New Estimator Of The Parameter Of Negative Exponential Failure Model

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Abstract :

The main aim of this paper is to give a comprehensive presentation of estimating methods namely Maximum likelihood , Bayes and proposed methods for the parameter of exponential failure model (using simulation).

Some new results are obtained and the new estimator was the best .

1-Introduction

The exponential distribution with mean θ provided a population model which is useful in many areas of statistics. The probability density function of random variable is given by

$$f(t;\theta)=\frac{1}{\theta}\exp\left(-\frac{t}{\theta}\right)$$

The only statistical theory that combines modeling inherent uncertainty and statistical uncertainty is Bayesian statistics . The theorem of Bayes provides a solution to how to learn from data . In the framework of estimating the parameters $\theta=(\theta_1,\ldots,\theta_d)$ of a probability distribution $L(t_1,\ldots,t_n|\theta)$, Bayes'

$$\Pi(\theta|t_1,\ldots,t_n)=\frac{L(t_1,\ldots,t_n|\theta)g(\theta)}{p(t_1,\ldots,t_n)}$$

$$=\frac{H(t_1,\ldots,t_n,\theta)}{p(t_1,\ldots,t_n)}$$

$$=\frac{H(t_1,\ldots,t_n,\theta)}{\int H(t_1,\ldots,t_n,\theta)d\theta}$$

Where $L(t_1,\ldots,t_n|\theta)$ = the likelihood function of the observations $(t_1,\ldots,t_n)$ when the parametric vector $\theta=(\theta_1,\ldots,\theta_d)$ is given,

$g(\theta)$ is the prior density of $\theta=(\theta_1,\ldots,\theta_d)$ before observing data $(t_1,\ldots,t_n)$,

$\Pi(\theta|t_1,\ldots,t_n)$ is the posterior density of $\theta=(\theta_1,\ldots,\theta_d)$ after observing data

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$(t_1,\ldots,t_n)$,

$p(t_1,\ldots,t_n)$=the marginal density of the observations $(t_1,\ldots,t_n)$.

2- Maximum Likelihood Estimation

Let $t_1,\ldots.t_n$ be set of random variables and a vector of unknown parameters $\theta=(\theta_1,\ldots,\theta_n)$,then $L(t,\theta)$ is the likelihood function such that

$$L(t,\theta)=\prod_{i=1}^{n} f(t_i,\theta)$$

$$\ln L(t,\theta)=\sum_{i=1}^{n}\ln f(t_i,\theta)$$

The ith element of the score vector is

$$U_i(\theta)=\frac{\partial \ln L(t,\theta)}{\partial \theta_i} \quad , i=1,2,\ldots p$$

Now we can find the maximum likelihood estimator. Let $(t_1,t_2,\text{----},t_n)$ be the set of random variable from exponential population with parameter θ.The probability density function of exponential distribution is given by :

$$f(t;\theta)=\frac{1}{\theta}\exp(-\frac{t}{\theta}) \quad ,\theta>0$$

The likelihood function is

$$L(t,\theta)=\prod_{i=1}^{n} f(t_i,\theta)$$

$$----=\prod_{i=1}^{n}\frac{1}{\theta}\exp\left(-\frac{t_i}{\theta}\right)$$

$$=\frac{1}{\theta^n}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)$$

$$=\theta^{-n}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)$$

$$\ln L(t,\theta)=-n\ln\theta-\frac{\sum_{i=1}^{n} t_i}{\theta}$$

The score vector is

$$U(\theta)=\frac{\partial \ln L(t,\theta)}{\partial \theta}$$

$$=\frac{-n}{\hat{\theta}}+\frac{\sum_{i=1}^{n} t_i}{\hat{\theta}^2}$$

Let $U(\theta)=0$, then the maximum likelihood estimator is

$$\hat{\theta}=\frac{\sum_{i=1}^{n} t_i}{n}=\bar{t}$$

Where,

$$E(\hat{\theta})=\frac{1}{n}\sum_{i=1}^{n} Et_i=\frac{1}{n}\sum_{i=1}^{n}\theta=\theta$$

$\hat{\theta}=\bar{t}$ is unbiased for θ , and

$$Var(\hat{\theta})=Var\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right)$$

$$=\frac{1}{n}\sum_{i=1}^{n} Var t_i=\frac{1}{n^2}n\theta^2=\frac{\theta^2}{n}$$

Therefore

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + [E(\hat{\theta}) - \theta]^2 = \frac{\theta^2}{n}$$

3- Bayes Estimation

Consider the one parameter exponential distribution

$$f(t;\theta) = \frac{1}{\theta}\exp(-\frac{t}{\theta})$$

We find Jeffery prior by :

$g(\theta) \propto \sqrt{I(\theta)}$,where

$$\sqrt{I(\theta)} = \sqrt{Fisher \ldots information} ,$$

and

$$\boxed{I(\theta) = -nE\left(\frac{\partial^2 \ln f(t,\theta)}{\partial\theta^2}\right)}$$

$$f(t;\theta) = \frac{1}{\theta}\exp(-\frac{t}{\theta})$$

$$\ln f(t,\theta) = -\ln\theta - \frac{t}{\theta}$$

$$\frac{\partial \ln f(t,\theta)}{\partial\theta} = -\frac{1}{\theta} + \frac{t}{\theta^2}$$

$$\frac{\partial^2 \ln f(t,\theta)}{\partial\theta^2} = \frac{1}{\theta^2} - \frac{2t}{\theta^3}$$

$$E\left(\frac{\partial^2 \ln f(t,\theta)}{\partial\theta^2}\right) = \frac{1}{\theta^2} - \frac{2}{\theta^2} = \frac{-1}{\theta^2}$$

$$I(\theta) = -n\, E\left(\frac{\partial^2 \ln f(t,\theta)}{\partial\theta^2}\right)$$

$$= \frac{n}{\theta^2}$$

Using Jeffery proposal that

$g(\theta) \propto \sqrt{I(\theta)}$,

Implies $g(\theta) \propto \frac{\sqrt{n}}{\theta}$

$$g(\theta) = k\frac{\sqrt{n}}{\theta}$$

$$L(t_1,\ldots,t_n|\theta) = \prod_{i=1}^{n} f(t_i|\theta)$$

$$= \frac{1}{\theta^n}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)$$

The joint probability density function $f(t_1,t_2,\ldots,t_n, \theta)$ is given by

$$H(t_1,\ldots,t_n,\theta) = \prod_{i=1}^{n} f(t_i,\theta)g(\theta)$$

$$= L(t_1,\ldots,t_n|\theta)g(\theta)$$

$$= \frac{1}{\theta^n}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)\frac{k\sqrt{n}}{\theta}$$

$$= \frac{k\sqrt{n}}{\theta^{n+1}}\exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right)$$

The marginal probability density function of $(t_1,\ldots,t_n)$ is given by

$$p(t_1,\dots,t_n)=\int H(t_1,\dots,t_n,\theta)d\theta$$

$$=\int_0^\infty \frac{k\sqrt{n}}{\theta^{n+1}}\exp\left(-\frac{\sum_{i=1}^n t_i}{\theta}\right)d\theta$$

$$=\left(k\sqrt{n}\right)\int_0^\infty \left(\frac{\sum_{i=1}^n t_i}{y}\right)^{-n-1}$$

$$\exp(-y)\left(\frac{\sum_{i=1}^n t_i}{y^2}\right)dy$$

$$=\left(k\sqrt{n}\right)\left(\sum_{i=1}^n t_i\right)^{-n}\int_0^\infty y^{n-1}$$
$$\exp(-y)dy$$

$$=\frac{\left(k\sqrt{n}\right)(n-1)!}{\left(\sum_{i=1}^n t_i\right)^n}$$

And the conditional probability density function of θ given the data $(t_1,\dots,t_n)$ is given by

$$\Pi(\theta|t_1,\dots,t_n)=\frac{H(t_1,\dots,t_n,\theta)}{p(t_1,\dots,t_n)}$$

$$=\frac{\frac{k\sqrt{n}}{\theta^{n+1}}\exp\left(-\frac{\sum_{i=1}^n t_i}{\theta}\right)}{\frac{k\sqrt{n}}{\left(\sum_{i=1}^n t_i\right)^n}(n-1)!}$$

$$=\frac{\exp\left(-\frac{\sum_{i=1}^n t_i}{\theta}\right)}{\theta^{n+1}}\frac{\left(\sum_{i=1}^n t_i\right)^n}{(n-1)!}$$

using squared error loss function

$$\ell(\hat{\theta},\theta)=c(\hat{\theta}-\theta)^2$$

Leads to the Risk function, $R(\hat{\theta},\theta)$

$$=R(\hat{\theta},\theta)=E\ell(\hat{\theta},\theta)$$

$$\int_0^\infty \ell(\hat{\theta},\theta)\Pi(\theta|t_1,\dots,t_n)d\theta$$

$$=\int_0^\infty c(\hat{\theta}-\theta)^2\frac{\exp\left(-\frac{\sum_{i=1}^n t_i}{\theta}\right)\left(\sum_{i=1}^n t_i\right)^n}{\theta^{n+1}(n-1)!}$$

$$= c\hat{\theta}^2 - 2c\hat{\theta}\int_0^\infty \frac{\left(\sum_{i=1}^{n} t_i\right)^n}{(n-1)!}\theta^{-n} \exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right) d\theta + \psi(\theta)$$

$$\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}} = 2c\hat{\theta} - 2c\frac{\left(\sum_{i=1}^{n} t_i\right)^n}{(n-1)!}\int_0^\infty \theta^{-n} \exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right) d\theta + zero$$

Let $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}} = 0$, then

$$\hat{\theta}_B = \frac{\left(\sum_{i=1}^{n} t_i\right)^n}{(n-1)!}\int_0^\infty \theta^{-n} \exp\left(-\frac{\sum_{i=1}^{n} t_i}{\theta}\right) d\theta$$

$$= \frac{\left(\sum_{i=1}^{n} t_i\right)^n}{(n-1)!}\int_0^\infty \left(\frac{\sum_{i=1}^{n} t_i}{y}\right)^{-n} \exp(-y)\left(\frac{\sum_{i=1}^{n} t_i}{y^2}\right) dy$$

$$= \frac{\left(\sum_{i=1}^{n} t_i\right)}{(n-1)!}\int_0^\infty y^{n-2}\exp(-y)dy$$

$$= \frac{\left(\sum_{i=1}^{n} t_i\right)(n-2)!}{(n-1)!}$$

$$\hat{\theta}_B = \frac{\sum_{i=1}^{n} t_i}{n-1}$$

Where,

$$E(\hat{\theta}_B) = \frac{1}{n-1}\sum_{i=1}^{n} Et_i$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\theta = \frac{n}{n-1}\theta$$

$\hat{\theta}_B$ is a biased estimator for θ , and

$$biased = E(\hat{\theta}_B) - \theta$$

$$= \frac{n}{n-1}\theta - \theta = \frac{1}{n-1}\theta$$

Similarly ,

$$Var(\hat{\theta}_B) = \frac{1}{(n-1)^2}\sum_{i=1}^{n} Var(t_i)$$

$$= \frac{1}{(n-1)^2}\sum_{i=1}^{n}\theta^2 = \frac{n}{(n-1)^2}\theta^2$$

and

$$MSE(\hat{\theta}_B) = \frac{n}{(n-1)^2}\theta^2 + \frac{1}{(n-1)^2}\theta^2$$

$$= \frac{n+1}{(n-1)^2}\theta^2$$

4- Proposed Estimator

Using both the M.L.E. and Bayesian estimator for the mean θ ,we propose the following new estimator :

$$\tilde{\theta} = p\hat{\theta}_{M.L.E} + (1-p)\hat{\theta}_B$$

And we try to find the value of p which minimizes MSE($\tilde{\theta}$) , where

$$\tilde{\theta} - \theta = p\hat{\theta}_{M.L.E} + (1-p)\hat{\theta}_B - \theta$$

$$\tilde{\theta} - \theta = p[\hat{\theta}_{M.L.E} - \theta] + (1-p)[\hat{\theta}_B - \theta]$$

$$E(\tilde{\theta} - \theta)^2 = p^2 E[\hat{\theta}_{M.L.E} - \theta]^2 + (1-p)^2 E[\hat{\theta}_B - \theta]^2 + --2p(1-p)E\left[\hat{\theta}_{M.L.E} - \theta\right]\left[\hat{\theta}_B - \theta\right]$$

$$MSE(\tilde{\theta}) = p^2 MSE(\hat{\theta}_{M.L.E}) + (1-p)^2 MSE(\hat{\theta}_B) + --2p(1-p)E\left[\hat{\theta}_{M.L.E} - \theta\right]\left[\hat{\theta}_B - \theta\right]$$

$$MSE(\tilde{\theta}) = p^2 MSE(\hat{\theta}_{M.L.E}) + (1-p)^2 MSE(\hat{\theta}_B) + ----2p(1-p)[E(\hat{\theta}_{M.L.E}\hat{\theta}_B) - E(\hat{\theta}_{M.L.E}\theta) - E(\hat{\theta}_B\theta) + E\theta^2]$$

$$\frac{\partial MSE(\tilde{\theta})}{\partial p} = 2pMSE(\hat{\theta}_{M.L.E}) - 2(1-p)MSE(\hat{\theta}_B) + ----(2-4p)[E(\hat{\theta}_{M.L.E}\hat{\theta}_B) - E(\hat{\theta}_{M.L.E}\theta) - E(\hat{\theta}_B\theta) + E\theta^2]$$

let $\frac{\partial MSE(\tilde{\theta})}{\partial p} = 0$, Implies

$$p = \frac{MSE(\hat{\theta}_B) - E(\hat{\theta}_{M.L.E}\hat{\theta}_B) + E(\hat{\theta}_{M.L.E}\theta) + E(\hat{\theta}_B\theta) - E\theta^2}{MSE(\hat{\theta}_{M.L.E}) + MSE(\hat{\theta}_B) - 2E(\hat{\theta}_{M.L.E}\hat{\theta}_B) + 2E(\hat{\theta}_{M.L.E}\theta) + 2E(\hat{\theta}_B\theta) - 2E(\theta^2)}$$

$$p = \frac{\frac{n+1}{(n-1)^2}\theta^2 - E\left(\frac{\sum t_i}{n}\frac{\sum t_i}{n-1}\right) + E\left(\frac{\sum t_i}{n}\theta\right) + E\left(\frac{\sum t_i}{n-1}\theta\right) - E(\theta^2)}{\frac{\theta^2}{n} + \frac{n+1}{(n-1)^2}\theta^2 - 2E\left(\frac{(\sum t_i)^2}{n(n-1)}\right) + 2E\left(\frac{\sum t_i}{n}\theta\right) + 2E\left(\frac{\sum t_i}{n-1}\theta\right) - 2E(\theta^2)}$$

$$p=\frac{\dfrac{n+1}{(n-1)^2}\theta^2-\dfrac{n}{n-1}\theta^2+\theta^2+\dfrac{n}{n-1}\theta^2-\theta^2}{\dfrac{\theta^2}{n}+\dfrac{n+1}{(n-1)^2}\theta^2-\dfrac{2n\theta^2}{n-1}+2\theta^2+\dfrac{2n}{n-1}\theta^2-2\theta^2}$$

$$P=\frac{n^2+n}{2n^2-n+1}$$

$$\tilde{\theta}=\frac{n^2+n}{2n^2-n+1}\hat{\theta}_{M.L.E}+\left(1-\frac{n^2+n}{2n^2-n+1}\right)\hat{\theta}_B$$

5- Simulation

In this experiment ,we choose the samples sizes as n=15,25,50,70,100 , with varieties of parameter value namely θ = 0.5 , 1, 2, 4, 8, 15, 30 .With 1000 of replication .

To compare between the methods of estimator , use the MSE and $|MPE|$, where

$$MSE(\hat{\theta})=\frac{\sum_{i=1}^{R}\left(\hat{\theta}_i-\theta\right)^2}{R}$$

and

$$|MPE|(\hat{\theta})=\frac{\left[\sum_{i=1}^{R}\frac{|\hat{\theta}_i-\theta|}{\theta}\right]}{R}$$

The simulation program is written by using Matlab program . And the results are written in table (1).

Table(1): MSE and MPE of estimated parameter of exponential distribution

Sample sizes (n)	Values of the parameter θ	MSE			MPE		
		Bayes	Proposed	Best	Bayes	Proposed	Best
15	0.5	0.0227	0.0202	Proposed	0.2345	0.2226	Proposed
	1	0.0766	0.0679	Proposed	0.2172	0.2057	Proposed
	2	0.3373	0.3016	Proposed	0.2209	0.2126	Proposed
	4	1.2846	1.1374	Proposed	0.2194	0.2087	Proposed
	8	4.9084	4.3530	Proposed	0.2158	0.2055	Proposed
	15	17.1028	15.2151	Proposed	0.2193	0.2089	Proposed
	30	74.7319	65.5735	Proposed	0.2261	0.2144	Proposed
25	0.5	0.0112	0.0104	Proposed	0.1682	0.1633	Proposed
	1	0.0426	0.0398	Proposed	0.1625	0.1582	Proposed
	2	0.1798	0.1668	Proposed	0.1650	0.1602	Proposed
	4	0.6936	0.6469	Proposed	0.1636	0.1594	Proposed
	8	2.8374	2.6396	Proposed	0.1683	0.1634	Proposed
	15	9.9978	9.2820	Proposed	0.1657	0.1607	Proposed
	30	38.4331	35.5157	Proposed	0.1616	0.1559	Proposed
50	0.5	0.0055	0.0053	Proposed	0.1167	0.1150	Proposed
	1	0.0208	0.0202	Proposed	0.1149	0.1133	Proposed
	2	0.0890	0.0857	Proposed	0.1170	0.1152	Proposed
	4	0.3544	0.3430	Proposed	0.1167	0.1156	Proposed
	8	1.2572	1.2176	Proposed	0.1102	0.1089	Proposed
	15	4.8098	4.6292	Proposed	0.1130	0.1114	Proposed
70	0.5	0.0035	0.0034	Proposed	0.0920	0.0914	Proposed
	1	0.0140	0.0135	Proposed	0.0936	0.0922	Proposed
	2	0.0626	0.0609	Proposed	0.0981	0.0972	Proposed
	4	0.2357	0.2298	Proposed	0.0961	0.0952	Proposed
	8	0.8642	0.8423	Proposed	0.0931	0.0922	Proposed
	15	3.4595	3.3899	Proposed	0.0969	0.0963	Proposed
	30	13.8710	13.5366	Proposed	0.0969	0.0961	Proposed
100	0.5	0.0025	0.0025	Proposed	0.798	0.0791	Proposed
	1	0.0102	0.0100	Proposed	0.0801	0.0797	Proposed
	2	0.0408	0.0403	Proposed	0.794	0.0792	Proposed
	4	0.1702	0.1681	Proposed	0.0815	0.0812	Proposed
	8	0.6413	0.6318	Proposed	0.0782	0.0778	Proposed
	15	2.3619	2.3214	Proposed	0.0809	0.0803	Proposed
	30	8.8720	8.7077	Proposed	0.0784	0.0779	Proposed

6- Conclusion

This paper presented Bayes method and a new method (combining Bayes and Maximum likelihood estimators) for estimating the mean of exponential failure model . It has been shown from the computational results the proposed method is the best .

7- References

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المستخلص

الهدف الرئيسي لهذه الدراسة هو تقديم توضيح شامل لبعض طرق التقدير مثل طريقة الامكان الاعظم وطريقة بيز بالاضافة الى طرق مقترحه من اجل تقدير معلمه نموذج الفشل الاسي باستخدام المحاكات. تم الحصول على بعض النتائج الجديدة وان المقدر المقترح هو الافضل.