Open Access
Published Online First: November 2022

Baghdad Science Journal
2023, 20(3): 843-852

P-ISSN: 2078-8665
E-ISSN: 2411-7986

DOI: http://dx.doi.org/10.21123/bsj.2022.6888

Some Subclasses of Univalent and Bi-Univalent Functions Related to K-Fibonacci Numbers and Modified Sigmoid Function

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Received 31/12/2021, Revised 15/5/2022, Accepted 17/5/2022, Published Online First 20/11/2022
Published 1/6/2023

Abstract:

This paper is interested in certain subclasses of univalent and bi-univalent functions concerning to shell- like curves connected with k-Fibonacci numbers involving modified Sigmoid activation function $\theta(t)=\frac{2}{1+e^{-t}}$,t ≥ 0 in unit disk $|z|<1$. For estimating of the initial coefficients $|c_2|$, $|c_3|$, Fekete-Szegö inequality and the second Hankel determinant have been investigated for the functions in our classes.

Keywords: Bi-univalent function, Borel distribution, Fibonacci numbers, Modified sigmoid function, Univalent function.

Introduction

Let $\mathcal{J}$ be the class of normalized functions of the form

$f(z)=z+\sum_{m=2}^{\infty} c_m z^m,$ 1

which are analytic in $\mathfrak{U}=\{z\in\mathbb{C}:|z|<1\}$ and let $\mathcal{T}$ be the subclass of $\mathcal{J}$ consisting of functions univalent in $\mathfrak{U}$. It is well-known that according to Koebe-One Quarter Theorem, every function $f\in\mathcal{T}$ has an inverse f^{-1}holding

$z=f^{-1}(f(z)),(z\in\mathfrak{U})$

and

$w=f(f^{-1}(w))\ (|w|<r_0(f),\ r_0(f)\geq\frac{1}{4})$ where

$g(w)=f^{-1}(w)=w-c_2w^2+(2c_2^2-c_3)w^3-(5c_2^3-5c_2c_3+c_4)w^4+\cdots.$ 2

A function $f\in\mathcal{J}$ is said to be bi-univalent in $\mathfrak{U}$ if both f and f^{-1} are univalent in $\mathfrak{U}$. Let $\mathcal{B}$ denoted the class of normalized bi-univalent functions in $\mathfrak{U}$ given by Eq.1 [1]. Now, by recalling the principle of subordination for two analytic functions in $\mathfrak{U}$ and say that f is subordinate to g symbolically $f\prec g$ if there exists an analytic function $h(z)$ in $\mathfrak{U}$ with $h(0)=0$ and $|h(z)|<1, z\in\mathfrak{U}$ such that

$f(z)=g(h(z)),(z\in\mathfrak{U})$

if $g(z)\in\mathcal{J}$ of the form

$$g(z)=z+\sum_{m=2}^{\infty} d_m z^m,\qquad (z\in\mathfrak{U})$$

then the convolution of f and g is defined by

$$(f*g)(z)=z+\sum_{m=2}^{\infty} c_m d_m z^m,(z\in\mathfrak{U})$$

Magesh, Nirmala and Yamini [2] have calculated the k-Fibonacci number sequence $\{F_{k,m}\}_{m=0}^{\infty}$, $k\in\mathbb{R}^+$ defined by

$F_{k,m+1}=kF_{k,m}+F_{k,m-1},\ F_{k,0}=0,\ F_{k,1}=1,\ (m\in\mathbb{N}=1,2,\ldots)$ 3

and

$F_{k,m}=\frac{(k-\tau_k)^m-\tau_k^m}{\sqrt{k^2+4}}$ with $\tau_k=\frac{k-\sqrt{k^2+4}}{2},$ 4

where $F_{k,m}$ represents the mth element of the k-Fibonacci sequence [3].

Ozgur and Sokół [4] proved that if

$$\tilde{p}_k(z)=\frac{1+\tau_k^2z^2}{1-k\tau_kz-\tau_k^2z^2}=1+\sum_{m=1}^{\infty}\tilde{p}_{k,m}z^m, \qquad 5$$

then

$\tilde{p}_{k,m}=(F_{k,m-1}+F_{k,m+1})\,\tau_k^m,\ m\geq 1$

such that

$$\tilde{p}_k(z)=1+(F_{k,0}+F_{k,2})\tau_kz+(F_{k,1}+F_{k,3})\tau_k^2z^2+\cdots=1+k\tau_kz+(k^2+2)\tau_k^2z^2+(k^3+3k)\tau_k^3z^3+\cdots,$$

where $\tau_k = \frac{k-\sqrt{k^2+4}}{2}$, $(z \in \mathfrak{U})$
indeed, if $k = 1$, $\tau_k = \tau$ it is possible to write

$$F_m = \frac{\rho^m - \tau^m}{\sqrt{5}} = \frac{(1-\tau)^m - \tau^m}{\sqrt{5}}, \qquad m \in \mathbb{N}_0$$

such that $\rho = \frac{1+\sqrt{5}}{2} \approx 1.6180339887 \ldots$ is called the golden ratio and

$\tau = \frac{1-\sqrt{5}}{2} = 1 - \rho = -\frac{1}{\rho} \approx -0.6180339887 \ldots.$

Let $\mathcal{J}_\theta$ denote the class of functions of the form
$f_\theta(z) = z + \sum_{m=2}^{\infty} \frac{2}{1+e^{-t}} c_m z^m = z + \sum_{m=2}^{\infty} \theta(t) c_m z^m,$
where $\theta(t) = \frac{2}{1+e^{-t}}$, $t \geq 0$ is a modified sigmoid function. Obviously, $\theta(0) = 1$ and $\mathcal{J}_1 = \mathcal{J}$ [5].

A discrete random variable y has a Borel distribution if it takes the values 1,2,3,…with the probabilities

$$\frac{e^{-\lambda}}{1!}, \frac{2\lambda e^{-2\lambda}}{2!}, \frac{9\lambda^2 e^{-3\lambda}}{3!}, \ldots$$

where λ is the parameter, hence
Prob$(y = r) = \frac{(\lambda r)^{r-1} e^{-\lambda r}}{r!}$. $(r = 1,2,3, \ldots)$

The following power series whose coefficients are probabilities of the Borel distribution introduced in [6] as follows

$$\mathcal{W}(\lambda, z) = z + \sum_{m=2}^{\infty} \frac{(\lambda(m-1))^{m-2} e^{-\lambda(m-1)}}{(m-1)!} z^m, \quad (z \in \mathfrak{U}; 0 < \lambda \leq 1)$$

by the familiar Ratio Test, it implies that the radius of convergence of $\mathcal{W}(\lambda, z)$ is infinity.
Hence, the linear operator $\mathcal{BW}_\lambda$ defined as follows

$$\mathcal{BW}(\lambda, z) f_\theta(z) = \mathcal{W}(\lambda, z) * f_\theta(z) = z + \sum_{m=2}^{\infty} \frac{(\lambda(m-1))^{m-2} e^{-\lambda(m-1)}}{(m-1)!} \theta(t) c_m z^m,$$

where $(*)$ signal the Hadamard product(or convolution) of two series.
Now, it is possible to introduce a linear operator $\mathcal{K}_\lambda^\eta : \mathcal{J}_\theta \to \mathcal{J}_\theta$ for $f_\theta \in \mathcal{J}_\theta$ by

$$\mathcal{K}_\lambda^\eta f_\theta(z) = \mathcal{BW}(\lambda, z) * \mathcal{R}^\eta, \qquad 6$$

where $\mathcal{R}^\eta, (\eta = 0,1,2, \ldots)$ denote the Ruscheweyh derivative operator [7] defined by

$$\mathcal{R}^\eta f_\theta(z) = z + \sum_{m=2}^{\infty} \frac{\Gamma(\eta+m)}{\Gamma(\eta+1)\Gamma(m)} \theta(t) c_m z^m, \ (z \in \mathfrak{U})$$

from Eq.6, it follows that

$$\mathcal{K}_\lambda^\eta f_\theta(z) = z + \sum_{m=2}^{\infty} \frac{\Gamma(\eta+m)\big(\lambda(m-1)\big)^{m-2} e^{-\lambda(m-1)}}{\Gamma(\eta+1)\big(\Gamma(m)\big)^2} \theta(t) c_m z^m, \ (z \in \mathfrak{U})$$

for $0 < \lambda \leq 1$, $\eta = 0,1,2, \ldots$
which can be written as

$$\mathcal{K}_\lambda^\eta f_\theta(z) = z + \sum_{m=2}^{\infty} \mathcal{L}_m(m, \eta, \lambda) \theta(t) c_m z^m, \ (z \in \mathfrak{U})$$

such that

$$\mathcal{L}_m(m, \eta, \lambda) = \frac{\Gamma(\eta+m)\big(\lambda(m-1)\big)^{m-2} e^{-\lambda(m-1)}}{\Gamma(\eta+1)\big(\Gamma(m)\big)^2}. \qquad 7$$

Noonan and Thomas [8] announced the qth Hankel determinant for $q \geq 1$ and $m \geq 1$ as

$$H_q(m) = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+q-1} \\ c_{m+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ c_{m+q-1} & \cdots & \cdots & c_{m+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For example, Noor [9] determined the average of growth of $H_q(m)$ as $m \to \infty$ for functions given by Eq.1 with restricted boundary. Ehrenborg [10] studied the Hankel determinant of exponential polynomials. Also Hankel determinant was studied by different authors including Mehrok and Singh [11], Magesh and Yamini [12],Srivastava, Altinkaya and Yalcin [13] and Güney et al. [14]. Easily, one can observe that the Fekete-Szegö functional is $H_2(1)$. Fekete and Szegö [15] then further generalized the estimate of $|c_3 - \xi c_2^2|$ where ξ is real (for detail see [16,17,18,19]). For our discussion in this paper, the Hankel determinant in the case of $q = 2$ and $m = 2$ can be considered as

$\begin{vmatrix} c_2 & c_3 \\ c_3 & c_4 \end{vmatrix}.$

Definition1. A function $f(z) \in \mathcal{J}$ of the form Eq.1 belongs to the class $\mathcal{UN}_{\mathcal{B}}^{k}(\gamma, \alpha, \eta, \lambda, \theta(t), \widetilde{p}_k)$ if

$$\left[(1-\gamma+2\alpha)\frac{\mathcal{K}_\lambda^\eta f_\theta(z)}{z} + (\gamma-2\alpha)\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)' + \alpha z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)''\right] \prec \tilde{p}_k(z), (z \in \mathfrak{U})$$

where $0 \le \gamma,\ 0 \le \alpha,\ 0 < \lambda \le 1$, $\eta \in \mathbb{N}_0$ and $\theta(t) = \frac{2}{1+e^{-t}},\ t \ge 0$.

Definition2. A function $f(z) \in \mathcal{J}$ of the form Eq.1 belongs to the class $\mathcal{UV}_{\mathcal{B}}^{k}(\delta, \varrho, \eta, \lambda, \theta(t), \widetilde{p}_k)$ if

$$\left[1 + \frac{1}{\delta}\left(\frac{z^{1-\varrho}\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'}{\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)^{1-\varrho}} - 1\right)\right] \prec \tilde{p}_k(z), \quad (z \in \mathfrak{U})$$

for $\delta \in \mathbb{C}\backslash\{0\},\ \varrho \ge 0, 0 < \lambda \le 1$, $\eta \in \mathbb{N}_0$ and $\theta(t) = \frac{2}{1+e^{-t}},\ t \ge 0$.

Definition3. A function $f(z) \in \mathcal{J}$ of the form Eq.1 belongs to the class $\mathcal{BM}_{\mathcal{B}}^{k}(\vartheta, \eta, \lambda, \theta(t), \widetilde{p}_k)$ if it holds

$$\left[\frac{z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'}{\mathcal{K}_\lambda^\eta f_\theta(z)} - \frac{z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)' + \vartheta z^2\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)''}{(1-\vartheta)\mathcal{K}_\lambda^\eta f_\theta(z) + \vartheta z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'} + \frac{z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)''}{\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'} + 1\right] \prec \tilde{p}_k(z), \quad (z \in \mathfrak{U}) \qquad 8$$

and

$$\left[\frac{w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'}{\mathcal{K}_\lambda^\eta g_\theta(w)} - \frac{w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)' + \vartheta w^2\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)''}{(1-\vartheta)\mathcal{K}_\lambda^\eta g_\theta(w) + \vartheta w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'} + \frac{w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)''}{\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'} + 1\right] \prec \tilde{p}_k(w), \quad (w \in \mathfrak{U}) \qquad 9$$

where $0 \le \vartheta \le 1,\ 0 < \lambda \le 1$, $\eta \in \mathbb{N}_0$, $\theta(t) = \frac{2}{1+e^{-t}},\ t \ge 0$ and $g_\theta(w) = f_\theta^{-1}(w)$ is given by Eq.2.

In particular, for $\vartheta = 1$, the class $\mathcal{BM}_{\mathcal{B}}^{k}(\vartheta, \eta, \lambda, \theta(t), \widetilde{p}_k)$ reduces to the class $\mathcal{BM}_{\mathcal{B}}^{k}(\eta, \lambda, \theta(t), \widetilde{p}_k)$ of bi-starlike functions which satisfying

$\frac{z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'}{\mathcal{K}_\lambda^\eta f_\theta(z)} \prec \tilde{p}_k(z)$ and $\frac{w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'}{\mathcal{K}_\lambda^\eta g_\theta(w)} \prec \tilde{p}_k(w)$.

If choosing $\vartheta = 0$, the class $\mathcal{BM}_{\mathcal{B}}^{k}(\vartheta, \eta, \lambda, \theta(t), \widetilde{p}_k)$ reduces to the class $\mathcal{BM}_{\mathcal{B}}^{k}(\eta, \lambda, \theta(t), \widetilde{p}_k)$ of bi-convex functions which satisfying

$1 + \frac{z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)''}{\mathcal{K}_\lambda^\eta f_\theta(z)'} \prec \tilde{p}_k(z)$ and $1 + \frac{w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)''}{\mathcal{K}_\lambda^\eta g_\theta(w)'} \prec \tilde{p}_k(w)$.

Definition4. A function $f(z) \in \mathcal{J}$ of the form Eq.1 belongs to the class $\mathcal{BN}_{\mathcal{B}}^{k}(\gamma, \alpha, \eta, \lambda, \theta(t), \widetilde{p}_k)$ if it holds

$$\left[(1-\gamma+2\alpha)\frac{\mathcal{K}_\lambda^\eta f_\theta(z)}{z} + (\gamma-2\alpha)\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)' + \alpha z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)''\right] \prec \tilde{p}_k(z), \quad (z \in \mathfrak{U}) \qquad 10$$

and

$$\left[(1-\gamma+2\alpha)\frac{\mathcal{K}_\lambda^\eta g_\theta(w)}{w} + (\gamma-2\alpha)\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)' + \alpha w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)''\right] \prec \tilde{p}_k(w), \quad (w \in \mathfrak{U}) \qquad 11$$

where $0 \le \gamma,\ 0 \le \alpha,\ 0 < \lambda \le 1$, $\eta \in \mathbb{N}_0$, $\theta(t) = \frac{2}{1+e^{-t}},\ t \ge 0$ and $g_\theta(w) = f_\theta^{-1}(w)$ is given by Eq.2.

In particular,

i. If $\gamma = 1 + 2\alpha$, then the class $\mathcal{BN}_{\mathcal{B}}^{k}(\gamma, \alpha, \eta, \lambda, \theta(t), \widetilde{p}_k) = \mathcal{BN}_{\mathcal{B}}^{k}(\alpha, \eta, \lambda, \theta(t), \widetilde{p}_k)$ which satisfying

$$\left[\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)' + \alpha z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)''\right] \prec \tilde{p}_k(z), \ (z \in \mathfrak{U})$$

and

$$\left[\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)' + \alpha w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)''\right] \prec \tilde{p}_k(w). \ (w \in \mathfrak{U})$$

ii. When $\alpha=0$, then the class $\mathcal{BN}_{\mathcal{B}}^{k}(\gamma,\alpha,\eta,\lambda,\theta(t),\tilde{p}_k)=\mathcal{BN}_{\mathcal{B}}^{k}(\gamma,\eta,\lambda,\theta(t),\tilde{p}_k)$ which satisfying

$$\left[(1-\gamma)\frac{\mathcal{K}_\lambda^\eta f_\theta(z)}{z}+\gamma\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'\right]\prec\tilde{p}_k(z),\quad (z\in\mathfrak{U})$$

and

$$\left[(1-\gamma)\frac{\mathcal{K}_\lambda^\eta g_\theta(w)}{w}+\gamma\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'\right]\prec\tilde{p}_k(w).\quad (w\in\mathfrak{U})$$

iii. For $\gamma=1,\alpha=0$, then the class $\mathcal{BN}_{\mathcal{B}}^{k}(\gamma,\alpha,\eta,\lambda,\theta(t),\tilde{p}_k)=\mathcal{BN}_{\mathcal{B}}^{k}(\eta,\lambda,\theta(t),\tilde{p}_k)$ which satisfying $\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'\prec\tilde{p}_k(z)$ and $\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'\prec\tilde{p}_k(w)$. $(z,w\in\mathfrak{U})$

Definition5. A function $f(z)\in\mathcal{J}$ of the form Eq.1 belongs to the class $\mathcal{BV}_{\mathcal{B}}^{k}(\delta,\varrho,\eta,\lambda,\theta(t),\tilde{p}_k)$ if it holds

$$\left[1+\frac{1}{\delta}\left(\frac{z^{1-\varrho}\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'}{\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)^{1-\varrho}}-1\right)\right]\prec\tilde{p}_k(z),\quad (z\in\mathfrak{U})\qquad 12$$

and

$$\left[1+\frac{1}{\delta}\left(\frac{w^{1-\varrho}\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'}{\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)^{1-\varrho}}-1\right)\right]\prec\tilde{p}_k(w),\quad (w\in\mathfrak{U})\qquad 13$$

where $\delta\in\mathbb{C}\backslash\{0\}$, $\varrho\geq 0, 0<\lambda\leq 1$, $\eta\in\mathbb{N}_0$, $\theta(t)=\frac{2}{1+e^{-t}}$, $t\geq 0$ and $g_\theta(w)=f_\theta^{-1}(w)$ is given by Eq.2.

For $\varrho=0$, then the class $\mathcal{BV}_{\mathcal{B}}^{k}(\delta,\varrho,\eta,\lambda,\theta(t),\tilde{p}_k)=\mathcal{BV}_{\mathcal{B}}^{k}(\delta,\eta,\lambda,\theta(t),\tilde{p}_k)$ which satisfying

$$\left[1+\frac{1}{\delta}\left(\frac{z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'}{\mathcal{K}_\lambda^\eta f_\theta(z)}-1\right)\right]\prec\tilde{p}_k(z),\quad (z\in\mathfrak{U})$$

and

$$\left[1+\frac{1}{\delta}\left(\frac{w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'}{\mathcal{K}_\lambda^\eta g_\theta(w)}-1\right)\right]\prec\tilde{p}_k(w).\quad (w\in\mathfrak{U})$$

Lemma 1. [20] If $\mathcal{v}\in\mathcal{O}$, where $\mathcal{O}$ is the class of analytic functions holding $\mathrm{R}(\mathcal{v}(z))>0, z\in\mathfrak{U}$, with $\mathcal{v}(z)=1+\mathcal{v}_1z+\mathcal{v}_2z^2+\cdots$, then $|\mathcal{v}_i|\leq 2$ $,\forall i$.

Lemma 2. [21] If $\mathcal{v}(z)=1+\mathcal{v}_1z+\mathcal{v}_2z^2+\cdots$, and $\mathcal{v}(z)\prec\tilde{p}_k(z)$, then

$$|\mathcal{v}_1|\leq|\tau_k|,\ |\mathcal{v}_2|\leq 3\tau_k^2 \text{ and } |\mathcal{v}_3|\leq 4\tau_k^3.$$

In this paper, some subclasses of univalent and bi-univalent functions concerning to shell-like curves connected with k-Fibonacci numbers have been introduced. Also, estimates of the initial coefficients $|c_2|$, $|c_3|$ and Fekete-Szegö inequality have been obtained. Furthermore, the second Hankel determinant has been inspected for the functions in these classes.

Coefficient bounds and Fekete-Szegö inequality for the function classes $\mathcal{BM}_{\mathcal{B}}^{k}$, $\mathcal{BN}_{\mathcal{B}}^{k}$ and $\mathcal{BV}_{\mathcal{B}}^{k}$

Theorem 1.
For $0\leq\vartheta\leq 1$, $0<\lambda\leq 1$, $\eta\in\mathbb{N}_0$ and $\theta(t)=\frac{2}{1+e^{-t}}$, $t\geq 0$, if $f(z)$ of the form Eq.1 belongs to $\mathcal{BM}_{\mathcal{B}}^{k}(\vartheta,\eta,\lambda,\theta(t),\tilde{p}_k)$, then

$$|c_2|\leq\frac{k\sqrt{k}|\tau_k|}{\sqrt{|k^2\tau_k[2\mathcal{L}_3\theta(t)(3-2\vartheta)-\theta^2(t)\mathcal{L}_2^2(5-(\vartheta+1)^2)]+N_K(2-\vartheta)^2\mathcal{L}_2^2\theta^2(t)|}},\qquad 14$$

$$|c_3|\leq\frac{k|\tau_k|}{|2(3-2\vartheta)\mathcal{L}_3\theta(t)|}+\frac{k^3\tau_k^2}{|k^2\tau_k[2\mathcal{L}_3\theta(t)(3-2\vartheta)-\theta^2(t)\mathcal{L}_2^2(5-(\vartheta+1)^2)]+N_K(2-\vartheta)^2\mathcal{L}_2^2\theta^2(t)|},\qquad 15$$

and for $\xi\in\mathbb{R}$

$$|c_3-\xi c_2^2|\leq\begin{cases}\dfrac{|k||\tau_k|}{4(3-2\vartheta)\mathcal{L}_3\theta(t)}, & 0\leq T(\xi)\leq\dfrac{|k||\tau_k|}{8(3-2\vartheta)\mathcal{L}_3\theta(t)}\\ 4|T(\xi)||k||\tau_k|, & T(\xi)\geq\dfrac{|k||\tau_k|}{8(3-2\vartheta)\mathcal{L}_3\theta(t)}\end{cases},\qquad 16$$

where $N_K=k-(k^2+2)\tau_k$,

$$T(\xi)=\frac{(1-\xi)k^3\tau_k^2}{4k^2\tau_k[2\mathcal{L}_3\theta(t)(3-2\vartheta)-\theta^2(t)\mathcal{L}_2^2(5-(\vartheta+1)^2)]+4N_K(2-\vartheta)^2\mathcal{L}_2^2\theta^2(t)}$$

and $\mathcal{L}_m$ given by Eq.7.

Proof: Suppose that $f(z) \in$ Eq.9 $\mathcal{BM}_{\mathcal{B}}^{k}(\vartheta,\eta,\lambda,\theta(t),\tilde{p}_k)$, then have from Eq.8 and

$$\left[\frac{z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'}{\mathcal{K}_\lambda^\eta f_\theta(z)} - \frac{z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)' + \vartheta z^2\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)''}{(1-\vartheta)\mathcal{K}_\lambda^\eta f_\theta(z) + \vartheta z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'} + \frac{z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)''}{\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'} + 1\right] = \tilde{p}_k\big(u(z)\big), (z \in \mathfrak{U}) \qquad 17$$

and

$$\left[\frac{w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'}{\mathcal{K}_\lambda^\eta g_\theta(w)} - \frac{w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)' + \vartheta w^2\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)''}{(1-\vartheta)\mathcal{K}_\lambda^\eta g_\theta(w) + \vartheta w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'} + \frac{w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)''}{\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'} + 1\right] = \tilde{p}_k\big(q(w)\big), \ (w \in \mathfrak{U}) \qquad 18$$

where $g_\theta(w) = f_\theta^{-1}(w)$ is given by Eq.2 and $\tilde{p}_k$ given by Eq.5. Let $v(z) = 1 + v_1 z + v_2 z^2 + \cdots$, and $v \prec \tilde{p}_k$ .Then there exists an analytic function $u(z)$ such that $|u(z)| < 1$ in $\mathfrak{U}$ and $v(z) = \tilde{p}_k\big(u(z)\big)$. Therefore, the function

$$\mathcal{Y}(z) = \frac{1+u(z)}{1-u(z)} = 1 + u_1 z + u_2 z^2 + \cdots,$$

belongs to the class $\mathcal{O}$, then

$$u(z) = \frac{\mathcal{Y}(z)-1}{\mathcal{Y}(z)+1} = \frac{u_1}{2}z + \left(u_2 - \frac{u_1^2}{2}\right)\frac{z^2}{2} + \left(u_3 - u_1u_2 + \frac{u_1^3}{4}\right)\frac{z^3}{2} + \cdots,$$

and

$$\begin{aligned}\tilde{p}_k\big(u(z)\big) &= 1 + \tilde{p}_{k,1}\left(\frac{u_1 z}{2} + \left(u_2 - \frac{u_1^2}{2}\right)\frac{z^2}{2} + \left(u_3 - u_1u_2 + \frac{u_1^3}{4}\right)\frac{z^3}{2} + \cdots\right) \\ &+ \tilde{p}_{k,2}\left(\frac{u_1 z}{2} + \left(u_2 - \frac{u_1^2}{2}\right)\frac{z^2}{2} + \left(u_3 - u_1u_2 + \frac{u_1^3}{4}\right)\frac{z^3}{2} + \cdots\right)^2 \\ &+ \tilde{p}_{k,3}\left(\frac{u_1 z}{2} + \left(u_2 - \frac{u_1^2}{2}\right)\frac{z^2}{2} + \left(u_3 - u_1u_2 + \frac{u_1^3}{4}\right)\frac{z^3}{2} + \cdots\right)^3 + \cdots \\ &= 1 + \frac{\tilde{p}_{k,1}u_1 z}{2} + \left(\frac{1}{2}\left(u_2 - \frac{u_1^2}{2}\right)\tilde{p}_{k,1} + \frac{u_1^2}{4}\tilde{p}_{k,2}\right)z^2 \\ &+ \left(\frac{1}{2}\left(u_3 - u_1u_2 + \frac{u_1^3}{4}\right)\tilde{p}_{k,1} + \frac{1}{2}u_1\left(u_2 - \frac{u_1^2}{2}\right)\tilde{p}_{k,2} + \frac{u_1^3}{8}\tilde{p}_{k,3}\right)z^3 + \cdots. \qquad 19\end{aligned}$$

Furthermore, there exists an analytic function $q(w)$ with $|q(w)| < 1$ in $\mathfrak{U}$ such that $v(w) = \tilde{p}_k\big(q(w)\big)$. Therefore, the function

$$\mathcal{X}(w) = \frac{1+q(w)}{1-q(w)} = 1 + q_1 w + q_2 w^2 + \cdots,$$

belongs to the class $\mathcal{O}$, similarly

$$\begin{aligned}\tilde{p}_k(q(w)) = 1 &+ \frac{\tilde{p}_{k,1}q_1 w}{2} \\ &+ \left(\frac{1}{2}\left(q_2 - \frac{q_1^2}{2}\right)\tilde{p}_{k,1} + \frac{q_1^2}{4}\tilde{p}_{k,2}\right)w^2 \\ &+ \left(\frac{1}{2}\left(q_3 - q_1q_2 + \frac{q_1^3}{4}\right)\tilde{p}_{k,1} + \frac{1}{2}q_1\left(q_2 - \frac{q_1^2}{2}\right)\tilde{p}_{k,2} + \frac{q_1^3}{8}\tilde{p}_{k,3}\right)w^3 + \cdots. \qquad 20\end{aligned}$$

According to Eq.17, Eq.18, Eq.19 and Eq.20, getting

$$(2-\vartheta)\mathcal{L}_2\theta(t)c_2 = \frac{u_1 k\tau_k}{2}, \qquad 21$$

$$2(3-2\vartheta)\mathcal{L}_3\theta(t)c_3-(5-(\vartheta+1)^2)\theta^2(t)\mathcal{L}_2^2c_2^2=\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)k\tau_k+\frac{u_1^2}{4}(k^2+2)\tau_k^2, \quad 22$$

and

$$-(2-\vartheta)\mathcal{L}_2\theta(t)c_2=\frac{q_1k\tau_k}{2}, \quad 23$$

$$2\mathcal{L}_3\theta(t)(3-2\vartheta)(2c_2^2-c_3)-(5-(\vartheta+1)^2)\theta^2(t)\mathcal{L}_2^2c_2^2=\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)k\tau_k+\frac{q_1^2}{4}(k^2+2)\tau_k^2, \quad 24$$

from Eq.21 and Eq.23, get

$$u_1=-q_1, \quad 25$$

and

$$2(2-\vartheta)^2\mathcal{L}_2^2\theta^2(t)c_2^2=\frac{(u_1^2+q_1^2)k^2\tau_k^2}{4}, \quad 26$$

by adding Eq.22 and Eq.24, it implies that

$$[4\mathcal{L}_3\theta(t)(3-2\vartheta)-2\theta^2(t)\mathcal{L}_2^2(5-(\vartheta+1)^2)]c_2^2=\frac{1}{2}(u_2+q_2)k\tau_k-\frac{1}{4}\left(k\tau_k-(k^2+2)\tau_k^2\right)(u_1^2+q_1^2), \quad 27$$

substituting Eq.26 in Eq.27, it follows that

$$c_2^2=\frac{(u_2+q_2)k^3\tau_k^2}{4k^2\tau_k[2\mathcal{L}_3\theta(t)(3-2\vartheta)-\theta^2(t)\mathcal{L}_2^2(5-(\vartheta+1)^2)]+4N_K(2-\vartheta)^2\mathcal{L}_2^2\theta^2(t)}, \quad 28$$

where $N_K=k-(k^2+2)\tau_k$.
On using Lemma1, getting Eq.14.
Now, subtraction Eq.24 from Eq.22 and using Eq.25, can be obtained

$$c_3=c_2^2+\frac{(u_2-q_2)k\tau_k}{8(3-2\vartheta)\mathcal{L}_3\theta(t)}, \quad 29$$

then by Eq.28 and Lemma1, Eq.29 reduces to Eq.15.
From Eq.28 and Eq.29, for $\in\mathbb{R}$, get

$$c_3-\xi c_2^2=\frac{(u_2-q_2)k\tau_k}{8(3-2\vartheta)\mathcal{L}_3\theta(t)}+(1-\xi)\frac{(u_2+q_2)k^3\tau_k^2}{4k^2\tau_k[2\mathcal{L}_3\theta(t)(3-2\vartheta)-\theta^2(t)\mathcal{L}_2^2(5-(\vartheta+1)^2)]+4N_K(2-\vartheta)^2\mathcal{L}_2^2\theta^2(t)},$$

which can be expressed as

$$c_3-\xi c_2^2=\left[T(\xi)+\frac{k\tau_k}{8(3-2\vartheta)\mathcal{L}_3\theta(t)}\right]u_2+\left[T(\xi)-\frac{k\tau_k}{8(3-2\vartheta)\mathcal{L}_3\theta(t)}\right]q_2,$$

where

$$T(\xi)=\frac{(1-\xi)k^3\tau_k^2}{4k^2\tau_k[2\mathcal{L}_3\theta(t)(3-2\vartheta)-\theta^2(t)\mathcal{L}_2^2(5-(\vartheta+1)^2)]+4N_K(2-\vartheta)^2\mathcal{L}_2^2\theta^2(t)}.$$

Taking modulus, hence

$$|c_3-\xi c_2^2|\le\begin{cases}\dfrac{|k||\tau_k|}{4(3-2\vartheta)\mathcal{L}_3\theta(t)}, 0\le T(\xi)\le\dfrac{|k||\tau_k|}{8(3-2\vartheta)\mathcal{L}_3\theta(t)}\\[2ex] 4|T(\xi)||k||\tau_k|, T(\xi)\ge\dfrac{|k||\tau_k|}{8(3-2\vartheta)\mathcal{L}_3\theta(t)}.\end{cases}$$

Theorem 2.
For $0\le\gamma,\ 0\le\alpha,\ 0<\lambda\le1$, $\eta\in\mathbb{N}_0$, $\theta(t)=\frac{2}{1+e^{-t}}$, $t\ge0$, if $f(z)$ of the form Eq.1 belongs to $\mathcal{BN}_{\mathcal{B}}^k(\gamma,\alpha,\eta,\lambda,\theta(t),\widetilde{p}_k)$, then

$$|c_2|\le\frac{k\sqrt{k}|\tau_k|}{\sqrt{|k^2\tau_k(1+2\gamma+2\alpha)\mathcal{L}_3\theta(t)+N_K(1+\gamma)^2\mathcal{L}_2^2\theta^2(t)|}}, \quad 30$$

$$|c_3|\le\frac{k|\tau_k|}{|(1+2\gamma+2\alpha)\mathcal{L}_3\theta(t)|}+\frac{k^3\tau_k^2}{|k^2\tau_k(1+2\gamma+2\alpha)\mathcal{L}_3\theta(t)+N_K(1+\gamma)^2\mathcal{L}_2^2\theta^2(t)|}, \quad 31$$

and for $\xi\in\mathbb{R}$

$$\left|c_3-\xi c_2^2\right| \leq \begin{cases} \dfrac{|k||\tau_k|}{(1+2\gamma+2\alpha)\mathcal{L}_3\theta(t)}, 0 \leq |M(\xi)| \leq \dfrac{1}{(1+2\gamma+2\alpha)\mathcal{L}_3\theta(t)} \\ |k||\tau_k||M(\xi)|, \qquad |M(\xi)| \geq \dfrac{1}{(1+2\gamma+2\alpha)\mathcal{L}_3\theta(t)}, \end{cases} \quad 32$$

where $N_K=k-(k^2+2)\tau_k$,

$$M(\xi)=\frac{(1-\xi)k^3\tau_k^2}{k^2\tau_k(1+2\gamma+2\alpha)\mathcal{L}_3\theta(t)+N_K(1+\gamma)^2\mathcal{L}_2^2\theta^2(t)}$$

and $\mathcal{L}_m$ given by Eq.7.

Proof: Suppose that $f(z) \in \mathcal{BN}_{\mathcal{B}}^k(\gamma,\alpha,\eta,\lambda,\theta(t),\tilde{p}_k)$, then have from Eq.10 and Eq.11

$$\left[(1-\gamma+2\alpha)\frac{\mathcal{K}_\lambda^\eta f_\theta(z)}{z}+(\gamma-2\alpha)\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'+\alpha z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)''\right]=\tilde{p}_k\big(u(z)\big), (\, z\in\mathfrak{U}) \quad 33$$

and

$$\left[(1-\gamma+2\alpha)\frac{\mathcal{K}_\lambda^\eta g_\theta(w)}{w}+(\gamma-2\alpha)\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'+\alpha w\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)''\right]=\tilde{p}_k\big(q(w)\big), \ (\, w\in\mathfrak{U}) \quad 34$$

According to Eq.33, Eq.34, Eq.19 and Eq.20, give

$$(1+\gamma)\mathcal{L}_2\theta(t)c_2=\frac{u_1k\tau_k}{2}, \quad 35$$

$$(1+2\gamma+2\alpha)\mathcal{L}_3\theta(t)c_3=\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)k\tau_k+\frac{u_1^2}{4}(k^2+2)\tau_k^2 \quad 36$$

and

$$-(1+\gamma)\mathcal{L}_2\theta(t)c_2=\frac{q_1k\tau_k}{2}, \quad 37$$

$$(1+2\gamma+2\alpha)\mathcal{L}_3\theta(t)(2c_2^2-c_3)=\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)k\tau_k+\frac{q_1^2}{4}(k^2+2)\tau_k^2. \quad 38$$

The results Eq.30-Eq.32 of the Theorem 2 follow from Eq.35-Eq.38 by applying steps as in Theorem1 with respect to Eq.21-Eq.24.

Theorem 3.

For $\delta\in\mathbb{C}\backslash\{0\}$, $\varrho\geq 0$, $0<\lambda\leq 1$, $\eta\in\mathbb{N}_0$, $\theta(t)=\frac{2}{1+e^{-t}}$, $t\geq 0$, if $f(z)$ of the form Eq.1 belongs to $\mathcal{BV}_{\mathcal{B}}^k(\delta,\varrho,\eta,\lambda,\theta(t),\tilde{p}_k)$, then

$$|c_2|\leq\frac{\delta k\sqrt{2k}|\tau_k|}{\sqrt{|\delta k^2\tau_k(\varrho+2)[(\varrho-1)\theta^2(t)\mathcal{L}_2^2+2\theta(t)\mathcal{L}_3]+2N_K(\varrho+1)^2\mathcal{L}_2^2\theta^2(t)|}}, \quad 39$$

$$|c_3|\leq\frac{\delta k|\tau_k|}{|(\varrho+2)\theta(t)\mathcal{L}_3|}+\frac{2\delta^2k^3\tau_k^2}{|\delta k^2\tau_k(\varrho+2)[(\varrho-1)\theta^2(t)\mathcal{L}_2^2+2\theta(t)\mathcal{L}_3]+2N_K(\varrho+1)^2\mathcal{L}_2^2\theta^2(t)|}, \quad 40$$

and for $\xi\in\mathbb{R}$

$$\left|c_3-\xi c_2^2\right|\leq\begin{cases}\dfrac{|k||\delta||\tau_k|}{(\varrho+2)\theta(t)\mathcal{L}_3}, \ 0\leq|\xi-1|\leq\dfrac{L}{2|k|^2|\delta||\tau_k|(\varrho+2)\theta(t)\mathcal{L}_3} \\ \dfrac{2|1-\xi|\delta^2k^3\tau_k^2}{L}, \qquad |\xi-1|\geq\dfrac{L}{2|k|^2|\delta||\tau_k|(\varrho+2)\theta(t)\mathcal{L}_3}\end{cases}, \quad 41$$

such that $N_K=k-(k^2+2)\tau_k$,
L= $\delta k^2\tau_k(\varrho+2)[(\varrho-1)\theta^2(t)\mathcal{L}_2^2+2\theta(t)\mathcal{L}_3]+2N_K(\varrho+1)^2\mathcal{L}_2^2\theta^2(t)$
and $\mathcal{L}_m$ given by Eq.7.

Proof: Suppose that $f(z) \in \mathcal{BV}_{\mathcal{B}}^k(\delta,\varrho,\eta,\lambda,\theta(t),\tilde{p}_k)$, then from Eq.12 and Eq.13 obtaining

$$\left[1+\frac{1}{\delta}\left(\frac{z^{1-\varrho}\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'}{\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)^{1-\varrho}}-1\right)\right]=\tilde{p}_k\big(u(z)\big), \quad (z\in\mathfrak{U}) \qquad 42$$

and

$$\left[1+\frac{1}{\delta}\left(\frac{w^{1-\varrho}\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)'}{\left(\mathcal{K}_\lambda^\eta g_\theta(w)\right)^{1-\varrho}}-1\right)\right]=\tilde{p}_k\big(q(w)\big). \quad (w\in\mathfrak{U}) \qquad 43$$

According to Eq.42, Eq.43, Eq.19 and Eq.20, give

$$\frac{1}{\delta}(\varrho+1)\mathcal{L}_2\theta(t)c_2=\frac{u_1 k\tau_k}{2}, \qquad 44$$

$$\frac{1}{\delta}(\varrho+2)\left[\mathcal{L}_3\theta(t)c_3+\frac{(\varrho-1)}{2}\theta^2(t)\mathcal{L}_2^2c_2^2\right]=\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)k\tau_k+\frac{u_1^2}{4}(k^2+2)\tau_k^2, \qquad 45$$

and

$$-\frac{1}{\delta}(\varrho+1)\mathcal{L}_2\theta(t)c_2=\frac{q_1 k\tau_k}{2}, \qquad 46$$

$$\frac{1}{\delta}(\varrho+2)\left[\theta(t)\mathcal{L}_3(2c_2^2-c_3)+\frac{(\varrho-1)}{2}\theta^2(t)\mathcal{L}_2^2c_2^2\right]=\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)k\tau_k+\frac{q_1^2}{4}(k^2+2)\tau_k^2. \qquad 47$$

The results Eq.39-Eq.41 of the Theorem3 follow from Eq.44-Eq.47 by applying steps as in Theorem1 with respect to Eq.21-Eq.24.

Second Hankel determinant for the function classes $\mathcal{UN}_{\mathcal{B}}^k$ and $\mathcal{UV}_{\mathcal{B}}^k$

Theorem 4. Let $f(z)\in\mathcal{UN}_{\mathcal{B}}^k(\gamma,\alpha,\eta,\lambda,\theta(t),\tilde{p}_k)$. Then

$$\left|c_2c_4-c_3^2\right|\le\frac{[4\mathcal{L}_3^2(1+2\gamma+2\alpha)^2+9\mathcal{L}_2\mathcal{L}_4(1+\gamma)(1+3\gamma+6\alpha)]}{\theta^2(t)\mathcal{L}_2\mathcal{L}_3^2\mathcal{L}_4(1+\gamma)(1+3\gamma+6\alpha)(1+2\gamma+2\alpha)^2}\tau_k^4, \qquad 48$$

where $\mathcal{L}_m$ given by Eq.7 for $0\le\gamma,\ 0\le\alpha,\ 0<\lambda\le 1$, $\eta\in\mathbb{N}_0$, $\theta(t)=\frac{2}{1+e^{-t}},\ t\ge 0$.

Proof: Since $f(z)\in\mathcal{UN}_{\mathcal{B}}^k(\gamma,\alpha,\eta,\lambda,\theta(t),\tilde{p}_k)$, so definition1 , gets

$$(1-\gamma+2\alpha)\frac{\mathcal{K}_\lambda^\eta f_\theta(z)}{z}+(\gamma-2\alpha)\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'+\alpha z\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)''=v(z)=1+v_1z+v_2z^2+\cdots, \qquad 49$$

expanding and equating the coefficients in Eq.49, it follows that

$$c_2=\frac{v_1}{\theta(t)\mathcal{L}_2(1+\gamma)},$$

$$c_3=\frac{v_2}{\theta(t)\mathcal{L}_3(1+2\gamma+2\alpha)},$$

and

$$c_4=\frac{v_3}{\theta(t)\mathcal{L}_4(1+3\gamma+6\alpha)},$$

thus, Lemma 2, gets

$$\left|c_2c_4-c_3^2\right|\le\frac{[4\mathcal{L}_3^2(1+2\gamma+2\alpha)^2+9\mathcal{L}_2\mathcal{L}_4(1+\gamma)(1+3\gamma+6\alpha)]}{\theta^2(t)\mathcal{L}_2\mathcal{L}_3^2\mathcal{L}_4(1+\gamma)(1+3\gamma+6\alpha)(1+2\gamma+2\alpha)^2}\tau_k^4.$$

If $\alpha=\eta=\gamma=0$ and $\lambda=1$,Theorem 4 gives the next Corollary.

Corollary 1. If $(z)\in\mathcal{UN}_{\mathcal{B}}^k(\theta(t),\tilde{p}_k)$, then

$$\left|c_2c_4-c_3^2\right|\le\frac{35e^4}{3\theta^2(t)}\tau_k^4,$$

where $\theta(t)=\frac{2}{1+e^{-t}},\ t\ge 0$.

Theorem 5. Let $f(z)\in\mathcal{UV}_{\mathcal{B}}^k(\delta,\varrho,\eta,\lambda,\theta(t),\tilde{p}_k)$. Then

$$\left|c_2c_4-c_3^2\right|\le\frac{4\delta^2\tau_k^4}{\theta^2(t)\mathcal{L}_2\mathcal{L}_4(\varrho+3)(\varrho+1)}+\left[\frac{(2\varrho+3)}{\mathcal{L}_2\mathcal{L}_4(\varrho+3)}+\frac{1}{\mathcal{L}_3^2}\right]\frac{3\delta^3\tau_k^4(1-\varrho)}{\theta^2(t)(\varrho+1)^2(\varrho+2)}+\left[\frac{(2\varrho+3)(1-\varrho)+2}{2\mathcal{L}_2\mathcal{L}_4(\varrho+3)}+\frac{(1-\varrho)}{4\mathcal{L}_3^2}\right]\frac{\delta^4\tau_k^4(1-\varrho)}{\theta^2(t)(\varrho+1)^4}+\frac{9\delta^2\tau_k^4}{\theta^2(t)\mathcal{L}_3^2(\varrho+1)^2}, \qquad 50$$

where $\mathcal{L}_m$ given by Eq.7 for $\delta \in \mathbb{C}\backslash\{0\}$, $\varrho \geq 0$, $0 < \lambda \leq 1$, $\eta \in \mathbb{N}_0$, $\theta(t) = \frac{2}{1+e^{-t}}$, $t \geq 0$.

Proof: Since $f(z) \in \mathcal{UV}_{\mathcal{B}}^k(\delta, \varrho, \eta, \lambda, \theta(t), \widetilde{p}_k)$, so definition2 , gives

$$1 + \frac{1}{\delta}\left(\frac{z^{1-\varrho}\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)'}{\left(\mathcal{K}_\lambda^\eta f_\theta(z)\right)^{1-\varrho}} - 1\right) = \upsilon(z) = 1 + \upsilon_1 z + \upsilon_2 z^2 + \cdots, \quad 51$$

expanding and equating the coefficients in Eq.51, it implies that

$$c_2 = \frac{\delta\upsilon_1}{\theta(t)\mathcal{L}_2(\varrho+1)},$$

$$c_3 = \frac{\delta\upsilon_2}{\theta(t)\mathcal{L}_3(\varrho+2)} + \frac{\delta^2\upsilon_1^2(1-\varrho)}{2\theta(t)\mathcal{L}_3(\varrho+1)^2},$$

And

$$c_4 = \frac{\delta\upsilon_3}{\theta(t)\mathcal{L}_4(\varrho+3)} + \frac{\delta^2\upsilon_1\upsilon_2(2\varrho+3)(1-\varrho)}{\theta(t)\mathcal{L}_4(\varrho+3)(\varrho+1)(\varrho+2)} + \left[\frac{(2\varrho+3)(1-\varrho)+2}{2}\right]\frac{\delta^3\upsilon_1^3(1-\varrho)}{\theta(t)\mathcal{L}_4(\varrho+1)^3(\varrho+3)},$$

thus, Lemma 2 gives

$$\left|c_2c_4 - c_3^2\right| \leq \frac{4\delta^2\tau_k^4}{\theta^2(t)\mathcal{L}_2\mathcal{L}_4(\varrho+3)(\varrho+1)} + \left[\frac{(2\varrho+3)}{\mathcal{L}_2\mathcal{L}_4(\varrho+3)} + \frac{1}{\mathcal{L}_3^2}\right]\frac{3\delta^3\tau_k^4(1-\varrho)}{\theta^2(t)(\varrho+1)^2(\varrho+2)} + \left[\frac{(2\varrho+3)(1-\varrho)+2}{2\mathcal{L}_2\mathcal{L}_4(\varrho+3)} + \frac{(1-\varrho)}{4\mathcal{L}_3^2}\right]\frac{\delta^4\tau_k^4(1-\varrho)}{\theta^2(t)(\varrho+1)^4} + \frac{9\delta^2\tau_k^4}{\theta^2(t)\mathcal{L}_3^2(\varrho+2)^2}.$$

For $\varrho = \eta = 0$ and $\lambda = \delta = 1$,Theorem 5 gives the next Corollary.

Corollary 2. If $f(z) \in \mathcal{UV}_{\mathcal{B}}^k(\theta(t), \widetilde{p}_k)$, then

$$\left|c_2c_4 - c_3^2\right| \leq \frac{58e^4}{9\theta^2(t)}\tau_k^4,$$

such that $\theta(t) = \frac{2}{1+e^{-t}}$, $t \geq 0$.

Conclusion

In present paper**,** two subclasses $\mathcal{UN}_{\mathcal{B}}^k$ and $\mathcal{UV}_{\mathcal{B}}^k$ of univalent functions and some subclasses of bi-univalent functions have been introduced and studied involving convolution operator and k- Fibonacci numbers. For these univalent subclasses of functions, the second Hankel determinant has been obtained by taking advantage of the initial estimates of coefficient.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Anbar University.

Authors' contributions statement:

A M Rashid conceived the idea of the manuscript, interpreted all the results and provided the revision and proofreading. A R S. J collected an acquisition of all data and structured the drafting of the MS and the design for the manuscript

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بعض الفئات الفرعية لدوال أحادية التكافؤ وثنائية التكافؤ المتعلقة بأرقام فيبوناتشي K ودالة السيني المعدلة

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الخلاصة:

يهتم هذا البحث بفئات فرعية معينة من الدوال احادية التكافؤ وثنائية التكافؤ فيما يتعلق بمنحنيات تشبه الصدفة المرتبطة بأرقام فيبوناتشي k تتضمن دالة التنشيط السيني المعدلة $t \geq 0$, $\theta(t) = \frac{2}{1+e^{-t}}$في قرص الوحدة1 $|z| <$. تقديرات المعاملات الاولية $|c_2|$, $|c_3|$تم التحقق من عدم المساواة Fekete-Szeg$\ddot{o}$ ومحدد هانكل الثاني للدوال في فئاتنا.

الكلمات المفتاحية. دالة ثنائية التكافؤ, توزيع بوريل, أرقام فيبوناتشي ,تعديل الدالة السينية, دالة أحادية التكافؤ.