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New sizes of complete $(k, 4)$-arcs in $PG(2, 17)$

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Abstract:

In this paper, the packing problem for complete $(k, 4)$-arcs in $PG(2, 17)$ is partially solved. The minimum and the maximum sizes of complete $(k, 4)$-arcs in $PG(2, 17)$ are obtained. The idea that has been used to do this classification is based on using the algorithm introduced in Section 3 in this paper. Also, this paper establishes the connection between the projective geometry in terms of a complete $(k, 4)$-arc K in $PG(2, 17)$ and the algebraic characteristics of a plane quartic curve over the field F_{17} represented by the number of its rational points and inflexion points. In addition, some sizes of complete $(k, 6)$-arcs in the projective plane of order thirteen are established, namely for k = 53, 54, 55, 56.

Keywords: Complete arc, Group of complete (k, n)-arc, Inequivalent secant distribution, PG(2,17), PG(2,13).

Introduction:

In the projective plane $PG(2, q)$ over the finite field F_q, a (k, r)-arc is a set of k points no $r + 1$ of which are collinear, containing r collinear points. A (k, r)-arc is complete if it is not contained in a $(k + 1, r)$-arc [1, 2]. The main goals of algebraic geometry are to study the properties of algebraic varieties, such as curves and surfaces. Over a finite field, incidence properties of their rational points are of interest [3]. Therefore, some specific questions arise, typified by those in this paper.

(i) What are the sizes of complete $(k, 4)$-arcs in $PG(2, 17)$?

(ii) What is the largest number of rational points of a quartic curve on a $(k, 4)$-arc K?

(iii) What is the number N of rational points on such a quartic curve?

(iv) What is the number of inflexion points on such a curve?

The following background in this study is of the references [4-10].

Preliminaries on arcs and algebraic curves:

Largest arc

Given q and r, the largest value of k for which a complete (k, r)-arc in $PG(2, q)$ exists is denoted by $m_r(2, q)$. In [4], it was shown that $m_2(2, 17) = 18$, and $m_3(2, 17) \leq 33$, $m_4(2, 17) \leq 52$.

Points on a curve

Let $F \in F_q[X, Y, Z]$; then $V(F) = \{(x, y, z) \in PG(2, q) \mid F(x, y, z) = 0\}$. An algebraic curve in $PG(2, q)$ is a pair $C = C(F) = (F, V(F))$; that is, a polynomial together with the points that are its zeros.

A point $P = (x, y, z)$ on a plane curve C is a singular point of C if the following holds:

$$\frac{\partial F}{\partial X}(x, y, z) = \frac{\partial F}{\partial Y}(x, y, z) = \frac{\partial F}{\partial Z}(x, y, z) = 0.$$

The Hessian of a curve $C = C(F)$ is the curve $H = C(H)$ defined by the polynomial

$$H(X, Y, Z) = \det\begin{bmatrix} F_{XX} & F_{XY} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{bmatrix}$$

A point $P = (x, y, z)$ is an inflexion of the curve C if P is non-singular and the tangent at P meets C at least three times at P. For q odd, a condition for $P = (x, y, z)$ to be an inflexion is that $H(x, y, z) = 0$.

Approach Method

The method that has been adopted to classify certain sizes of complete (k, 4)-arcs in $PG(2, 17)$ and starts by choosing a set of six points K_6=

$\{P_1, P_2, P_3, P_4, P_5, P_6\}$ from the plane, $PG(2,17)$. This set contains four collinear points. So, K_6 is defined to be the first arc as input in the complete arcs construction algorithm:

(1) Input (k, n)-arcs.
(2) For each arc do the steps below:
(3) Calculate the n-secants of (k, n)-arc.
(4) Calculate the points on the n-secants of (k, n)-arc.
(5) Calculate the set $KK = \{PG(2,17) \setminus n - \text{secants}\}$.
(6) If $KK \neq 0$, then add every point $P \in KK$ independently to (k, n)-arc to establish $(k + 1, n)$-arcs. Otherwise, $KK = 0$. Then (k, n)-arc is complete.
(7) End.
The outcome is $(k + 1, n)$-arcs.
(8) Now, input $(k + 1, n)$-arcs.
(9) Calculate the secants distributions of $(k + 1, n)$-arcs. Then, calculate the inequivalent classes of $(k + 1, n)$-arcs.
(10) Stop.

Sizes of complete $(\boldsymbol{k}, 4)$-arcs in $\boldsymbol{PG}(2,17)$

In this section, the spectrum of certain sizes of complete $(k, 4)$-arcs in $PG(2,17)$ obtained by the algorithm above is given. The arcs of minimum and maximum size that have been found are a complete (31, 4)-arc and a complete (48, 4)-arc, as shown in Table 1.

Table 1. Complete $(\boldsymbol{k}, 4)$-arcs in $\boldsymbol{PG}(2, 17)$

$\boldsymbol{k}$	Number
31	4
32	13
33	40
34	748
35	1023
36	807
37	619
38	455
39	305
40	274
41	108
42	25
48	1

Table 1 gives the following theorems.

Theorem 1

In $PG(2,17)$, the minimum size of a complete $(k, 4)$-arc is at most 31.

Theorem 2

In $PG(2,17)$, the maximum size of a complete $(k, 4)$-arc is at least 48.

Remark 1

(1) The problem of finding the precise values of the minimum and the maximum sizes of complete $(k, 4)$-arcs in $PG(2,17)$ is so hard especially when the value of q is large, the reason is related with the increasing number of the constructed arcs in each structure that reaches hundreds of millions and more. These numbers require super computers and large storage memory to accommodate it. Therefore, in Table 1, the possible sizes of inequivalent complete $(k, 4)$-arcs are given.
(2) The phrase 'inequivalent complete arcs' refers to inequivalent arcs that have distinct classes of secant distributions [11].
(3) The programming language that used in this study is gap[12].

Table 2. Timing (msec) of $(\boldsymbol{k}, 4)$-arcs

$(\boldsymbol{k}, 4)$-arcs	Construction
(31, 4)-arcs	17312902
(32, 4)-arcs	24000133
(33, 4)-arcs	24918765
(34, 4)-arcs	26634992
(35, 4)-arcs	27534305
(36, 4)-arcs	27018654
(37, 4)-arcs	28943351
(38, 4)-arcs	29624222
(39, 4)-arcs	30915953
(40, 4)-arcs	28843213
(41, 4)-arcs	766443
(42, 4)-arcs	663296
(48, 4)-arcs	1702

Algebraic properties of complete $(\boldsymbol{k}, 4)$-arcs in $\boldsymbol{PG}(2,17)$

In this section, examples of a non-singular quartic curve associated with each complete $(k, 4)$-arc in $PG(2,17)$ are given in Table 2. For k = 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42 and for k = 48, algebraic properties of a complete k-arc are given in Table 3. The number of rational points of a curve C that lie on a $(k, 4)$-arc K is denoted by $K(C)$. The number of inflexions on C is denoted by I(C). From Table 3, the number of rational points of the corresponding quartic curve that lie on the complete $(k, 4)$-arc for k = 32, 33, 34, 35, 36, 37, 39, 40, 42, 48 is 15. For k = 31, 38 it is 16, and for k = 41 it is 17. There are six curves C, having at least one inflexion.

Table 3. Algebraic properties of $C_i, 1 \le i \le 13$

k	C_i	F	$K(C)$	N_1	$I(C)$
31	C_1	$X^3Y+X^3Z+10X^2Y^2+8X^2YZ+7X^2Z^2+6XY^3+15XY^2Z+11XYZ^2+2Y^3Z+2Y^2Z^2+5YZ^3$	16	29	1
32	C_2	$X^4-X^3Y+X^3Z+14X^2Y^2+10X^2YZ+9X^2Z^2+5XY^3+8XY^2Z+5XYZ^2+XZ^3+15Y^4+8Y^3Z+9Y^2Z^2+10YZ^3$	15	28	1
33	C_3	$X^4+11X^3Y+10X^3Z+11X^2Y^2+14X^2YZ+3X^2Z^2+13XY^3+XY^2Z+2XYZ^2u+5XZ^3+15Y^4+5Y^3Z+15Y^2Z^2+12YZ^3+11Z^4$	15	28	0
34	C_4	$X^4+7X^3Y+11X^3Z+4X^2Y^2+13X^2YZ+X^2Z^2+12XY^3+12XY^2Z+XYZ^2+10Y^4+8Y^3Z+7Y^2Z^2+15YZ^3$	15	27	1
35	C_5	$X^4+13X^3Y+4X^3Z+9X^2Y^2+5X^2YZ+11X^2Z^2+10XY^3+6XY^2Z+7XYZ^2+3XZ^3+Y^4+8Y^3Z-Y^2Z^2+5YZ^3+6Z^4$	15	32	0
36	C_6	$X^4+13X^3Y+X^2Y^2+9X^2YZ+6X^2Z^2+8XY^3+6XYZ^2+5XZ^3+11Y^4+12Y^3Z+9Y^2Z^2+YZ^3-Z^4$	15	31	0
37	C_7	$X^4+13X^3Y+X^2Y^2+9X^2YZ+6X^2Z^2+8XY^3+6XYZ^2+5XZ^3+11Y^4+12Y^3Z+9Y^2Z^2+YZ^3-Z^4$	15	31	0
38	C_8	$X^4+14X^3Y+5X^3Z+14X^2Y^2+5X^2YZ+X^2Z^2+4XY^3+7XY^2Z+9XYZ^2+13XZ^3+4Y^4+8Y^3Z-Y^2Z^2+14YZ^3+13Z^4$	16	29	0
39	C_9	$X^4+7X^3Z+13X^2Y^2+13X^2YZ+13X^2Z^2+3XY^3+11XY^2Z+6XYZ^2+9XZ^3+14Y^2Z^2+12YZ^3$	15	27	2
40	C_{10}	$X^4+8X^3Y+7X^3Z+8X^2Y^2+3X^2YZ+2X^2Z^2+11XY^3+3XY^2Z-XYZ^2+10XZ^3+2Y^4+12Y^3Z+8Y^2Z^2+15Z^4$	15	28	1
41	C_{11}	$X^3Y+10X^3Z+13X^2Y^2+10X^2YZ+6X^2Z^2+3XY^3+13XY^2Z+11XYZ^2-XZ^3+11Y^2Z^2+8YZ^3$	17	35	0
42	C_{12}	$X^4+3X^3Y-X^3Z+2X^2Y^2+5X^2YZ+7X^2Z^2+15XY^3-XY^2Z+12XYZ^2+6XZ^3+13Y^4+5Y^3Z+12Y^2Z^2+6YZ^3$	15	27	2
48	C_{13}	$X^4+11X^3Y+12X^3Z+11X^2Y^2+5X^2YZ+8X^2Z^2+11XY^3+3XY^2Z+3XYZ^2+11XZ^3+7Y^3Z+15Y^2Z^2+4YZ^3$	15	30	0

Complete $(\boldsymbol{k}, 6)$-arcs in $\boldsymbol{PG}(2,13)$

In this section, the field is changed from F_{17} to F_{13}. Complete $(k, 6)$-arcs for four values of k, namely, k = 53, 54, 55, 56, are given in Table. 4, with their secant distribution ($\beta_6, \beta_5, \beta_4, \beta_3, \beta_2, \beta_1, \beta_0$).

Table 4. Complete (*k*, 6)-arcs

Size	Complete (k; 6)-arc	Secant distribution
53	(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), (2, 1, 0), (5, 1, 0), (11, 1, 0), (6, 1, 0), (7, 6, 1), (8, 2, 1), (9, 3, 1), (2, 10, 1), (10, 7, 1), (12, 9, 1), (6, 4, 1), (11, 10, 1), (10, 5, 1), (8, 8, 1), (11, 9, 1), (2, 5, 1), (4, 3, 1), (4, 12, 1), (1, 6, 1), (3, 7, 1), (3, 10, 1), (7, 0, 1), (9, 4, 1), (2, 3, 1), (3, 1, 1), (0, 2, 1), (6, 0, 1), (7, 7, 1), (1, 12, 1), (10, 6, 1), (6, 9, 1), (4, 4, 1), (5, 6, 1), (9, 1, 1), (0, 5, 1), (8, 0, 1), (8, 5, 1), (12, 12, 1), (8, 3, 1), (3, 4, 1), (3, 0, 1), (6, 2, 1), (4, 11, 1), (10, 4, 1), (1, 2, 1), (4, 2, 1), (9, 6, 1), (0, 11, 1), (6, 3, 1)	(49, 31, 36, 31, 24, 8, 4)
54	(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), (2, 1, 0), (5, 1, 0), (11, 1, 0), (6, 1, 0), (7, 6, 1), (8, 2, 1), (9, 3, 1), (2, 10, 1), (10, 7, 1), (12, 9, 1), (6, 4, 1), (11, 10, 1), (10, 5, 1), (8, 8, 1), (11, 9, 1), (2, 5, 1), (4, 3, 1), (4, 12, 1), (1, 6, 1), (3, 7, 1), (3, 10, 1), (7, 0, 1), (9, 4, 1), (2, 3, 1), (3, 1, 1), (0, 2, 1), (6, 0, 1), (7, 7, 1), (1, 12, 1), (10, 6, 1), (6, 9, 1), (4, 4, 1), (5, 6, 1), (9, 1, 1), (0, 5, 1), (8, 0, 1), (8, 5, 1), (12, 12, 1), (8, 3, 1), (3, 4, 1), (3, 0, 1), (1, 3, 1), (12, 4, 1), (0, 6, 1), (6, 10, 1), (5, 12, 1), (1, 11, 1), (0, 11, 1), (5, 1, 1), (12, 7, 1)	(51, 31, 42, 30, 14, 9, 6)
55	(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), (2, 1, 0), (5, 1, 0), (11, 1, 0), (6, 1, 0), (7, 6, 1), (8, 2, 1), (9, 3, 1), (2, 10, 1), (10, 7, 1), (12, 9, 1), (6, 4, 1), (11, 10, 1), (10, 5, 1), (8, 8, 1), (11, 9, 1), (2, 5, 1), (4, 3, 1), (4, 12, 1), (1, 6, 1), (3, 7, 1), (3, 10, 1), (7, 0, 1), (9, 4, 1), (2, 3, 1), (3, 1, 1), (0, 2, 1), (6, 0, 1), (7, 7, 1), (1, 12, 1), (10, 6, 1), (6, 9, 1), (4, 4, 1),	(55, 36, 31, 33, 15, 7, 6)

56	(5, 6, 1), (9, 1, 1), (0, 5, 1), (8, 0, 1), (8, 5, 1), (12, 12, 1), (8, 3, 1), (3, 4, 1), (3, 0, 1), (1, 11, 1), (4, 6, 1), (7, 9, 1), (2, 9, 1), (5, 12, 1), (1, 2, 1), (4, 2, 1), (0, 11, 1), (7, 3, 1), (12, 7, 1) (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), (2, 1, 0), (5, 1, 0), (11, 1, 0), (6, 1, 0), (7, 6, 1), (8, 2, 1), (9, 3, 1), (2, 10, 1), (10, 7, 1), (12, 9, 1), (6, 4, 1), (11, 10, 1), (10, 5, 1), (8, 8, 1), (11, 9, 1), (2, 5, 1), (4, 3, 1), (4, 12, 1), (1, 6, 1), (3, 7, 1), (3, 10, 1), (7, 0, 1), (9, 4, 1), (2, 3, 1), (3, 1, 1), (0, 2, 1), (6, 0, 1), (7, 7, 1), (1, 12, 1), (10, 6, 1), (6, 9, 1), (4, 4, 1), (5, 6, 1), (9, 1, 1), (0, 5, 1), (8, 0, 1), (8, 5, 1), (12, 12, 1), (8, 3, 1), (3, 4, 1), (3, 0, 1), (4, 2, 1), (7, 9, 1), (10, 12, 1), (0, 5, 1), (1, 11, 1), (4, 6, 1), (7, 3, 1), (5, 12, 1), (12, 7, 1), (1, 2, 1), (2, 9, 1)	(57, 33, 30, 35, 15, 8, 5)

Remark 2

From Table 4, the stabilizer groups of the above complete $(k;6)$-arcs are the identity groups.

Conclusion:

In this paper, the packing problem for complete (k, 4)-arcs in PG(2, 17) and complete (k, 6)-arcs in PG(2, 13) is discussed. New sizes of these complete arcs of degrees four and six are given.

Here, the computation approach that has been used in this study gives the following results:

(1) The sizes of complete (k; 4)-arcs in PG(2; 17) that obtained are 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42,48.

(2) For each complete (k; 4)-arc in PG(2; 17), an associated quartic curve with its algebraic properties in terms of the number of rational points and number of infexion points are given. From Table 3, the number of rational points of the corresponding quartic curve that lie on the complete (k; 4)-arc for k = 32; 33; 34; 35; 36; 37; 39; 40; 42; 48 is 15 and for k = 31; 38 it is 16 then for k = 41 it is 17. Also, there are six curves C1; C, having at least one inflexion.

(3) Some sizes of complete (k; 6)-arcs for the value of k = 53; 54; 55; 56 are established. Also, the stabilizer of these complete arcs is calculated.

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Author's declaration:

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Mustansiriyah University.

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احجام جديد للأقواس الكاملة $(k,4)$ في الفضاء الاسقاطي $PG(2,17)$

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الخلاصة:

في هذا البحث, تم حل مشكلة الأقواس الكاملة من الدرجة الرابعة في الفضاء الاسقاطي PG(2,17) جزئيا. حيث تم اعتمدت الفكرة التي تم استخدامها للقيام بهذا التصنيف على استخدام الخوارزمية المقدمة في القسم 3. في الفضاء الاسقاطي PG(2,17) الحصول على الحد الادنى والحد الاقصى لأقواس الدرجة الرابعة. ايضا، ان هذه الدراسة تحدد العلاقة بين الهندسة الاسقاطية من حيث الاقواس الكاملة في الفضاء الاسقاطي F_{17} والخصاص الجبرية للمنحني الرباعي في الحقل متمثلة بعدد من النقاط المنطقية ونقاط الانقلاب، بالإضافة إلى ذلك، تم بناء بعض احجام الاقواس الكاملة من الدرجة السادسة k =53,54,55,56. عندما PG(2,13) في المستوى الاسقاطي

الكلمات المفتاحية : القوس الكامل, زمر الاقواس الكاملة (k,n) , توزيعات القاطع المختلفة PG(2,17) , PG(2,13).