F}(m,n)) + d(Tm, \mathcal{F}(t,s)} + \beta \frac{d(Tm, \mathcal{F}(m,n))[1 + d(Tt, \mathcal{F}(t,s)]}{d(Tm,Tt) + d(Tt, \mathcal{F}(m,n)) + d(Tm, \mathcal{F}(t,s)} + \delta d(Tm, Tt)$$

$$\leq (\alpha + \beta + \delta) max \left\{ \frac{d(Tm, \mathcal{F}(m,n))[1 + d(Tt, \mathcal{F}(t,s))]}{1 + d(Tm, Tt)}, \frac{(Tm, \mathcal{F}(m,n))d(Tt, \mathcal{F}(t,s))}{d(Tm,Tt)}, d(Tm, Tt) \right\}$$

$$\leq kM_d(m,n,r,s)$$

Where $k = \alpha + \beta + \delta \in [0,1)$. Therefore, it yields Corollary 5.

Example 1: The following example supports our Theorem with $L = 0$. Let $M = \{(1,0)(0,1), (0,1/2), (1/2,0)\}$ with the Euclidean distance d_2, and $\leq := \{(r,r): r \in M\}$. Also, consider $F: M \times M \to M$ and $T: M \to M$ given by

$$F(r,s) = \begin{cases} (1,0) & if \quad r = (1,0) \\ (0,1) & otherwise \end{cases} \text{ and } Tr = \begin{cases} (1/2,0) & if \quad (0,1/2) \\ r & otherwise \end{cases}$$

Moreover, take $\phi, \psi: [0,\infty] \to [0,\infty]$ such that $\psi(t) = 16t^2$ and $\phi(t) = 16t^2/5$ First, F and T are trivial continuous, and T is monotone F-nondecreasing. Let $r,s \in M$ with $Tr \leq Ts, Ts \leq Tr$ and $r = (1/2,0)$, $s = (0,1/2)$. Then

$$d_2\left(F((1/2,0),(0,1/2)), F((0,1/2),(1/2,0))\right) = 0 \qquad 33$$

$$M_{d_2}((1/2,0),(0,1/2),(0,1/2),(1/2,0))$$

$$= max \left\{ \frac{d\left(F((1/2,0),(0,1/2)), T(1/2,0)\right) d\left(F((0,1/2),(1/2,0)), T(0,1/2)\right)}{d(T(1/2,0),T(0,1/2)) + d\left(F((0,1/2),(1/2,0)), T(1/2,0)\right) + d\left(F((1/2,0),(0,1/2)), T(0,1/2)\right)}, d(T(1/2,0), T(0,1/2)) \right\}$$

$$= max \left\{ \frac{d((0,1),(1/2,0))d((0,1),(1/2,0))}{d((1/2,0),(1/2,0)) + d((0,1),(1/2,0)) + d((0,1),(1/2,0))}, d((1/2,0),(1/2,0)) \right\}$$

Where $d((0,1),(1/2,0)) = \sqrt{(0 - 1/2)^2 + (1 - 0)^2} = \sqrt{5}/2$ and $d((1/2,0),(1/2,0)) = 0$, then

$$= max \left\{ \frac{(\sqrt{5}/2)(\sqrt{5}/2)}{(\sqrt{5}/2) + (\sqrt{5}/2)}, 0 \right\} = \sqrt{5}/4 \qquad 34$$

From (33) and (34), this implies,

$$\psi\left(d_2\left(F((1/2,0),(0,1/2)), F((0,1/2),(1/2,0))\right)\right) = 0$$
$$\leq \psi\left(M_{d_2}((1/2,0),(0,1/2),(0,1/2),(1/2,0))\right) - \phi\left(M_{d_2}((1/2,0),(0,1/2),(0,1/2),(1/2,0))\right)$$
$$= 5 - 1 = 4$$

Thus condition (7) is held, then $Couple\mathrm{C}(\mathcal{F}, T) \neq \emptyset$ such that $F((0,1),(1,0)) = T(0,1) = (0,1)$ and $F((1,0),(0,1)) = T(1,0) = (1,0)$.

It can be remarked that F and T are single-valued maps and for multivalued maps, see [15-17].

For further research work extensions in partially ordered complete b-metric spaces, see [18,19].

Conclusions:

This work has proven some coupled coincidence fixed point theorems for maps satisfying a contractive condition with rational expression in partially ordered metric spaces, which generalize similar theorems in the case of standard metric spaces.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Basrah.

Authors' contributions statement:

A. M. H. proposed the idea of a fixed point for two mappings in partially ordered metric spaces, proved theorems 1 and 3, and obtained certain findings as a result. She also reviewed the work and submitted the necessary files for publication. A. A. K. proposed to include the concept of rational expression in partly ordered metric space, and he proved Theor ms 2 and 4 as well as obtaining certain results. He also revised the references to reflect his findings. Both authors discussed the findings and made contributions to the final copy of the manuscript.

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النقاط الصامدة المتطابقة المزدوجة مع الصفة الترتيبية

علي عبدالهادي كاظم **امل محمد هاشم طعمة البطاط**

قسم الرياضيات، كلية العلوم، جامعة البصرة، البصرة، العراق.

الخلاصة:

الغرض من هذا البحث استعراض واثبات النقاط الصامدة الثنائية للدوال الذاتية التي تحقق الشرط $(\phi - \psi)$ مع التعابير النسبية في الفضاءات المترية الكاملة المرتبة جزئيا والتي تتضمن دوال المسافة بخاصية ال (MP). تعمل النتائج التي حصلنا عليها على تحسين وتوحيد العديد من النتائج في مبرهنات النقطة الصامدة الثنائية وتعميم بعض النتائج الحديثة في الفضاء المتري المرتب جزئيا. تم إعطاء مثال لاظهار صحة نتيجتنا الرئيسية .

الكلمات المفتاحية: النقطة الصامدة المشتركة, النقطة الصامدة المزدوجة, الفضاء المتري, الصفة الرتيبية المزدوجة, مجموعة مرتبة, مجموعة مرتبة جزئيا, التعبير النسبي,