Some New Fixed Point Theorems in Weak Partial Metric Spaces

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Abstract:
The main objective of this work is to introduce and investigate fixed point (F. p) theorems for maps that satisfy contractive conditions (η − ρ) in weak partial metric spaces (W.P.M.S.), and give some new generalization of the fixed point theorems of Mathews and Heckmann. Our results extend, and unify a multitude of (F. p) theorems and generalize some results in (W.P.M.S). An example is given as an illustration of our results.

Keywords: Coincidence points, Fixed point, Partial metric, Weak partial metric, Weakly Compatible.

Introduction:
A partial metric space (P.M.S.) is a generalization of standard metric space developed by Matthews in 1994 as an extension of standard metric space (M.S), in which self-distance might not be equal to zero. The notion of (P.M.S) plays an important part in the theory of computation. Numerous articles have been published on fixed points for maps satisfying some contractive conditions in (P.M.S), and also for generalizing contractions. In 1999, Heckmann developed the notion of (W.P.M.S). which is a generalization of (P.M.S.) by omitting the small self-distance axiom. Some results for mappings in (W.P.M.S) have been obtained, also many authors proved fixed point (F.P) results for maps satisfying implicit relations. The main purpose of this paper is to study fixed point under (η − ρ) contractive conditions in weak partial metric space (W.P.M.S).

Preliminaries
Definition 1.1: A (P.M.) on M ≠ φ, is a function P: M² → R⁺ = [0, ∞) satisfying the following axioms,
(P₁) μ = η ⇔ P(μ, μ) = P(μ, η) = P(η, η) (T₀ - separation axiom)
(P₂) P(μ, μ) ≤ P(μ, η), (non-negatively and small self–distance)
(P₃) P(μ, η) = P(η, μ), (symmetry)
(P₄) P(μ, η) ≤ P(μ, τ) + P(τ, η) − P(τ, τ), (triangular inequality)
for all μ, η, τ ∈ M. Then (M, P) is said to be a partial metric space (for short P.M.S).

Remark 1.1: Clearly P(μ, η) = 0 ⇒ μ = η by using (P₁) and (P₂). But the reverse is false in general. For each partial metric P on the set M, the function dₚ: M² → R⁺ is defined by every P on set M generates a Topology P on set M whose base is the collection of open P-ball (Bₚ(μ, r), μ ∈ M, r > 0), where Bₚ(μ, r) = {η ∈ M; P(μ, η) < P(μ, μ) + r}. For all μ ∈ M and r > 0.

Remark 2.1: If P is (P.M.). on M, then the functions dₚ, dₒ: M² → R⁺ given by

dₚ(μ, η) = 2P(μ, η) − P(μ, μ) − P(η, η)
dₒ(μ, η) = P(μ, η) − min{P(μ, μ), P(η, η)}

are ordinary metrics on M. Note that dₚ, dₒ are equivalent on M.

Definition 2.1: Let (M, P) be a (P.M.) then

1. A sequence {rₙ} such that [M, P] converges to a point r ∈ M if and only if limₙ→∞ P(rₙ, r) = P(r, r)

2. A sequence {rₙ} in a P.M. (M, P) is called a Cauchy if and only if limₙ→∞ P(rₙ, rₙ) exists (and is finite).

3. If every Cauchy sequence{rₙ} in M converges, (with respect to the topology P), to a member r ∈ M ∈ limₙ→∞ P(rₙ, rₙ) = P(r, r) then (M, P) is complete.
Lemma 1  
Let $(M, P)$ be any P.M.S. Then
1. A sequence $\{n\}$ is Cauchy in a P.M.S $\iff \{n\}$ is a Cauchy in a metric space $(M, d_\omega)$.
2. A P.M.S $(M, P)$ is complete $\iff (M, d_\omega)$ is complete. In addition to that
$$\lim_{n \to \infty} d_\omega(n, r) = 0 \iff \rho(r, r) = \lim_{n \to \infty} \rho(n, r) = \lim_{m,n \to \infty} \rho(n, r).$$

Definition 3  
A weak partial metric space (for short W.P.M.S) on a nonempty set $M$ is a function $\rho : M^2 \to \mathbb{R}^2$ satisfying the following axioms for all $\mu, \nu, \tau \in M$:
\begin{align*}
(WP_1) & \mu = \eta \iff \rho(\mu, \eta) = \rho(\eta, \eta) \quad (T \text{-separation}), \\
(WP_2) & \rho(\mu, \eta) = \rho(\eta, \mu) \quad \text{(symmetry)}, \\
(WP_3) & \rho(\mu, \eta) \leq \rho(\mu, \tau) + \rho(\tau, \eta) - \rho(\tau, \tau) \quad \text{(modified triangular inequality)}.
\end{align*}

Also, Heckmann showed that if $\rho$ is a(W.P.M.S) on $M, \forall \mu, \nu, \tau \in M$ then the following property is satisfied:
$$\rho(\mu, \eta) \geq \frac{\rho(\mu, \eta) + \rho(\eta, \eta)}{2} \quad \forall \mu, \nu, \tau \in M.$$ 

It is clear that (P.M.S) implies (WPMS), but the reverse is not true in general.

Example 1  
Let $M = [0, \infty)$ and $\rho(\mu, \eta) = \frac{\mu + \eta}{2}$, then $(M, \rho)$ is a W.P.M.S space and is not a P.M.S.

Lemma 2  
Let $(M, \rho)$ be a (W.P.M.S). Then
(a) A sequence $\{n\}$ is Cauchy sequence in $(M, \rho)$ if $\{n\}$ is a Cauchy in $(M, d_\omega)$.
(b) A (W.P.M.S) is complete $\iff (M, d_\omega)$ is complete. In addition to that
$$\lim_{n \to \infty} d_\omega(n, r) = 0 \iff \rho(r, r) = \lim_{n \to \infty} \rho(n, r) = \lim_{m,n \to \infty} \rho(n, r).$$

Remark 3  
Note that $(M, d_\omega)$ is a standard $(M, \rho)$.

Definition 4  
The mappings $\alpha$ and $\beta : M \to M$ are called commuting maps if $\forall \alpha \in M, \beta \alpha = \alpha \beta$.

Definition 5  
Let $\alpha$ and $\beta : M \to M$ be mappings on $(M, \rho)$ if $\rho(\alpha \beta) = \beta \alpha$ for some $\beta \in M$, then $\beta$ is referred to as a coincidence point and $\beta \alpha$ is referred as a point of coincidence. The pair $(\alpha, \beta)$ is weakly compatible ($W.C.$) if $\alpha \beta = \beta \alpha$.

Remark 4  
It is remarked to point out that the definition provided above is taken from the definition in standard metric space $(M, d)$.

Definition 6  
The mappings $\alpha$ and $\beta : M \to M$ are called weak* compatible ($W^* \alpha \beta$) if they commute at one of their coincidence points that is if $\exists \beta \in M, \alpha \beta = \beta \alpha$ then $\alpha \beta \alpha = \beta \alpha \beta$.

The following example shows that weak* compatible maps are more general than weakly compatible maps.

Example 2  
Let $\alpha \beta = \frac{\mu^2}{4}$ and $\beta \alpha = \mu^4$ for $\mu \in [0, \frac{1}{2}]$. Then $\alpha$ and $\beta$ have two coincidence points $0$ and $\frac{1}{2}$.

Clearly, they commute at $0$ but not at $\frac{1}{2}$.

Definition 7  
A continuous non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(\tau) = 0$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a lower semi-continuous with $\phi(t) > 0 \forall t > 0$.

Main Result:

Theorem 1:  
Suppose that $(M, \rho)$ are a complete (WPMS), and $\alpha, \beta : M \to M$ are mappings such $\alpha M \subseteq \beta M$,
$$\psi(\rho(\alpha \eta, \alpha \eta)) \leq \psi(M(\rho(\mu, \eta) - \phi \left( M(\rho(\mu, \eta) \right),$$
\begin{align*}
\forall & \mu, \eta \in M \text{ and where:} \\
M(\rho(\mu, \eta)) & = M(\rho(\mu, \eta)), \quad \psi(M(\rho(\mu, \eta) - \phi \left( M(\rho(\mu, \eta) \right)),
\end{align*}
\begin{align*}
\text{Further, if } & \alpha \text{ and } \beta \text{ are } (w^*, c). \\
\text{Then } & \alpha \text{ and } \beta \text{ possess a point of coincidence,} \\
\text{if and only if } & \alpha \text{ and } \beta \text{ are } (w^*, c).
\end{align*}

Then $\alpha$ and $\beta$ possess a unique common fixed point.

Proof:  
Let $\xi_0$ construct the sequences $\{\alpha \xi_n\}$ and $\{\xi_n\} \subseteq M$ in the following manner.

Since $\alpha M \subseteq \beta M$, choose $\xi_1 \in M$ such that $\beta \xi_1 = \alpha \xi_0 \text{ and } \beta \xi_2 = \alpha \xi_1$.

Inductively,
$$\beta \xi_{n+1} = \alpha \xi_n \quad \forall n \geq 0$$

is obtained.

If $\alpha \xi_n = \beta \xi_{n+1}$ for some $n \in N$, then $\alpha \xi_n = \alpha \xi_{n+1} = \beta \xi_n$, and $\xi_n \in M$ is a coincidence point of $\alpha$ and suppose that $\alpha \xi_n \neq \beta \xi_{n+1} \forall n \geq 0$.

Using condition 1, this implies,
$$\psi(M(\rho(\xi_{n-1}, \xi_n))) \leq \psi(M(\rho(\xi_{n-1}, \xi_n))) - \phi \left( M(\rho(\xi_{n-1}, \xi_n) \right),$$
\begin{align*}
\text{Where,} \\
M(\rho(\xi_n, \xi_{n-1})) &= M(\rho(\xi_n, \xi_{n-1})), \\
\psi(M(\rho(\xi_{n-1}, \xi_n))) &\leq \frac{1}{2} \rho(\beta \xi_{n-1}, \alpha \xi_{n-1}) + \rho(\beta \xi_{n-1}, \alpha \xi_{n-1}),
\end{align*}
\begin{align*}
\text{By the} \quad (WP_3), \text{ it follows that,} \\
\rho(\alpha \xi_{n-1}, \alpha \xi_{n-1}) &\leq \frac{1}{2} \rho(\alpha \xi_{n-1}, \alpha \xi_{n-1}) + \rho(\alpha \xi_{n-1}, \alpha \xi_{n-1}),
\end{align*}
\begin{align*}
\text{But} \quad (WP_3), \quad \lim_{n \to \infty} \rho(\alpha \xi_{n-1}, \alpha \xi_{n-1}) &\leq \frac{1}{2} \rho(\alpha \xi_{n-1}, \alpha \xi_{n-1}) + \rho(\alpha \xi_{n-1}, \alpha \xi_{n-1}),
\end{align*}
\begin{align*}
\text{If,} \quad M(\rho(\xi_{n-1}, \xi_n)) &\leq \rho(\alpha \xi_{n-1}, \alpha \xi_{n-1}) \\
\text{Then by} \quad \text{inequality (2) implies,} \\
\rho(\alpha \xi_{n-1}, \alpha \xi_{n-1}) &\leq \rho(\alpha \xi_{n-1}, \alpha \xi_{n-1}).
\end{align*}
\[\psi(\alpha \xi_{n+1}, \alpha \xi_n) \leq \psi(\alpha \xi_n, \alpha \xi_{n-1}) - \varphi(\alpha \xi_{n-1}, \alpha \xi_n) \]

since \( \psi(t) > 0, \forall t > 0 \), and \( \psi \) is non-decreasing function. \( \psi(\alpha \xi_{n-1}, \alpha \xi_n) < \psi(\alpha \xi_n, \alpha \xi_{n-1}) \), which is contradiction to our assumption, Hence \( M_\rho(\alpha \xi_{n-1}, \alpha \xi_n) = \rho(\alpha \xi_{n-2}, \alpha \xi_n) \) and by use of inequality it yields

\[\psi(\alpha \xi_{n-1}, \alpha \xi_n) \leq \psi(\alpha \xi_n, \alpha \xi_{n-1}) - \varphi(\alpha \xi_{n-2}, \alpha \xi_n)\]

Since \( \psi(t) > 0, \forall t > 0 \) and \( \psi \) is non-decreasing function, this implies

\[\psi(\alpha \xi_{n-1}, \alpha \xi_n) \leq \psi(\alpha \xi_n, \alpha \xi_{n-1})\]

Therefore, \( \rho(\alpha \xi_{n-1}, \alpha \xi_n) \) is a decreasing sequence.

Thus, there exists \( \delta \geq 0 \) such that,

\[
\lim_{n \to \infty} \rho(\alpha \xi_{n-1}, \alpha \xi_n) = \delta
\]

Now to show that \( \delta = 0 \). Suppose \( \delta > 0 \). Then, making the limit of supremum in \( n \to \infty \) in the inequality, \( \psi(\delta) \leq \psi(\delta) - \varphi(\delta) < \psi(\delta) \) is obtained.

Which is a contradiction to our assumption since \( \psi(\delta) > 0 \). Therefore \( \delta = 0 \) and so

\[
\lim_{n \to \infty} \rho(\alpha \xi_{n-1}, \alpha \xi_n) = 0
\]

By weak small self-distance property

\[
\rho(\alpha \xi_{n-1}, \alpha \xi_n) \geq \frac{1}{2} \left[ \rho(\alpha \xi_n, \alpha \xi_{n-1}) + \rho(\alpha \xi_{n-1}, \alpha \xi_n) \right] = 0
\]

Now it can be concluded that \( \{\alpha \xi_n\} \) is a Cauchy sequence in \( (M, d_\omega) \). Let us assume otherwise. Then \( \exists \varepsilon > 0, \exists \forall n \in \mathbb{N} \) such that \( j < m(j) < n(j) \) and

\[
d_\omega(\alpha \xi_{n(j) + 1}, \alpha \xi_{m(j) + 1}) \geq \varepsilon
\]

Pick out \( n(j) \) in such a way that it is the smallest integer with \( n(j) > m(j) \) satisfying inequality (8).

Hence,

\[
d_\omega(\alpha \xi_{m(j) + 1}, \alpha \xi_{n(j) + 1}) < \varepsilon
\]

Now by using the inequalities (8),(9) and the triangular inequality of \( d_\omega \)

\[
\varepsilon \leq d_\omega(\alpha \xi_{m(j) + 1}, \alpha \xi_{n(j) + 1}) \\
\leq d_\omega(\alpha \xi_{m(j) + 1}, \alpha \xi_{m(j)}) + d_\omega(\alpha \xi_{m(j) + 1}, \alpha \xi_{n(j) + 1}) \\
\leq d_\omega(\alpha \xi_{m(j)}, \alpha \xi_{m(j + 1)}) + d_\omega(\alpha \xi_{m(j + 1)}, \alpha \xi_{n(j + 1)}) \\
\leq 2d_\omega(\alpha \xi_{m(j)}, \alpha \xi_{m(j + 1)}) + d_\omega(\alpha \xi_{m(j + 1)}, \alpha \xi_{n(j + 1)}) + \\

\leq 3d_\omega(\alpha \xi_{m(j)}, \alpha \xi_{m(j + 1)}) + d_\omega(\alpha \xi_{m(j + 1)}, \alpha \xi_{n(j + 1)}) \\
\leq 3d_\omega(\alpha \xi_{m(j) + 1}, \alpha \xi_{n(j) + 1}) + d_\omega(\alpha \xi_{n(j) + 1}, \alpha \xi_{n(j + 1)})
\]

Letting \( j \to \infty \) and by the property of \( \psi \) and \( \varphi \) in the above inequality,

\[
\psi(\varepsilon) = \psi(\max(\varepsilon, 0, 0, \varepsilon)) - \varphi(\max(\varepsilon, 0, 0, \varepsilon)) < \psi(\varepsilon), \psi(\varepsilon) < \psi(\varepsilon)
\]

Hence from condition (1) and (10) \( \psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon) < \psi(\varepsilon) \), a contradiction to our assumption \( \varepsilon > 0 \).

Thus \( \{\alpha \xi_n\} \) is a Cauchy sequence in \( (M, d_\omega) \). Then

\[
\lim_{n \to \infty} d_\omega(\alpha \xi_n, \alpha \xi_m) = 0
\]

and \( \lim_{n \to \infty} d_\omega(\alpha \xi_n, \alpha \xi_m) = 0 \). By completeness of \( (M, \rho) \) and theorem 1 the sequence \( \{\alpha \xi_n\} \) converges in \( (M, d_\omega) \), thus \( \exists \alpha \xi_n \in M \) such that \( \lim_{n \to \infty} d_\omega(\alpha \xi_n, \alpha \xi_m) = 0 \). Then

\[
\rho(u, u) = \lim_{n \to \infty} \rho(\alpha \xi_n, u) = \lim_{n \to \infty} \rho(\alpha \xi_n, \alpha \xi_m) = 0
\]
Also, the subsequences \( \{a\xi_{nj}\} \) and \( \{a\xi_{mj}\} \) are convergent to \( u, \alpha_\omega \in M \supset u = a_\omega \).

Now to show that \( \beta_\omega = a_\omega = u, \)
\[
\rho(\beta_\omega, a_\omega) \leq \rho(\beta_\omega, a_{\xi_{n+1}}) + \rho(a_{\xi_{n+1}}, a_\omega)
\]
\[
= \rho(a_{\xi_{n+1}}, a_{\xi_{n+1}}) \leq \rho(\beta_\omega, a_{\xi_{n+1}}) + \rho(a_{\xi_{n+1}}, a_\omega)
\]
\[
\psi(\rho(a_{\xi_{mj}+1}, a_\omega)) \leq \psi(M_\rho(\xi_{mj}+1, a_\omega))
\]
\[
- \varphi(M_\rho(\xi_{mj}+1, a_\omega))
\]

Where,
\[
M_\rho(\xi_{mj}+1, a_\omega) = \max(\rho(\beta, a_\omega), \rho(\beta_\omega, a_{\xi_{mj}+1}), 
\]
\[
\rho(\beta_\omega, a_\omega), 1/2 \rho(\beta_\omega, a_{\xi_{mj}+1}, a_\omega) + \rho(\beta_\omega, a_m).
\]
\[
\rho(\alpha_{\xi_{mj}}, a_{\xi_{mj}+1}) = \rho(\alpha_{\xi_{mj}}, a_{\xi_{mj}+1})
\]
\[
Letting j \to \infty it can be concluded that,
\[
\rho(\alpha_{\xi_{mj}+1}, a_\omega) = \max(\rho(\beta_\omega, a_\omega), \rho(u, u))
\]
\[
\rho(\alpha_\omega, a_\omega)
\]

Therefore, as \( j \to \infty \) condition (1) reduce to
\[
\psi(\rho(a_{\xi_{mj}+1}, a_\omega)) = \psi(\rho(\beta_\omega, a_\omega)) - \varphi(\rho(\beta_\omega, a_\omega))
\]
\[
\psi(\rho(\beta_\omega, a_\omega)) < \psi(\rho(\beta_\omega, a_\omega))
\]
\[
since \varphi(t) > 0 \text{ if } t > 0, \text{ and } \psi(t) \text{ is non- increasing this implies,}
\]
\[
\rho(\beta_\omega, a_\omega) = 0 \Rightarrow a_\omega = \beta_\omega
\]

Thus, it follows that \( \alpha_\omega = \beta_\omega = u, u \) is a point of coincidence. If the mapping \( \alpha \) and \( \beta \) are weak* compatible , then \( \beta_\omega = a_\omega = u \) since, thus \( \beta_\omega = \alpha_\omega = u \) i.e. \( u \) is a common fixed point of \( \alpha \) and \( \beta \), the uniqueness of common fixed point of \( \alpha \) and \( \beta \), follows from condition 1. If not assume that \( \alpha \) another fixed point \( \sigma \) exists \( \beta \sigma = \alpha_\omega = \sigma \).

Then,
\[
\psi(\rho(\sigma, a_\omega)) = \psi(\rho(a_\sigma, a_\sigma))
\]
\[
\leq \psi(M_\rho(\sigma, a_\omega)) - \varphi(M_\rho(\sigma, a_\omega))
\]

Where,
\[
M_\rho(\sigma, a_\omega) = \max(\rho(\beta_\sigma, a_\sigma), \rho(\beta_\sigma, a_\omega), \rho(\beta_\sigma, a_\sigma),
\]
\[
1/2 [\rho(\beta_\sigma, a_\omega) + \rho(\beta_\sigma, a_\sigma)]
\]
\[
= \max(\rho(a_\omega, a_\sigma), 0, 0, 1/2 [\rho(\sigma, a_\sigma) + \rho(\sigma, a_\omega)])
\]
\[
so,
\]
\[
\psi(\rho(\sigma, a_\omega)) = \psi(\rho(a_\sigma, a_\sigma))
\]
\[
\leq \psi(\rho(\sigma, a_\omega)) - \varphi(\rho(\sigma, a_\omega))
\]
\[
< \psi(\rho(\sigma, a_\omega)).
\]

This implies \( u = \sigma \).

**Corollary 1** Let \( (M, \rho) \) be a (C. WPMS). Assume that \( \alpha, \beta: M \to M \) are mappings \( \exists \psi(\rho(\alpha_\mu, a_\eta)) \leq \theta \max(\rho(\mu, \eta), \rho(\mu, a_\mu), \rho(\eta, a_\eta), 1/2 \rho(\mu, \eta) + \rho(\eta, a_\eta)) \)

For all \( \mu, \eta \in M \), and. Where, \( \theta: \Re^+ \to \Re^+ \) is continuous and \( \alpha(t) \) is a continuous and \( \beta \) is \( \beta^* \) compatible. Then \( \alpha \) and \( \beta \) possess common fixed point.

**Proof:** Take \( \psi(t) = \tau \) and \( \varphi(t) = t - \theta(t) \). Then by Theorem 1 implies the condition 12.

**Corollary 2** Let \( (M, \rho) \) be a (C. WPMS). Suppose that \( \alpha: M \to M \) be a mapping such that \( \psi(\rho(a_\mu, a_\eta)) \leq \psi(M_\rho(\mu, \eta)) \psi(M_\rho(\mu, \eta)) \)

For all \( \mu, \eta \in M \), where
\[
M_\rho(\mu, \eta) = \max(\rho(\mu, \eta), \rho(\mu, a_\eta), \rho(\eta, a_\eta), 1/2 [\rho(\mu, a_\eta) + \rho(\eta, a_\eta)])
\]

Where \( \psi(t) \) and \( \varphi(t) \) are altering distance function and. Then \( \alpha \) has a unique fixed point.

**Proof:** Take \( \alpha = \beta \). Then by Theorem 1, implies the condition 13. \( \alpha \) possesses a unique fixed point.

**Corollary 3** Let \( (M, \rho) \) be a (C. WPMS). Suppose that \( \alpha: M \to M \) be a mapping such that \( \rho(\alpha_\mu, a_\eta) \leq \delta(\mu, \eta) + \rho(\mu, a_\mu) + \rho(\eta, a_\eta) \)

For all \( \mu, \eta \in M \), \( 0 \leq \delta \leq \frac{1}{3} \) then \( \alpha \) possesses a unique F.P.

**Proof:** Take \( \psi(t) = \tau, \varphi(t) = (1 - 3\delta)t \) and \( \alpha = \beta \) Then by Theorem 1, \( \alpha \) possesses a unique F.P.

**Corollary 4** Let \( (M, \rho) \) be a (C. WPMS). Suppose that \( \alpha: M \to M \) be a mapping such that \( \rho(\alpha_\mu, a_\eta) \leq k(\mu, \eta, \rho(\mu, a_\mu), \rho(\eta, a_\eta)) \)

For all \( \mu, \eta \in M, k \in [0, 1) \), then \( \alpha \) possesses a unique fixed point.

**Proof:** Take \( \psi(t) = \tau, \varphi(t) = (1 - k)t \) for \( k \in (0, 1) \) and \( \alpha = \beta \), then by theorem 1, \( \alpha \) possesses a unique F.P.

An example is given to illustrate our main result.

**Example 3** Let \( M = [0, 1] \) and \( \rho: M \to M \) be a WPMS. Then \( (M, \rho) \) is a WPMS. Define \( \alpha \) and \( \beta: M \to M \) such that \( \alpha_\mu = \mu \) and \( \beta_\mu = \mu \). Let \( \psi(t) = t, \varphi(t) = \frac{2}{3}t \). Then for all \( \mu, \eta \in M \), it yields,
\[ \psi(\alpha \mu, \alpha \eta) = \psi\left(\frac{\mu}{3}, \frac{\eta}{3}\right) = \psi\left(\frac{1}{2} \left(\mu + \eta \frac{1}{3}\right)\right) = \frac{1}{3} \rho(\mu, \eta) \]

\[ \leq \rho(\mu, \eta) - \frac{2}{3} \rho(\mu, \eta) \]

\[ = \psi(M_p(\mu, \eta) - \varphi(M_p(\mu, \eta))) = \frac{1}{3} \rho(\mu, \eta) \]

\[ M_p(\mu, \eta) = \max \left\{ \rho(\beta \mu, \beta \eta), \rho(\beta \mu, \alpha \mu), \rho(\beta \eta, \alpha \eta) \right\} \]

\[ = \max \left\{ \frac{1}{2} \left[ \rho(\beta \mu, \alpha \mu) + \rho(\beta \eta, \alpha \eta) \right] \right\} \]

\[ = \max \left\{ \frac{1}{2} \left[ \rho(\mu, \eta) + \rho(\eta, \mu) \right] \right\} \]

\[ \leq \rho(\mu, \eta) \]

\[ \leq \frac{2}{3} \rho(\mu, \eta) \]

\[ = \frac{1}{3} \rho(\mu, \eta) \]

Therefore, all conditions of Theorem 1 are satisfied. Since \( \alpha 0 = \beta 0 = 0 \), and \( \alpha, \beta \) are weak* compatible then \( \alpha \) and \( \beta \) possess a unique common F.P \( d(0) = 0 \).

It can be remarked that \( \alpha \) and \( \beta \) are single valued maps and for multivalued maps see 18.

**Conclusions:**

In this paper, the theorems of coincidence and fixed point for two maps satisfying a generalized contractive condition 1 in a weak partial metric space are proven as a generalization of partial metric space and the standard metric space in the sense that the self-distance of any point need not equal to zero.

**Authors’ declaration:**

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Basrah.

**Authors’ contributions statement:**

A. M. H. was in charge of developing the idea of fixed point theorem in weak partial metric space which is a generalization of partial metric space. A. T. H. verified the analytical procedures used in the research and she proved an example to support the results. Both of the authors discussed the findings and contributed to the final draft of the paper.

**References**

حول مبرهنات النقطة الصامدة في الفضاءات المتعبة الجزئية الضعيفة

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الخلاصة:
الهدف الرئيسي لهذا البحث هو استعراض ودراسة بعض النقطة الصامدة للدوال التي تحقق الشرط \(\phi-\psi\) في الفضاءات المتفرقة الجزئية الضعيفة. يتم إعطاء بعض التعميمات الجديدة لمبرهنات النقطة الصامدة لكل من مانوس ومكمه. إن النتائج التي حصلنا عليها هي توسيع وتكييف العديد من النتائج في مبرهنات النقطة الصامدة وتعتيم بعض النتائج الحديثة في الفضاءات المتفرقة الجزئي الضعيف كما أطعنا مثال لتوضيح نتائجنا.

الكلمات المفتاحية: النقاط المتداخلة، المتفرقة الجزئي الضعيف، النقاط الصامدة المتفرقة.