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A novelty Multi-Step Associated with Laplace Transform Semi Analytic Technique for Solving Generalized Non-linear Differential Equations

Khalid Hammood AL-Jizani *Ahmed Hanoon Abud*

Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq
*Corresponding author: khalid.math79@uomustansiriyah.edu.iq
E-mail address: ahmedhanoon@uomustansiriyah.edu.iq

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Abstract:

In this work, a novel technique to obtain an accurate solutions to nonlinear form by multi-step combination with Laplace-variational approach (MSLVIM) is introduced. Compared with the traditional approach for variational it overcome all difficulties and enable to provide us more an accurate solutions with extended of the convergence region as well as covering to larger intervals which providing us a continuous representation of approximate analytic solution and it give more better information of the solution over the whole time interval. This technique is more easier for obtaining the general Lagrange multiplier with reduces the time and calculations. It converges rapidly to exact formula with simply computable terms with few time. To investigate of this technique, selected examples to show the ability, validity, accurately and effectiveness.

Keywords: General Lagrange multiplier, Laplace transform, Laplace-variational, MSLVIM, Multi-step,

Introduction:

The exact solutions in many problems epically for non-linear cases are difficult to obtain, therefore many approximate analytical methods appeared for this purpose [1-4] and also [5-8] as well as [9-10]. In this work, a novel technique, which combining a multi-step associated with Laplace transformation based on variational approach (MSLVIM) is presented to obtain an accurate solutions for nonlinear quadratic form. Many researchers recently applied variational approach (VIM) for obtaining the analytic solutions with different types of equations including partial, algebraic, differential, diffusion, delay and integro-differential equations [11-15]. The application of Laplace with VIM is found to find the approximate solution of telegraph equation [16]. The approximate solution of nonlinear gas dynamics with VIM Equation is conducted [17]. Laplace transformation implemented to make VIM easier for getting the analytic solution [18]. The analytic solution of time-fractional Fornberg–Whitham Equation is obtained by VIM with Laplace transform and comparison with Adomian decomposition method is conducted [19]. A multi-step with homotopy is applied to obtain the analytic solution for solving fractional-order model for HIV [20]. Authors in [21], implemented a multi-step to get the approximate solutions for different types of equations. A Laplace variational iteration approach strategy for solving differential equations is discussed [22]. Homotopy perturbation associated with Laplace transform and comparison with the variational iteration approach is considered [23]. Variational Iteration with the Laplace transform for Modified Fractional Derivatives with Non-singular Kernel is presented [24]. Variational iteration method associated with the multistage for solving nonlinear system ordinary differential equations is studied [25].

In this letter, a MSLVIM presents. The mainly thrust for this technique is to construct a correction functional and to overcome all difficulties which face us related to integration when one obtain the Lagrange multiplier and to get an accurate solution for large intervals with more extended of the convergence region which the traditional approach fail normally. It converges rapidly to exact formula with simply computable terms with reduce and few time. As well as covering to larger intervals which providing us a continuous representation of approximate analytic solution therefore it capable to give better information of the solution over the whole time interval. The implementation of technique makes for a number of examples and the results confirm the simplicity, suitability and effectiveness of this technique using only few terms of the iterative scheme. In the next section present Laplace-variational iteration method (LVIM).

LVIM for Nonlinear Differential Equations

Consider the general nonlinear differential equation

$$Lp(t)+Np(t)=q(t), \qquad (1)$$

subject to the initial conditions

$$p^{(k)}=c_k,\ k=0,1,...,m-1$$

where L, N are n-th linear and analytic nonlinear terms respectively, $q(t)$ is analytic known function.
Taking the Laplace transform to the both sides of Eq.1, one get

$$\ell[Lp(t)+Np(t)-q(t)]=0, \qquad (2)$$

By using the differential property of the Laplace transform for the linear operator $L=\frac{d^m}{dt^m}$,
yield to

$$S^m P(s)-\sum_{k=1}^{m} S^{m-k}p(0)+\ell[Np(t)-q(t)], \qquad (3)$$

where $P(s)$ is the Laplace transform to $p(t)$. According to the VIM, the correctional function of Eq. 3 can be constructed as

$$P_{n+1}(s)=P_n(s)+\lambda(s)[S^m P_n(s)-\sum_{k=1}^{m} S^{m-k}p^{(k-1)}(0)+\ell[Np_n-q(t)], \qquad (4)$$

where $\lambda(s)$ is the Lagrange multiplier, by considering $\ell[Np_n-q(t)]$ as a restricted variation, then one have

$$P_{n+1}(s)=P_n(s)+S^m\lambda(s)P_n(s), \qquad (5)$$

Therefore, it can easily derive the Lagrange multiplier as

$$\lambda(s)=\frac{-1}{S^m},$$

As a result, Eq. 4 can be written as

$$P_{n+1}(s)=P_n(s)-\frac{1}{S^m}\left[S^m P_n(s)-\sum_{k=1}^{m} S^{m-k}p^{(k-1)}(0)+\ell[Np_n+q(t)]\right], \qquad (6)$$

Now, by taking the inverse Laplace to the both sides of Eq. 6, then the solution (1) is given by

$$p(t)=\lim_{n\to\infty} p_n.$$

Multi-Step LVIM for Non-linear Differential Equations

Like other analytical methods such as HAM, ADM and DTM, the VIM has some drawbacks. The series solutions are usually valid in any every small region and it converges slowly or completely diverges in some cases. To overcome this problem and shortcoming, multi-step methods presented in many works [1, 20, 21].

In this work, a new multi-step technique based on a combination with Laplace transform with standard VIM presents. This technique does not require calculating the integral at each step of the process. Moreover and importantly, the analytic solution is valid for a long time interval.

Let $[0,T]$ be the interval for which it is required to obtain the analytic solution of Eq. 1. The basic idea of MSLVIM is divided the interval $[0,T]$ into subintervals

$[t_{j-1},t_j]$, $j=1,2,...,J$, such that $t=0$, $t_J=T$

and the step size $h=\frac{T}{J}$.

Now, to find the analytic solution over the interval $[0,t_1]$, need to apply LVIM on Eq. 1 using $p_1^{(k)}(0)=c_k$ as the initial guess. So, one get $p_1(t)=\lim_{m\to\infty} p_{1,m}$ along
$t\in[0,t_1]$.

Following the some process of each intervals $[t_{j-1},t_j]$ using the initial solution
$p_j^{(k)}(t_{j-1})=p^{(k)}{}_{j-1}(t_{j-1})$, to generate a sequence of solutions as:

$$p(t)=\begin{cases} p_1(t), t\in[t_0,t_1] \\ p_2(t), t\in[t_1,t_2] \\ . \\ . \\ . \\ p_J(t), t\in[t_{J-1},t_J]. \end{cases} \qquad (7)$$

Numerical Simulation

In this section, apply the MSLVIM technique as explains above on nonlinear differential equations to investigate the validation and efficiency for obtaining the analytic solutions.

Example 1

Consider the nonlinear differential equation

$$p' = 1 - p^2, \tag{8}$$

with initial condition

$$p(0) = 0,$$

where the exact solution is given by

$$p(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

Now, apply the MSLVIM as it explained to Eq. 8 for getting the analytic solution, taking the Laplace transform to the both sides of Eq. 8 with given initial
Condition as

$$SP(s) - p(0) + \ell\left[p^2 - 1\right] = 0, \tag{9}$$

Consequently, the correction functional can be constructed as

$$P_{n+1}(s) = P_n(s) + \lambda(s)\left[SP_n(s) - p(0) + \ell\left[p_n^{\ 2} - 1\right]\right], \tag{10}$$

Substituting $\lambda(s) = \frac{-1}{s}$, and simplifying

$$P_{n+1}(s) = \frac{-1}{s}\left[-p(0) + \ell\left[p^2{}_n - 1\right]\right], \tag{11}$$

Now, to construct the MSLVIM series solution on $[0, t_1]$, take the inverse Laplace transform to Eq. 11, using the initial condition $p(0) = 0$, which gives

$$P_1(t) = t + \frac{1}{2}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \ldots$$

for each subinterval $[t_{j-1}, t_j]$, $p_j(t_j) = p_{j-1}(t_j)$ is used as a new initial solution and the process repeating again to construct the series of solution at each subinterval as in Eq. 7. Table 1 demonstrate the accuracy and efficiency of MSLVIM the exact solution compared with the standard VIM. Figure 1. show the standard VIM with the exact solution, while figure 2. show the accuracy and efficiency of MSLVIM with the exact solution.

Table 1. Comparison between MSLVIM and VIM with the exact solution

t	*MSLVIM*	*VIM*	*Exact Solution*
0	0	0	0
0.5	5.61789×10^{-8}	1.38244×10^{-5}	0.462117
1.	5.75252×10^{-7}	4.42789×10^{-3}	0.761594
1.5	9.63629×10^{-7}	9.14481×10^{-2}	0.905148
2.	8.0718×10^{-7}	5.72336×10^{-1}	0.964028
2.5	4.94298×10^{-7}	1.07521	0.986614
3.	2.58443×10^{-7}	7.41306	0.995055
3.5	1.23692×10^{-7}	304.379	0.998178
4.	5.6072×10^{-8}	4250.67	0.999329
4.5	2.45184×10^{-8}	35669.9	0.999753
5.	1.04511×10^{-8}	218682.	0.999909

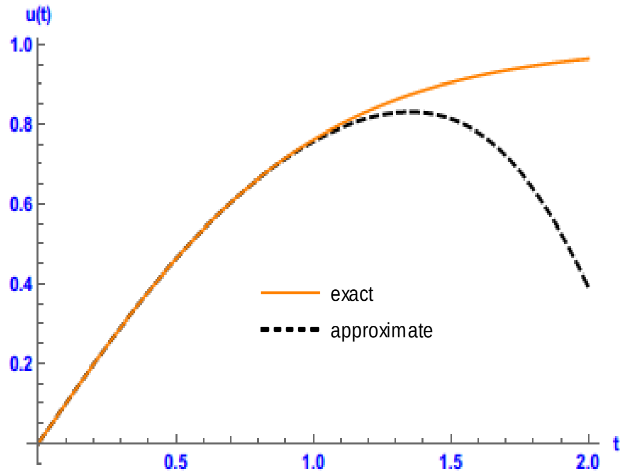

Figure 1. Comparison between VIM with the exact solution

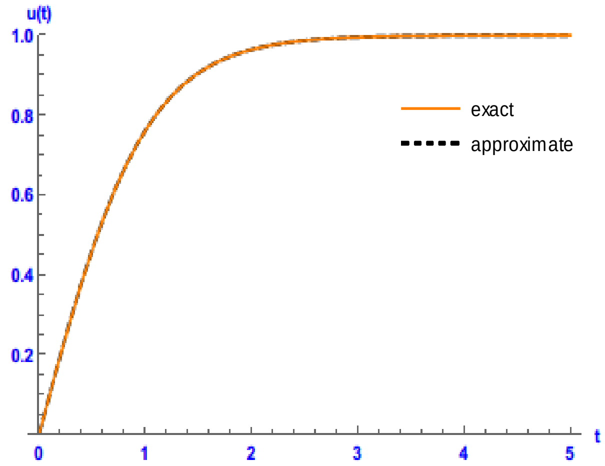

Figure 2. Comparison between MSLVIM technique with the exact solution

Example 2

Consider the nonlinear differential equation

$$p' = 4p - p^3(t), \tag{12}$$

with initial condition

$$p(0) = 0.5,$$

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where the exact solution is given by

$$p(t) = \frac{2e^{4t}}{\sqrt{15+e^{8t}}}.$$

Now, apply the MSLVIM as it explains in example (1), to Eq. 12 for getting the analytic solution. Table 2 demonstrate the accuracy, validity and the efficiency of this technique with the exact solution compared with the standard VIM. Figure 3. show the standard VIM with the exact solution, while figure 4. show the accuracy and efficiency of MSLVIM with the exact solution.

Table 2. Comparison between MSLVIM and VIM with the exact solution

t	*MSLVIM*	*VIM*	*Exact Solution*
0	0	0	0.5
0.5	9.31559×10^{-5}	3.64735×10^{-3}	1.77141
1.	9.63383×10^{-5}	9.7993×10^{-1}	1.99499
1.5	4.2961×10^{-6}	2.25137	1.99991
2.	1.26466×10^{-7}	1.56038×10^{8}	2.
2.5	3.21505×10^{-9}	8.09032×10^{12}	2.
3.	7.57915×10^{-11}	2.38688×10^{16}	2.
3.5	1.70646×10^{-12}	1.519410^{19}	2.
4.	3.73426×10^{-14}	3.58116×10^{21}	2.
4.5	9.82444×10^{-16}	4.15481×10^{23}	2.
5.	0	2.8196×10^{25}	2.

It can observe from the above examples, the results are more accurate, effective and more extended of the convergence region for whole time interval compared with the standard VIM.

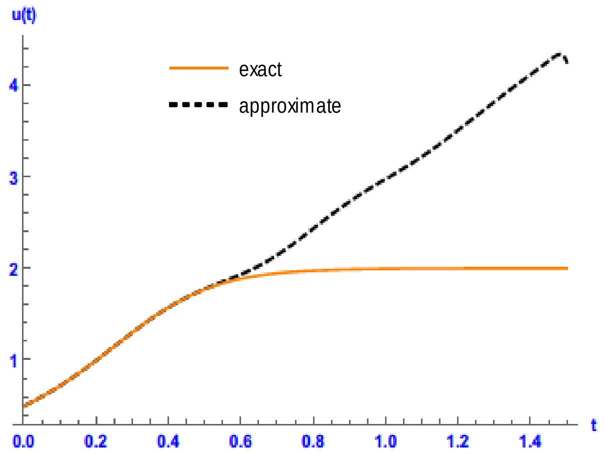

Figure 3. Comparison between VIM with the exact solution

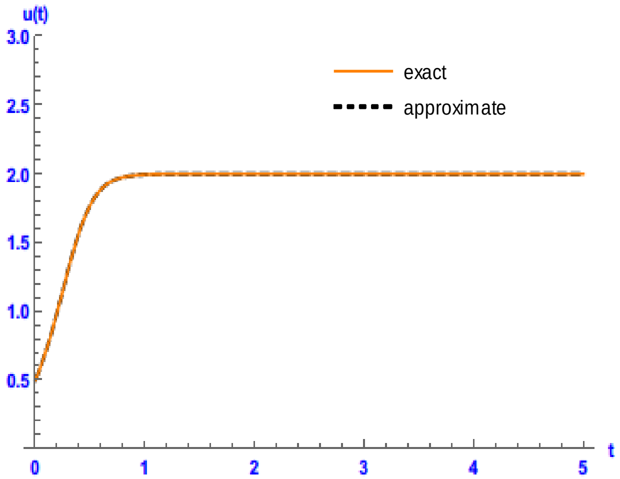

Figure 4. Comparison between MSLVIM technique with the exact solution

Conclusion:

In this letter, a new MSLVIM technique for solving generalized non-linear differential equations is proposed. Selected examples are conducted confirm that this technique an efficiency alternative and in certain cases an enhance and support of the VIM strategy. A multi-Step VIM is a reliable modification of the VIM which more improving an accurate solutions and extended of the convergence region for whole time interval compared to the series solution when the standard VIM failed for the larger time interval. This technique provide instant and visual symbol terms of approximately and analytically solutions for non-linear differential equations. A Laplace transform correction functional presents to enable us for expressing the integral in many cases in the form of detour. Laplace transform made the variational approach easier to tackling. A particularly for obtaining the general Lagrange multiplier which is very hard in some cases of non-linear form. A validity for the proposed technique has been shown successfully by applying it for non-linear form. Results indicate that this technique is effective and applicable to different types of non-linear problems in addition its reliable and promising compared with the existing approaches. Finally, we can use this technique for solving many types of fractional, partial and delay differential equations to get more an accurate results with more extended of the convergence region for future work.

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Authors' Declaration:

- Conflicts of Interest: None.

- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Mustansiriyah University.

Authors' Contributions Statement:

Kh. H. AL. and A. H. A. contributed to the design and implementation of the research, to the analysis of the results and to the writing of the manuscript.

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طريقة الملتي ستيب مع طريقة تحويلات لابلاس لشبه الطرق التقريبية التحليلية لحل المعادلات العامة التفاضلية الغيرخطية

خالد حمود الجيزاني **أحمد حنون عبود**

قسم الرياضيات، كلية العلوم، الجامعة المستنصرية، بغداد، العراق.

الخلاصة:

في هذا البحث تناولنا تقنية جديدة للحصول على الحل الاكثر دقة خلال استخدام طريقتي الملتي ستيب مع تحويلات لابلاس مدمجة مع طريقة الفيريشنال . الطريقة التقليدية او الكلاسيكية للفيرسشنال تعطي نتائج جيدة لكن افترة صغيرة لا تتجاوز الفترة بين الصفر و الواحد في كل الاحوال بينما التقنية المقترحة تتخطى هذه الصعوبات ولها القدرة بتزويدنا على حل اكثر دقة مع توسيع منطقة التقارب لكل الفترة المعطاة وبالتالي ستزودنا هذه التقتية ب حل مستمر يمكننا من دراسة السلوك الخاص للحل لكل التفرة المعطاة. هذه التقنية لها القدرة على ايجاد متعدد لاكرانج (لايمدا) بسهولة جدا . اضافة الى انه هذه التقنية تقلل الوقت والحسابات المعقدة . ميزة اخرى ان الحل الناتج يقترب بسرعة للحل المظبوط بتكرار و وقت قليل . للتحقق من هذه التقنية اخترنا مجموعة امثلة لعرض القدرة و الفعالية و الدقة لهذه التقنية.

الكلمات المفتاحية: متعددة لاكرانج، تحويل لابلاس، لابلاس مع الفاريشنل، طريقة الملتي ستيب مع تحويل لابلاس مع الفاريشنل، طريقة الملتي ستيب.