New sizes of complete \((k, 4)\)-arcs in \(PG(2, 17)\)

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Abstract:
In this paper, the packing problem for complete \((k, 4)\)-arcs in \(PG(2, 17)\) is partially solved. The minimum and the maximum sizes of complete \((k, 4)\)-arcs in \(PG(2, 17)\) are obtained. The idea that has been used to do this classification is based on using the algorithm introduced in Section 3 in this paper. Also, this paper establishes the connection between the projective geometry in terms of a complete \((k, 4)\)-arc \(K\) in \(PG(2, 17)\) and the algebraic characteristics of a plane quartic curve over the field \(F_{17}\) represented by the number of its rational points and inflexion points. In addition, some sizes of complete \((k, 6)\)-arcs in the projective plane of order thirteen are established, namely for \(k = 53, 54, 55, 56\).

Keywords: Complete arc, Group of complete \((k, n)\)-arc, Inequivalent secant distribution, PG(2,17), PG(2,13).

Introduction:
In the projective plane \(PG(2, q)\) over the finite field \(F_q\), a \((k, r)\)-arc is a set of \(k\) points no \(r + 1\) of which are collinear, containing \(r\) collinear points. A \((k, r)\)-arc is complete if it is not contained in a \((k + 1, r)\)-arc \(^1, 2\). The main goals of algebraic geometry are to study the properties of algebraic varieties, such as curves and surfaces. Over a finite field, incidence properties of their rational points are of interest \(^3\). Therefore, some specific questions arise, typified by those in this paper.
(i) What are the sizes of complete \((k, 4)\)-arcs in \(PG(2, 17)\)?
(ii) What is the largest number of rational points of a quartic curve on a \((k, 4)\)-arc \(K\)?
(iii) What is the number \(N\) of rational points on such a quartic curve?
(iv) What is the number of inflexion points on such a curve?

The following background in this study is of the references \(^4\) \(^10\).

Preliminaries on arcs and algebraic curves:

Largest arc
Given \(q\) and \(r\), the largest value of \(k\) for which a complete \((k, r)\)-arc in \(PG(2, q)\) exists is denoted by \(m_r(2, q)\). In \(^4\), it was shown that \(m_2(2, 17) = 18\), and \(m_3(2, 17) \leq 33, m_4(2, 17) \leq 52\).

Points on a curve
Let \(F \in F_q [X, Y, Z]\); then \(V(F) = \{(x, y, z) \in PG(2, q) | F(x, y, z) = 0\}\). An algebraic curve in \(PG(2, q)\) is a pair \(C = C(F) = (F, V(F))\); that is, a polynomial together with the points that are its zeros.

A point \(P = (x, y, z)\) on a plane curve \(C\) is a singular point of \(C\) if the following holds:
\[
\frac{\partial F}{\partial x}(x, y, z) = \frac{\partial F}{\partial y}(x, y, z) = \frac{\partial F}{\partial z}(x, y, z) = 0.
\]

The Hessian of a curve \(C = C(F)\) is the curve \(H = C(H)\) defined by the polynomial
\[
H(X, Y, Z) = \det \begin{bmatrix} F_{XX} & F_{XY} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{bmatrix}
\]
A point \(P = (x, y, z)\) is an inflexion of the curve \(C\) if \(P\) is non-singular and the tangent at \(P\) meets \(C\) at least three times at \(P\). For \(q\) odd, a condition for \(P = (x, y, z)\) to be an inflexion is that \(H(x, y, z) = 0\).

Approach Method
The method that has been adopted to classify certain sizes of complete \((k, 4)\)-arcs in \(PG(2, 17)\) and starts by choosing a set of six points \(K_6=\)
(\{P_1, P_2, P_3, P_4, P_5, P_6\}) from the plane, PG(2, 17). This set contains four collinear points. So, \(K_6\) is defined to be the first arc as input in the complete arcs construction algorithm:

1. Input \((k, n)\)-arcs.
2. For each arc do the steps below:
3. Calculate the \(n\)-secants of \((k, n)\)-arc.
4. Calculate the points on the \(n\)-secants of \((k, n)\)-arc.
5. Calculate the set \(KK = \{PG(2, 17) \setminus n - \text{secants}\}\).
6. If \(KK \neq 0\), then add every point \(P \in KK\) independently to \((k, n)\)-arc to establish \((k + 1, n)\)-arcs. Otherwise, \(KK = 0\). Then \((k, n)\)-arc is complete.
7. End.
8. The outcome is \((k + 1, n)\)-arcs.
9. Now, input \((k + 1, n)\)-arcs.
10. Calculate the secants distributions of \((k + 1, n)\)-arcs. Then, calculate the inequivalent classes of \((k + 1, n)\)-arcs.
11. Stop.

Sizes of complete \((k, 4)\)-arcs in \(PG(2, 17)\)

In this section, the spectrum of certain sizes of complete \((k, 4)\)-arcs in \(PG(2, 17)\) obtained by the algorithm above is given. The arcs of minimum and maximum size that have been found are a complete \((31, 4)\)-arc and a complete \((48, 4)\)-arc, as shown in Table 1.

<table>
<thead>
<tr>
<th>(k)</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>4</td>
</tr>
<tr>
<td>32</td>
<td>13</td>
</tr>
<tr>
<td>33</td>
<td>40</td>
</tr>
<tr>
<td>34</td>
<td>748</td>
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<tr>
<td>35</td>
<td>1023</td>
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<td>36</td>
<td>807</td>
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<td>37</td>
<td>619</td>
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<tr>
<td>38</td>
<td>455</td>
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<td>305</td>
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<td>40</td>
<td>274</td>
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<td>41</td>
<td>108</td>
</tr>
<tr>
<td>42</td>
<td>25</td>
</tr>
<tr>
<td>48</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1 gives the following theorems.

**Theorem 1**

In \(PG(2, 17)\), the minimum size of a complete \((k, 4)\)-arc is at most 31.

**Theorem 2**

In \(PG(2, 17)\), the maximum size of a complete \((k, 4)\)-arc is at least 48.

**Remark 1**

1. The problem of finding the precise values of the minimum and the maximum sizes of complete \((k, 4)\)-arcs in \(PG(2, 17)\) is so hard especially when the value of \(q\) is large, the reason is related with the increasing number of the constructed arcs in each structure that reaches hundreds of millions and more. These numbers require super computers and large storage memory to accommodate it. Therefore, in Table 1, the possible sizes of inequivalent complete \((k, 4)\)-arcs are given.
2. The phrase 'inequivalent complete arcs' refers to inequivalent arcs that have distinct classes of secant distributions 11.
3. The programming language that used in this study is gap 12.

**Table 2. Timing (msec) of \((k, 4)\)-arcs**

<table>
<thead>
<tr>
<th>((k, 4))-arcs</th>
<th>Construction</th>
</tr>
</thead>
<tbody>
<tr>
<td>((31, 4))-arcs</td>
<td>17312902</td>
</tr>
<tr>
<td>((32, 4))-arcs</td>
<td>24000133</td>
</tr>
<tr>
<td>((33, 4))-arcs</td>
<td>24918765</td>
</tr>
<tr>
<td>((34, 4))-arcs</td>
<td>26634992</td>
</tr>
<tr>
<td>((35, 4))-arcs</td>
<td>27534305</td>
</tr>
<tr>
<td>((36, 4))-arcs</td>
<td>27018654</td>
</tr>
<tr>
<td>((37, 4))-arcs</td>
<td>28943351</td>
</tr>
<tr>
<td>((38, 4))-arcs</td>
<td>29624222</td>
</tr>
<tr>
<td>((39, 4))-arcs</td>
<td>30915953</td>
</tr>
<tr>
<td>((40, 4))-arcs</td>
<td>28843213</td>
</tr>
<tr>
<td>((41, 4))-arcs</td>
<td>766443</td>
</tr>
<tr>
<td>((42, 4))-arcs</td>
<td>663296</td>
</tr>
<tr>
<td>((48, 4))-arcs</td>
<td>1702</td>
</tr>
</tbody>
</table>

Algebraic properties of complete \((k, 4)\)-arcs in \(PG(2, 17)\)

In this section, examples of a non-singular quartic curve associated with each complete \((k, 4)\)-arc in \(PG(2, 17)\) are given in Table 2. For \(k = 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42\) and for \(k = 48\), algebraic properties of a complete \(k\)-arc are given in Table 3. The number of rational points of a curve \(C\) that lie on a \((k, 4)\)-arc \(K\) is denoted by \(K(C)\). The number of inflexions on \(C\) is denoted by \(I(C)\). From Table 3, the number of rational points of the corresponding quartic curve that lie on the complete \((k, 4)\)-arc for \(k = 32, 33, 34, 35, 36, 37, 39, 40, 42, 48\) is 15. For \(k = 31, 38\) it is 16, and for \(k = 41\) it is 17. There are six curves \(C\), having at least one inflexion.
In this section, the field is changed from \( F \) to \( F_1 \). Complete \((k, 6)\)-arcs for four values of \( k \), namely, \( k = 53, 54, 55, 56 \), are given in Table 4, with their secant distribution \((\beta_0, \beta_3, \beta_5, \beta_3, \beta_2, \beta_1, \beta_0)\).

<table>
<thead>
<tr>
<th>Size</th>
<th>Complete ((k, 6))-arc</th>
<th>Secant distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>53</td>
<td>(1, 0, 0), (0, 0, 0), (0, 1, 0), (1, 1, 1), (2, 1, 0), (5, 1, 0), (11, 1, 0), (6, 1, 0), (7, 6, 1), (8, 2, 1), (9, 3, 1), (2, 10, 1), (10, 7, 1), (12, 9, 1), (6, 4, 1), (11, 10, 1), (10, 5, 1), (8, 8, 1), (11, 9, 1), (2, 5, 1), (4, 3, 1), (4, 12, 1), (1, 6, 1), (3, 7, 1), (3, 10, 1), (7, 0, 1), (9, 4, 1), (2, 3, 1), (3, 1, 0), (2, 0, 1), (6, 0, 1), (7, 7, 1), (1, 12, 1), (10, 6, 1), (6, 9, 1), (4, 4, 1), (5, 6, 1), (9, 1, 1), (0, 5, 1), (8, 0, 1), (8, 5, 1), (12, 12, 1), (8, 3, 1), (3, 4, 1), (3, 0, 1), (6, 2, 1), (4, 11, 1), (10, 4, 1), (1, 2, 1), (4, 2, 1), (9, 6, 1), (0, 11, 1), (6, 3, 1)</td>
<td>(49, 31, 36, 31, 24, 8, 4)</td>
</tr>
<tr>
<td>54</td>
<td>(1, 0, 0), (0, 0, 0), (0, 1, 0), (1, 1, 1), (2, 1, 0), (5, 1, 0), (11, 1, 0), (6, 1, 0), (7, 6, 1), (8, 2, 1), (9, 3, 1), (2, 10, 1), (10, 7, 1), (12, 9, 1), (6, 4, 1), (11, 10, 1), (10, 5, 1), (8, 8, 1), (11, 9, 1), (2, 5, 1), (4, 3, 1), (4, 12, 1), (1, 6, 1), (3, 7, 1), (3, 10, 1), (7, 0, 1), (9, 4, 1), (2, 3, 1), (3, 1, 0), (2, 0, 1), (6, 0, 1), (7, 7, 1), (1, 12, 1), (10, 6, 1), (6, 9, 1), (4, 4, 1), (5, 6, 1), (9, 1, 1), (0, 5, 1), (8, 0, 1), (8, 5, 1), (12, 12, 1), (8, 3, 1), (3, 4, 1), (3, 0, 1), (6, 2, 1), (4, 11, 1), (10, 4, 1), (1, 2, 1), (4, 2, 1), (9, 6, 1), (0, 11, 1), (6, 3, 1)</td>
<td>(51, 31, 42, 30, 14, 9, 6)</td>
</tr>
<tr>
<td>55</td>
<td>(1, 0, 0), (0, 0, 0), (0, 1, 0), (1, 1, 1), (2, 1, 0), (5, 1, 0), (11, 1, 0), (6, 1, 0), (7, 6, 1), (8, 2, 1), (9, 3, 1), (2, 10, 1), (10, 7, 1), (12, 9, 1), (6, 4, 1), (11, 10, 1), (10, 5, 1), (8, 8, 1), (11, 9, 1), (2, 5, 1), (4, 3, 1), (4, 12, 1), (1, 6, 1), (3, 7, 1), (3, 10, 1), (7, 0, 1), (9, 4, 1), (2, 3, 1), (3, 1, 0), (2, 0, 1), (6, 0, 1), (7, 7, 1), (1, 12, 1), (10, 6, 1), (6, 9, 1), (4, 4, 1), (5, 6, 1), (9, 1, 1), (0, 5, 1), (8, 0, 1), (8, 5, 1), (12, 12, 1), (8, 3, 1), (3, 4, 1), (3, 0, 1), (6, 2, 1), (4, 11, 1), (10, 4, 1), (1, 2, 1), (4, 2, 1), (9, 6, 1), (0, 11, 1), (5, 1, 1), (12, 7, 1)</td>
<td>(55, 36, 31, 33, 15, 7, 6)</td>
</tr>
</tbody>
</table>
Remark 2
From Table 4, the stabilizer groups of the above complete \((k; 6\)-arcs are the identity groups.

Conclusion:
In this paper, the packing problem for complete \((k, 4\)-arcs in \(PG(2, 17)\) and complete \((k, 6\)-arcs in \(PG(2, 13)\) is discussed. New sizes of these complete arcs of degrees four and six are given. Here, the computation approach that has been used in this study gives the following results:

1. The sizes of complete \((k; 4\)-arcs in \(PG(2; 17)\) that obtained are 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 48.
2. For each complete \((k; 4\)-arc in \(PG(2; 17)\), an associated quartic curve with its algebraic properties in terms of the number of rational points and number of inflexion points are given. From Table 3, the number of rational points of the corresponding quartic curve that lie on the complete \((k; 4\)-arc for \(k = 32; 33; 34; 35; 36; 37; 39; 40; 42; 48\) is 15 and for \(k = 31\); 38 it is 16 then for \(k = 41\) it is 17. Also, there are six curves \(C_1\); \(C\), having at least one inflexion.
3. Some sizes of complete \((k; 6\)-arcs for the value of \(k = 53; 54; 55; 56\) are established. Also, the stabilizer of these complete arcs is calculated.

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Author's declaration:
- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Mustansiriyah University.

References:
في الفضاء الإسقاطي $(k, 4)$ احجام جديد للأقواس الكاملة $(2,17)$

زينب شهاب حامد
قسم الرياضيات، كلية العلوم، الجامعة المستنصرية، بغداد، العراق

الخلاصة:
في هذا البحث، تم حل مشكلة الأقواس الكاملة من الدرجة الرابعة في الفضاء الإسقاطي. حيث تم اعتماد الفكرة التي PG$(2,17)$ في هذا البحث، تم حل مشكلة الأقواس الكاملة من الدرجة الرابعة في الفضاء الإسقاطي. تم استخدام الخوارزمية المقدمة في القسم 3. أيضاً، أن هذه الدراسة تحدد العلاقة بين PG$(2,17)$ وPG$(2,13)$ من حيث الأقواس الكاملة في الفضاء الإسقاطي وخصائص الجبرية للمنحني الرباعي في الحقل المتماثل. بالإضافة إلى ذلك، تم بناء بعض احجام الأقواس الكاملة من الدرجة السادسة، $k=53,54,55,56$ عندما PG$(2,13)$ والمستوي الإسقاطي.

الكلمات المفتاحية: القوس الكامل، زمر الأقواس الكاملة $(k, n)$، توزيعات القاطع المختلفة (PG$(2,17)$, PG$(2,13)$).