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Nonlinear Ritz Approximation for the Camassa-Holm Equation by Using the Modify Lyapunov-Schmidt method

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Abstract:

In this work, the modified Lyapunov-Schmidt reduction is used to find a nonlinear Ritz approximation of Fredholm functional defined by the nonhomogeneous Camassa-Holm equation and Benjamin-Bona-Mahony. We introduced the modified Lyapunov-Schmidt reduction for nonhomogeneous problems when the dimension of the null space is equal to two. The nonlinear Ritz approximation for the nonhomogeneous Camassa-Holm equation has been found as a function of codimension twenty-four.

Key words: Bifurcation of Solutions, Benjamin-Bona-Mahony equation, Camassa-Holm equation, Caustic, Modify Lyapunov-Schmidt method.

Introduction:

There are a lot of mathematical, physical, chemical, and engineering phenomena that are shown as nonlinear problems so can be described these problems as a nonlinear Fredholm operator.

$$g(x,\gamma)=\varphi, x\in S\subseteq X, \varphi\in Y, \gamma\in R^n \qquad 1$$

When g is a smooth Fredholm map with zero indexes and S is an open subset of Banach spaces. One of them is Y. Write the other one as X. To solve these problems may be used the method of reduction to the dimensional equation by solving this equation,

$$\theta(\xi,\gamma)=\beta, \xi\in E, \beta\in N, \qquad 2$$

When E and N are smooth manifolds of finite dimensional and $\theta: R^n\rightarrow R$ is a smooth function. The Lyapunov-Schmidt method can reduce Eq. 1 to Eq. 2, in which Eq. 2 has the same properties as Eq. 1, in particular topological properties (multiplicity) and analytical properties (bifurcation diagram), which are found in [1]. So that to study Eq. 1 it is sufficient to study Eq. 2.

Nonlinear problems are one subject of the greatest important subjects of mathematical phenomena possess received a great interest in scientific research in the last decades because of their wide set of geometry and scientific applications. Many of these studies focus on getting the bifurcation solutions of some equations, especially nonlinear partial differential equations (PDEs) that occur in Engineering, Physics, or mathematics. Also, in the Lyapunov-Schmidt method, the solutions in unlimited dimensional spaces coincide with the solutions in limited dimensional spaces. Therefore, the method is an important method in modernistic Mathematics to find analytical solutions. Many researchers have dealt with this method; it was previously called the alternative method by the researcher Krasnoselskii 1956 [2] who used it to study Bifurcation for extremely without boundaries while the implicit function theory was unable to be used. Sapronov and his group. For example, in [3] used the homogeneous solution to have the linear Ritz approximation represented by the function $\mathcal{W}(\zeta,\lambda)$ of the functional in Eq.1. Lyapunov-Schmidt method was also used to study boundary value problems, which can be seen in [4-7]. Abdul Hussain, Mayada[8] and Mizeal[9], study a bifurcation equation for a nonlinear system given by two algebraic equations.

Abdul Hussain [10] introduces a general method for finding nonlinear Ritz approximation of nonlinear Fredholm functionals. He introduces an example for finding a nonlinear Ritz approximation of the functional corresponding to the Duffing equation. Also, Abdul Hussain, 2015 [10] used a modified Lyapunov-Schmidt method to get a nonlinear Ritz approximation of the functional corresponding to the following equation

$$\frac{d^4v}{dx^4}+\alpha\frac{d^2v}{dx^2}+\beta v+v+v^2+v^3=0,$$

with boundary conditions

$$v(0)=v(2\pi)=v''(0)=v''(2\pi)=0$$

it is shown that the nonlinear Ritz approximation is a function given by,

$$\begin{aligned}\widehat{W}(\xi,\delta)=c_1\xi^{20}+c_2\xi^{18}+c_3\xi^{16}+c_4\xi^{14}+c_5\xi^{12}\\ +\alpha_1\xi^{10}+\alpha_2\xi^{8}+\alpha_3\xi^{6}+c_6\xi^{4}\\ +\alpha_4\xi^{2}+\ O(|\zeta|^{20})\\ +\ O(|\zeta|^{20})O(|\delta|)\end{aligned}$$

where $\xi=(\xi_1,\xi_2),\delta=\{c_{1,2,3,4,5,6},\alpha_{1,2,3,4}\}$ such that c,α are parameters.

In [11] Murtada used Lyapunov-Schmidt reduction (LSR) to study bifurcation solutions and the bifurcation diagram of the following nonlinear system

$$\sqrt{2\pi}\lambda_1X_1-X_1X_2-X_2X_3-X_3X_4=0$$

$$\sqrt{2\pi}\lambda_2X_2-{X_1}^2-2X_1X_3-2X_2X_4=0$$

$$\sqrt{2\pi}\lambda_3X_3+3X_1X_2-3X_1X_4=0$$

$$\sqrt{\pi}\lambda_4X_4+2\sqrt{2}X_1X_3+\sqrt{2}{X_2}^2=0$$

In 2017 Rosen [12] has been studied to modify the Lyapunov-Schmidt method to find a nonlinear Ritz approximation for nonlinear Fredholm functional defined by the nonlinear fourth ODE. In his study, he considered the following cases,

1. $\breve{v}=D^{(2)}(\zeta)$,
2. $\breve{v}=D^{(2)}(\zeta)+D^{(3)}(\zeta)$,
3. $\breve{v}=D^{(2)}(\zeta)+D^{(3)}(\zeta)+D^{(4)}(\zeta)$,
4. $\breve{v}=D^{(2)}(\zeta)+D^{(3)}(\zeta)+D^{(4)}(\zeta)+D^{(5)}(\zeta)$.

where $D^{(k)}(\zeta)$ are homogeneous polynomials of degree $k=1,2,3,4,5$ and $\zeta\in$ R.

In the last years, Kadhim[13] studied the bifurcation solution of extremes of the functions of codimensions eight and five at the origin by using Lyapunov-Schmidt reduction (LSR). In previous works, the presence and absence of u shaped solutions were studied using the Lyapunov-Schmidt method and Ritz linear approximation. As for our work, we study the presence and the absence of $u+v$ solutions using the modified Lyapunov-Schmidt method and the nonlinear Ritz approximation.

The goal of this paper is to find the nonlinear Ritz approximation of the functional corresponding to the nonhomogeneous Camassa-Holm equation.

Materials and Methods:

Methods:

Proposition 1[4]. Suppose that the triple $\{p,\varphi,N\}$ is an elliptic finite dimensional reduction for the functional V on a set Ω from the smooth Banach manifold M. Then the marginal map φ locates a one-to-one corresponding between the critical points for the functional V and the critical points for the key function W.

Lyapunov-Schmidt reduction (LSR)

The **LSR** was first suggested by Schmidt 1908 [14]. He discovered this method to get the solutions to operator equations. It is a method employed to solve the problems that possess variational property and the problems that do unpossessed variational property [1]. Variational problems can be solved in other ways like Boubaker Polynomials[15], but **LSR** has been successfully exercised to solve different nonlinear partial differential equations, as well as it has succeeded in finding bifurcation solutions to the equations, for example, Zainab and Mudhir [16], they found the bifurcation solutions for the equation of sixth order with boundary conditions using the Lyapunov-Schmidt method in the variational case. This method gives as follows:

Let E and K are real Banach spaces and $G:E\to K$ be a nonlinear Fredholm operator with zero index, when G is defined by

$$G(z,\gamma)=0,\ \ z\in E,\qquad \gamma\in\mathbb{R}^n.$$

Written the spaces E and K as a direct sum,

$$E=W\oplus W^{\perp},$$

$$K=\widetilde{W}\oplus\ \widetilde{W}^{\perp}$$

where $\dim(W)=\dim(\widetilde{W}\ )=n$ are subspaces of E and K respectively, the orthogonal spaces of W and $\widetilde{W}$ in E and K are $W^{\perp}$ and $\widetilde{W}^{\perp}$ respectively. Wherefore exist projections $P:E\to W$ & $(I-P):E\to W^{\perp}$ defined by $Pz=w$ & $(I-P)z=v$. where $e_1,e_2,\dots,e_k$ a basis of space W, then $\forall\ z\in E$ is written in a unique way:

$$z=w+v,\qquad w\in W,\qquad v\in W^{\perp},$$

$$w=\sum_{i=1}^{K}x_ie_i\,.$$

In the same way, exists projections $Q:K\to\widetilde{W}$ and $(I-Q\ ):K\to\widetilde{W}^{\perp}$ defined by

$$QH(z,\gamma)=G_1(z,\gamma)\quad \&\quad (I-Q)H(z,\gamma)=G_2(z,\gamma)\,.$$

where $g_1,g_2,\dots,g_k$ is the basis for space $\widetilde{W}$ then

$$H(z,\lambda)=H_1(z,\gamma)+H_2(z,\gamma),$$

$$H_1(z,\gamma)\in\widetilde{W},\qquad H_2(z,\gamma)\in\widetilde{W}^{\perp}\,,$$

$$H_1(z,\gamma)=\sum_{i=1}^{k}v_i(z,\gamma)g_i\ ,\quad H_2(z,\gamma)\perp\widetilde{W}\ .$$

It concludes that,

$$H(z,\gamma)=QH(z,\gamma)+(I-Q)H(z,\gamma)=0.$$

Hence, the result from it

$$QH(z,\gamma)=0$$

$$(I-Q)H(z,\gamma)=0$$

or

$$QH(w+v,\gamma)=0$$

$$(I-Q)H(w+v,\gamma)=0.$$

From implicit function theorem, exists a map $\theta: W \to W^{\perp}$ that is smooth defined by, $\theta(w,\gamma)=v$ and

$$(I-Q)H(w+\theta(w,\gamma),\gamma)=0.$$

To get the solutions of the equation $H(z,\gamma)=0$ at the neighborhood about a point $z=b$ it is sufficient to get solutions to the equation,

$$QH(w+\theta(w,\gamma),\gamma)=0\,.$$

The above equation is called bifurcation equation[11].

Modify Lyapunov-Schmidt method for the nonhomogeneous nonlinear differential equations (MLSM)

Modify Lyapunov-Schmidt method is a procedure for obtaining the nonlinear Ritz approximation to a Fredholm functional. MLSM is similar to the Lyapunov-Schmidt reduction but the **MLSM** is based on finding the particular solution of the operator Eq. 1 in the nonhomogeneous cases as follows:

Suppose the nonlinear operator which is Fredholm with zero index $f: E \to F$ such that

$$f(u,\gamma)=\Psi,\ \gamma\in R^{n}, u\in\Lambda\subset E \qquad 3$$

Where E, F are real Banach space, $\Psi=\varepsilon\varphi$ (ε-small parameter) is a continuous function and $\Lambda\subseteq E$ is open. let's say the operator f possesses a variational property, this means, there is a functional $V:\Lambda\subset E\to R$, such that $f=grad_H V$ when Λ is a bounded domain. Written operator f as:

$$f(u,\gamma)=\mathrm{H}u+Nu=\Psi, \Psi\in \mathrm{F}$$

Where $H=\frac{\partial f}{\partial u}(u_0,\gamma)$ is Frechet derivative of the operator f about the point u_0 and its linear continuous Fredholm operator and N represents the nonlinear operator for f. Applied the LSR, we get the following decomposition

$$E=W\oplus W^{\perp},\ F=\widehat{W}\oplus\widehat{W}^{\perp}$$

where $W=ker\ H$ is the null space of the operator f, (here $dim\ W=dim\ \widehat{W}=2$) and $W^{\perp},\widehat{W}^{\perp}$ the orthogonal complements of the subspaces W, $\widehat{W}$ respectively. If e_1, e_2 is an orthonormal set in W such that $He_{\mathrm{i}}=\alpha_{\mathrm{i}}(\gamma)e_{\mathrm{i}}$, $\alpha_{\mathrm{i}}(\gamma)$ is a continuous function, where $i=1{,}2$ then $\forall\, u \in E$ can be expressed in the unique format,

$$u=w+v,\ \ w=\xi_1e_1+\ \xi_1e_2\in W,\ \ W\perp v \in W^{\perp},\ \xi_i=\langle u,e_i\rangle,$$

When $\langle .,.\rangle$ represents the inner product in Hilbert space $\mathcal{H}$. So there are projections $p: E\to W$ & $I-p: E\to W^{\perp}$ defined by $\omega=pu$ & $(I-p)u=v$. Similarly, there exist two projections $Q: F\to W$ and $I-Q: F\to\widehat{W}^{\perp}$ defined by

$$f(u,\gamma)=Qf(u,\gamma)+(I-Q)f(u,\gamma)$$

Or 4

$$f(\omega+v,\gamma)=Qf(\omega+v,\gamma)+(I-Q)f(\omega+v,\gamma)$$

And we get

$$Qf(\omega+v,\gamma)=\Psi_1,\quad \Psi_1\in W$$

$$(I-Q)f(\omega+v,\gamma)=\Psi_2,\quad \Psi_2\in\widehat{W}^{\perp}$$

Where $\Psi=\Psi_1+\Psi_2$, $\Psi_1=t_1e_1+t_2e_2$ and here assume that,

$$\Psi_2=a_1{t_1}^2+a_2t_1t_2+a_3{t_2}^2$$

where $\mathrm{a}_i, i=1{,}2{,}3$ are constants and $\mathrm{t}_i, i=1{,}2$ are parameters.

By implicit function theorem getting

$$M(\xi,\beta)=V(\theta(\xi,\beta),\beta),\qquad \xi=(\xi_1,\xi_2,\dots,\xi_n)^{\perp}$$

Where $deg\ M\geq 2$, the functional V has the linear Ritz approximation represent by a function M defined by

$$M(\xi,\beta)=V(\textstyle\sum_{i=1}^{n}\xi_ie_i,\beta)=M_0(\xi)+M_1(\xi,\beta) \qquad 5$$

Where $M_0(\xi)$ represents a homogenous polynomial with degree $n\geq 3$ s.t $M_0(0)\ =\ 0$ & $M_1(\xi,\beta)$ is a polynomial function of degree $<n$. If $q_1, q_2, \dots\ q_m$ are the coefficients to the quadratic terms for the function $M_1(\xi,\beta)$, then can be written the function $M_1(\xi,\beta)$ in the formula,

$$M_1(\xi,\beta)=M_2(\xi,\beta)+\sum_{k=1}^{m}q_k\xi_k^2$$

Where $deg\ M_2=\ d, 2\ <\ d\ <\ n.$

The functional V has a nonlinear Ritz approximation, it's a function M defined by

$$M(\xi,\beta)=V\left(\sum_{i=1}^{n}\xi_ie_i+\theta\left(\sum_{i=1}^{n}\xi_ie_i,\beta\right),\beta\right)$$

When $\theta(\omega,\beta)=v(\mathrm{x},\xi,\beta), v\in \mathrm{N}^{\perp}$. Taylor's expansion to the functions $\mu_{\mathrm{k}}(\xi)$ and $v(\mathrm{x},\xi,\beta)$ will be used to determine the nonlinear Ritz approximation for the functional V, by assuming as following:

$$q_k=\hat{q}_k+\mu_k(\xi)=\hat{q}_k+\sum_{i=2}^{r}D_k^{j}(\xi),$$

$$k=1,\dots,m$$

$$v(x,\xi,\beta)=\sum_{i=2}^{r}B^{j}(\xi)$$

Where $D_k^{(j)}(\xi)$ and $B^{(j)}(\xi)$ are polynomials with degree j which be homogenous, have coefficients μ_{ki} and $v_{ji}(x,\beta)$ respectively, $\xi=(\xi_1,\xi_2,\dots,\xi_n)$.since

$$Qf(u,\gamma)=\langle f(u,\gamma),e_1\rangle e_1+\langle f(u,\gamma),e_2\rangle e_2=\Psi_1$$

It follows that

$$\langle Hu+Nu,e_1\rangle e_1+\langle Hu+Nu,e_2\rangle e_2=\Psi_1$$

Hence

$$q_1\xi_1e_1+q_2\xi_2e_2+\langle Nu,e_1\rangle e_1+\langle Nu,e_2\rangle e_2 =\Psi_1,\qquad q_i=\alpha_i(\gamma)$$

$$q_1\xi_1 e_1 + q_2\xi_2 e_2 + \left[\int_\Omega N(w+v)e_1\right]e_1 + \left[\int_\Omega N(w+v)e_2\right]e_2 = \Psi_1, \tag{6}$$

From Eq. 4 it follows that

$$(I-Q)f(u,\gamma) = f(u,\gamma) - Qf(u,\gamma).$$

From $H(\omega+v) + N(\omega+v) = \Psi_2$ it follows that

$$Hv + N(w,v) + q_1\xi_1 e_1 + q_2\xi_2 e_2 = \Psi_2, \tag{7}$$

Substituting the values of $q_{\mathrm{i}}, \mu_{\mathrm{i}}(\xi)$ and $v(\mathrm{x},\xi,\delta)$ in Eq.6 and Eq.7 yields

$$\left[\hat{q}_1 + \sum_{j=2}^{r}(D_1^j(\xi) + D_2^j(\xi))\right]\xi_1 e_1 + \left[\hat{q}_2 + \sum_{j=2}^{r}(D_1^j(\xi) + D_2^j(\xi))\right]\xi_2 e_2 + \left[\int_\Omega N(q_1\xi_1 e_1 + q_2\xi_2 e_2 + \sum_{j=2}^{r} B^j(\xi)e_1\right]e_1 + \left[\int_\Omega N(q_1\xi_1 e_1 + q_2\xi_2 e_2 \ \sum_{j=2}^{r} B^j(\xi)e_2\right]e_2 = \Psi_1 \tag{8}$$

$$\mathrm{H}\left(\sum_{j=2}^{r} B^j(\xi)\right) + N\left(q_1\xi_1 e_1 + q_2\xi_2 e_2 + \sum_{j=2}^{r} B^j(\xi)\right) + \left[\hat{q}_1 + \sum_{j=2}^{r}(D_1^j(\xi) + D_2^j(\xi))\right]\xi_1 e_1 + \left[\hat{q}_2 + \sum_{j=2}^{r}(D_1^j(\xi) + D_2^j(\xi))\right]\xi_2 e_2 = \Psi_2 \tag{9}$$

To calculate the functions $v(\mathrm{x},\xi,\beta)$ & $\mu_{\mathrm{k}}(\xi)$ equate the coefficients of $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ in Eq.8 to find the value of μ_{ki} and after some calculation from Eq.9, it is getting a linear ODE in the variable $v_{ji}(x,\gamma)$. Solving the equation which appears one can get the value to $v_{ji}(x,\gamma)$.

In the following section, we give two examples to find a nonlinear Ritz approximation for the functional corresponding to the nonhomogeneous Camassa-Holm Equation and Benjamin-Bona-Mahony equation as an application of the Modify Lyapunov-Schmidt method given above.

Results:

Nonlinear Ritz Approximation for the Camassa-Holm Equation (CH)

This section applied **MLSM** given in the previous section for finding nonlinear Ritz approximation for the functional corresponding to the nonhomogeneous Camassa-Holm equation.

Camassa and Holm in 1993[17], used the Hamiltonian method to find a new model for a completely integrable shallow water wave equation,

$$z_t + 2Kz_x - z_{xxt} + 3zz_x = 2z_x z_{xx} + zz_{xxx}, \tag{10}$$

where t is the time, z is the speed of the fluid in x trend and K is a constant number. Eq. 10 is known as Camassa-Holm (CH) equation. Moreover, in newly years, Camassa-Holm was generalized to the following equation,

$$z_t + 2Kz_x - z_{xxt} + \frac{1}{2}[f(z)]_x = 2z_x z_{xx} + zz_{xxx}, \tag{11}$$

when $f(z)$ is a function of z and $[f(z)]_x$ is the derivative of f for x.

Eq. 10 can obtain from Eq.11 by putting $\alpha = 3$ and $\beta = 0$ in the function $f(z) = \alpha z^2 + \beta z^3$. Let $z(x,t) = w(y), y = x - \alpha t$, when α the wave velocity. Eq. 11 transformed to the following ordinary differential equation for a variable $w(y)$,

$$\alpha w'' + \beta w + \frac{3}{2}w^2 - \left(\frac{1}{2}(w')2 + ww''\right) = \psi \tag{12}$$

where $' = \frac{d}{dy}$ and α, β are parameters.

Abdul Hussain provided a model in [10] for finding non-linear approximation and bifurcation solutions of differential equations of the fourth-order. The present section includes an example for finding the bifurcation of Eq. 12 with the coming boundary conditions which satisfy Eq. 10,

$$w(0) = w(1) = 0,$$

where $w = w(y), y \in [0,1]$.

To obtain a nonlinear approximation for the Camassa-Holm equation. Firstly, write Eq. 12 as a nonlinear Fredholm operator as follows:

$$g(w,\gamma) = \alpha w'' + \beta w + \frac{3}{2}w^2 - \left(\frac{1}{2}(w')^2 + ww''\right) \tag{13}$$

when $g\colon \mathrm{E} \longrightarrow M$ is Fredholm operator which is nonlinear of index zero from Banach space E to Banach space M, where $E = C^2([0,1],\mathbb{R})$ is the space of all continuous functions that have derivative of order at most two, $M = C([0,1],\mathbb{R})$ is the space of every continuous function, $\gamma = (\alpha,\beta)$. The operator g own variational property, so there is a functional V defined by,

$$g(w,\gamma) = grad_H V(w,\gamma)$$

Where

$$\mathrm{V}(\mathrm{w},\lambda,\psi) = \frac{1}{2}\int_0^1 \left(-\alpha(w')^2 + \beta w^2 + w^3 + \mathrm{w}(w')^2 - \mathrm{w}\psi\right)dx$$

Where $grad_H V$ denotes the gradient of V. Every solution of Eq. 12 is a solution of operator equation[18],

$$g(w,\lambda) = \psi, \psi \in F \tag{14}$$

From the definition of Fréchet derivative have,

$$\begin{aligned} g(w+\varepsilon\hbar) = {} & \alpha(w+\varepsilon\hbar)'' + \beta(w+\varepsilon\hbar) \\ & + \frac{3}{2}(w+\varepsilon\hbar)^2 \\ & - \left(\frac{1}{2}((w+\varepsilon\hbar)')^2 \right. \\ & \left. + (w+\varepsilon\hbar)(w+\varepsilon\hbar)''\right) \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial \varepsilon} = {} & \alpha\hbar'' + \beta\hbar + 3(w+\varepsilon\hbar)\hbar \\ & - \left((w+\varepsilon\hbar)'\hbar' + (w+\varepsilon\hbar)\hbar'' \right. \\ & \left. + (w+\varepsilon\hbar)''\hbar\right) \end{aligned}$$

$$\left.\frac{\partial g}{\partial \varepsilon}\right|_{\varepsilon=0} = \alpha\hbar'' + \beta\hbar + 3\mathrm{w}\hbar - (\mathrm{w}'\hbar' + \mathrm{w}\hbar'' + \mathrm{w}''\hbar)$$

The Fréchet the derivative at the point $(0,\gamma)$ of the nonlinear operator $g(w,\gamma)$ has the form,

$$dg(0,\lambda)\hbar = \alpha\hbar'' + \beta\hbar$$

And hence the linearized equation identical to Eq. 10 is defined by,

$$A\hbar = 0, \hbar \in E$$

$$A = dg(0,\gamma) = \alpha\frac{\mathrm{d}^2}{dx^2} + \beta, x \in [0,1], \qquad 15$$

$$\hbar(0) = \hbar(1) = 0$$

Eq. 15 is called a linearized equation.

The solution of the linearized Eq.15 verification of the boundary conditions is get by,

$$e_p = a_p\, sin(p\pi x), p = 1,2,3, \ldots \qquad 16$$

Substituting Eq. 16 in Eq. 15 has a characteristic equation identical to the above solution in the form,

$$\beta - \alpha p^2\pi^2 = 0$$

The equation above gives in the characteristic lines ($\alpha\beta - plane$), wherefore, a point of characteristic lines it's the points of (α,β) such that Eq.10 own nontrivial solutions. Can be found at the bifurcation point[18] in the space of parameters (α,β) from the point of intersection of the $\alpha\beta - plane$. As a result, (0,0) is a bifurcation point for Eq.10. And localized parameters for α,β gives by,

$$\hat{\alpha} = 0 + \Gamma_1, \hat{\beta} = 0 + \Gamma_2.$$

where Γ_1, Γ_2 are parameters which small lead to the below modes over the bifurcation.

$$e_1 = \sqrt{2}\sin(\pi x), e_2 = \sqrt{2}\sin(2\pi x)$$

Where the norms of $e_1 and\ e_2$ in Hilbert space ($\mathcal{H} = L_2([0,1],R)$) are equal to one, and $a_1 = a_2 = \sqrt{2}$. This means that $e_1 and\ e_2$ are the orthonormal basis of null space $\mathrm{W} = \ker(H)$.

Can separate the space E into subspace W and it's an orthogonal complement,

$$E = W \oplus \hat{E}, \hat{E} = W^{\perp} \cap E = \{v \in E : v \perp W\}$$

Likewise, the space M separated to subspace N it's an orthogonal complement as follows

$$F = N \oplus \hat{F}, \qquad \hat{F} = \mathrm{N} \cap F = \{v \in F : v \perp N\}$$

For that, there exist projections $j: E \to W$ & $I - j: E \to \hat{E}$ such that $jw = u$ and $(I - j)w = v$, so $\forall\, w \in E$ represented as $w = u + v, u = \sum_{i=1}^{2} \xi_i e_i, W \perp v \in \hat{E}, \xi_i = \langle w, e_i\rangle$ by the same way there are projection $G: F \to N$ & $I - G: F \to \hat{F}$ in which

$$g(u,\gamma) = Gg(u,\gamma) + (I - G)g(u,\gamma) = \psi, \psi = (w,t), t = (t_1,t_2)$$

Accordingly, Eq.1 can be represented as follows,

$$Gg(u + v,\gamma) = \psi_1$$

$$(I - G)g(u + v,\gamma) = \psi_2$$

Such that $\psi_1 = e_1 t_1 + e_2 t_2$ *and* $\psi_2 = a_1{t_1}^2 + a_2 t_1 t_2 + a_3{t_2}^2$

where $\mathrm{a}_i, i = 1,2,3$ are constants and $\mathrm{t}_i, i = 1,2$ are parameters.

From implicit function theory, obtain a map $\theta: W \to \hat{E}$ that is smooth satisfy,

$$W(\xi,\Gamma,\psi) = V(\theta(\xi,\gamma),\Gamma,\psi), \Gamma = (\Gamma_1,\Gamma_2)$$

By finding the functions $v(x,\xi,\gamma) = O(\xi^2)$, $\mu(\xi) = O(\xi), \tilde{\mu}(\xi) = O(\xi), \xi = (\xi_1,\xi_2)$ can get the nonlinear Ritz approximation of $V(\theta(\xi,\gamma),\Gamma,\psi)$, when

$$q_1 = \widetilde{q_1} + \mu(\xi_1,\xi_2), q_2 = \widetilde{q_2} + \tilde{\mu}(\xi_1,\xi_2)$$

$$\left.\begin{aligned} v(x,\xi,\gamma) = v_0(x,\lambda)\xi_1^2 + v_1(x,\lambda)\xi_1\xi_2 + v_2(x,\lambda)\xi_2^2 + \cdots \\ \mu(\xi_1,\xi_2) = \mu_0\xi_1 + \mu_1\xi_2 \\ \tilde{\mu}(\xi_1,\xi_2) = \tilde{\mu}_0\xi_1 + +\tilde{\mu}_1\xi_2 \end{aligned}\right\}$$

written Eq. 14 as follows

$$g(u,\gamma) = Au + Tu = \psi,$$

When $Aw = \alpha\frac{d^2w}{dx^2} + \beta w$ represents a linear part while $Tw = \frac{3}{2}w^2 - \left(\frac{1}{2}(w')^2 + ww''\right)$ is the nonlinear part of Eq. 13. Since

$$Qf(w,\lambda) = \sum_{i=1}^{2}\langle f(w,\lambda), e_i\rangle e_i = \psi_1,$$

obtaining

$$\sum_{i=1}^{2}\langle A(w) + T(w), e_i\rangle e_i = \sum_{i=1}^{2}\left(\int_0^{\pi}(A(w)e_i + T(w)e_i)dx\right)e_i = \psi_1.$$

Thus,

$$(q_1\xi_1 + \frac{3}{2}\int_0^1(\xi_1e_1 + \xi_2e_2 + v)^2 e_1dx - \frac{1}{2}\int_0^1((\xi_1e_1 + \xi_2e_2 + v)')^2 e_1dx - \int_0^1(\xi_1e_1 + \xi_2e_2 + v)(\xi_1e_1 + \xi_2e_2 + v)''e_1dx)e_1 + (q_2\xi_2 + \frac{3}{2}\int_0^1(\xi_1e_1 + \xi_2e_2 + v)^2 e_2dx - \frac{1}{2}\int_0^1((\xi_1e_1 + \xi_2e_2 + v)')^2 e_2dx - \int_0^1(\xi_1e_1 + \xi_2e_2 + v)(\xi_1e_1 + \xi_2e_2 + v)''e_2dx)e_2 = t_1e_1 + t_2e_2 \qquad 18$$

And

$$\alpha v'' + \beta v + \frac{3}{2}(\xi_1e_1 + \xi_2e_2 + v)^2 - \frac{1}{2}((\xi_1e_1 + \xi_2e_2 + v)')^2 - (\xi_1e_1 + \xi_2e_2 + v)(\xi_1e_1 + \xi_2e_2 + v)'' + q_1\xi_1e_1 + q_2\xi_2e_2 = a_1{t_1}^2 + a_2t_1t_2 + a_3{t_2}^2 \qquad 19$$

by substituting $q_1 = \widetilde{q_1} + \mu(\xi_1,\xi_2)$ and $q_2 = \widetilde{q_2} + \tilde{\mu}(\xi_1,\xi_2)$ in Eq.18 and Eq.19, obtaining

$$\Big[\left(\widetilde{q_1} + \mu(\xi_1,\xi_2)\right)\xi_1 + \frac{3}{2}\xi_1^2\int_0^1 e_1^3dx + 3\xi_1\xi_2\int_0^1 e_1^2e_2dx + \frac{3}{2}\xi_2^2\int_0^1 e_1e_2^2dx - \frac{1}{2}\xi_1^2\int_0^1 e_1{e'}_1^2dx - \xi_1\xi_2\int_0^1 e_1e_1'{e'}_2dx - \frac{1}{2}\xi_2^2\int_0^1 e_1{e'}_2^2dx - \xi_1^2\int_0^1 {e_1}^2{e_1}''dx - \xi_1\xi_2\int_0^1 e_1^2{e''}_2dx - \xi_1\xi_2\int_0^1 e_1e_2e_1''dx - \xi_2^2\int_0^1 e_1e_2{e''}_2dx\Big]e_1 + \Big[\left(\widetilde{q_2} + \tilde{\mu}(\xi_1,\xi_2)\right)\xi_2 + \frac{3}{2}\xi_1^2\int_0^1 e_1^2e_2dx + 3\xi_1\xi_2\int_0^1 e_2^2e_1dx + \frac{3}{2}\xi_2^2\int_0^1 e_2^3dx - \frac{1}{2}\xi_1^2\int_0^1 e_2{e'}_1^2dx - \xi_1\xi_2\int_0^1 e_2e_1'{e'}_2dx - \frac{1}{2}\xi_2^2\int_0^1 e_2{e'}_2^2dx -$$

$$\xi_1^2\int_0^1 e_1e_2{e_1}''dx-\xi_1\xi_2\int_0^1 e_2^2{e''}_1dx-\xi_1\xi_2\int_0^1 e_1e_2e_2''dx-\xi_2^2\int_0^1 {e_2}^2{e''}_2dx\Big]e_2=t_1e_1+t_2e_2 \qquad 20$$

$$\alpha v''+\beta v+\frac{3}{2}(\xi_1^2e_1^2+2e_1e_2\xi_1\xi_2+\xi_2^2e_2^2+2ve_1\xi_1+2ve_2\xi_2+v^2)-\frac{1}{2}(\xi_1^2{e'}_1^2+2{e'}_1{e'}_2\xi_1\xi_2+\xi_2^2{e'}_2^2+2v'{e'}_1\xi_1+2v'{e'}_2\xi_2+{v'}^2)-(\xi_1^2e_1{e_1}''+\xi_1\xi_2e_1e_2''+v''e_1\xi_1+{e''}_1e_2\xi_1\xi_2+\xi_2^2e_2e_2''+v''e_2\xi_2+v{e''}_1\xi_1+v{e''}_2\xi_2+vv'')+\big(\widetilde{q_1}+\mu(\xi_1,\xi_2)\big)\xi_1e_1+\big(\widetilde{q_2}+\tilde{\mu}(\xi_1,\xi_2)\big)\xi_2e_2=a_1{t_1}^2+a_2t_1t_2+a_3{t_2}^2 \qquad 21$$

The functions $v(x,\xi,\lambda),\mu(\xi)\ and\ \tilde{\mu}(\xi)$ in Eq. 17 determine by finding the coefficients $\mu_0,\mu_1,,\tilde{\mu}_0,\tilde{\mu}_1,,v_0,v_1,and\ v_2$ in Eq. 20, 21.

By equating the coefficients of ξ_1^2 in Eq. 20 and 21, then getting two equations,

$$\Big[\mu_0+\frac{3}{2}\int_0^1 e_1^3dx-\frac{1}{2}\int_0^1 e_1{e'}_1^2dx-\int_0^1 {e_1}^2{e_1}''dx\Big]e_1+\Big[\frac{3}{2}\int_0^1 e_1^2e_2dx-\frac{1}{2}\int_0^1 e_2{e'}_1^2dx-\int_0^1 e_1e_2{e_1}''dx\Big]e_2=0 \qquad 22$$

$$\alpha v_0''+\beta v_0+\frac{3}{2}e_1^2-\frac{1}{2}{e'}_1^2-e_1{e_1}''+\mu_0e_1=0. \quad 23$$

Eq.22 gives $\mu_0=-\frac{1}{6}\frac{(-9\pi^5-4\pi^2\sqrt{2}+24\sqrt{2})}{\pi}$ substitute for this value in ODE Eq.23

$$\alpha v_0''+\beta v_0+\frac{3}{2}e_1^2-\frac{1}{2}{e'}_1^2-e_1{e_1}''-\frac{1}{6}\frac{(-9\pi^5-4\pi^2\sqrt{2}+24\sqrt{2})}{\pi}e_1=0$$

And then have

$$v_0=\frac{3(\pi^2+1)}{-4\alpha\pi^2+\beta}cos(\pi x)^2+\frac{(9\pi^5\sqrt{2}+8\pi^2-48)}{6\pi(-\pi^2\alpha+\beta)}sin(\pi x)+\frac{(2\pi^2-3)\beta+(2\pi^4+6\pi^2)\alpha}{\beta(-4\pi^2\alpha+\beta)}$$

Now, to find coefficients of $\xi_1\xi_2$,

$$\Big[\mu_1+3\int_0^1 e_1^2e_2dx-\int_0^1 e_1e_1'{e'}_2dx-\int_0^1 e_1^2{e''}_2dx-\int_0^1 e_1e_2e_1''dx\Big]e_1+\Big[\tilde{\mu}_0+3\int_0^1 e_2^2e_1dx-\int_0^1 e_2e_1'{e'}_2dx-\int_0^1 e_2^2{e''}_1dx-\int_0^1 e_1e_2e_2''dx\Big]e_2=0 \qquad 24$$

24

$$\alpha v_1''+\beta v_1+3e_1e_2-{e'}_1{e'}_2-e_1e_2''-{e''}_1e_2+\mu_1e_1+\tilde{\mu}_0e_2=0 \qquad 25$$

From Eq.24 get $\mu_1=0$, and $\tilde{\mu}_0=-\frac{16}{5}\frac{(3\pi^2+2)\sqrt{2}}{\pi}$, so that, Eq.(25) becomes

$$\alpha v_1''+\beta v_1+3e_1e_2-{e'}_1{e'}_2-e_1e_2''-{e''}_1e_2-\frac{16}{5}\frac{(3\pi^2+2)\sqrt{2}}{\pi}e_2=0$$

The solution of *ODE* gives the function v_1 as follows

$$v_1=-\frac{28\left(\pi^2+\frac{3}{7}\right)}{9\pi^2\alpha+\beta}cos(\pi\, x)^3+\frac{64(3\pi^2+2)}{5\pi(-4\pi^2\alpha+\beta)}sin(\pi\, x)cos(\pi\, x)+\frac{(48\pi^4+36\pi^2)\alpha+(-24\pi^2+12)\beta}{(-\pi^2\alpha+\beta)(-9\pi^2\alpha+\beta)}cos(\pi\, x)$$

Equating the coefficients of ξ_2^2, have

$$\Big[\frac{3}{2}\int_0^1 e_1e_2^2dx-\frac{1}{2}\int_0^1 e_1{e'}_2^2dx-\int_0^1 e_1e_2{e''}_2dx\Big]e_1+\Big[\tilde{\mu}_1+\frac{3}{2}\int_0^1 e_2^3dx-\frac{1}{2}\int_0^1 e_2{e'}_2^2dx-\int_0^1 {e_2}^2{e''}_2dx\Big]e_2=0$$

$$\alpha v_2''+\beta v_2+\frac{3}{2}e_2^2-\frac{1}{2}{e'}_2^2-e_2e_2''+\tilde{\mu}_1e_2=0, 26$$

hence $\tilde{\mu}_1=0$ that implies Eq. 26, becomes

$$\alpha v_2''+\beta v_2+\frac{3}{2}e_2^2-\frac{1}{2}{e'}_2^2-e_2e_2''=0,$$

and the solution for this equation

$$v_2=-\frac{3(4\pi^2+1)}{32\pi^2\alpha+2\beta}cos(4\pi x)-\frac{2}{\beta}\left(\pi^2+\frac{3}{4}\right)$$

So, the nonlinear approximation for Eq. 12 was found by substituting the values of $\mu_0,\mu_1,,\tilde{\mu}_0,\tilde{\mu}_1,,v_0,v_1,and\ v_2$ in Eq. 17,

$$w(x,\xi)=\sqrt{2}\xi_1\sin(\pi x)+\sqrt{2}\xi_2\sin(2\pi x)+\Big[\frac{3(\pi^2+1)}{-4\alpha\pi^2+\beta}\cos(\pi x)^2+\frac{(9\pi^5\sqrt{2}+8\pi^2-48)}{6\pi(-\pi^2\alpha+\beta)}\sin(\pi x)+\frac{(2\pi^2-3)\beta+(2\pi^4+6\pi^2)\alpha}{\beta(-4\pi^2\alpha+\beta)}\Big]\xi_1^2+\Big[-\frac{28\left(\pi^2+\frac{3}{7}\right)}{9\pi^2\alpha+\beta}\cos(\pi x)^3+\frac{64(3\pi^2+2)}{5\pi(-4\pi^2\alpha+\beta)}\sin(\pi x)\cos(\pi x)+\frac{(48\pi^4+36\pi^2)\alpha+(-24\pi^2+12)\beta}{(-\pi^2\alpha+\beta)(-9\pi^2\alpha+\beta)}\cos(\pi x)\Big]\xi_1\xi_2+\Big[-\frac{3(4\pi^2+1)}{32\pi^2\alpha+2\beta}cos(4\pi x)-\frac{2}{\beta}\left(\pi^2+\frac{3}{4}\right)\Big]\xi_2^2 \qquad 27$$

$$q_1=\widetilde{q_1}-\frac{1}{6}\frac{(-9\pi^5-4\pi^2\sqrt{2}+24\sqrt{2})}{\pi}\xi_1,$$

$$q_2=\widetilde{q_2}-\frac{16}{5}\frac{(3\pi^2+2)\sqrt{2}}{\pi}\xi_1$$

Eq.27 is a solution of the functional $V(u,\lambda)$. which is represent the nonlinear Ritz approximation of V.

From the above results we deduced the following theorem

Theorem 1. The nonlinear Ritz approximation of the functional

$$\mathrm{V}\,(\mathrm{w},\gamma,\psi)=\frac{1}{2}\int_0^1(-\alpha(\mathrm{w}')^2+\beta\mathrm{w}^2+\mathrm{w}^3+w(\mathrm{w}')-\mathrm{w}\psi)dx.$$

is given by the function

$$W(\xi,\delta)=U(\xi,\delta)+o(|\xi|^6)+O(|\xi|^6)O(|\delta|),$$

$$\begin{aligned}U(\xi,\delta)=\gamma_1{\xi_1}^6+\gamma_2{\xi_2}^6+\gamma_3{\xi_1}^4{\xi_2}^2+\gamma_4{\xi_1}^2{\xi_2}^4\\+\gamma_5{\xi_1}^5+\gamma_6\xi_1{\xi_2}^4\\+\gamma_7{\xi_1}^3{\xi_2}^2+\gamma_8{\xi_1}^4+\gamma_9{\xi_2}^4\\+\gamma_{10}{\xi_1}^2{\xi_2}^2+\gamma_{11}{\xi_1}^3+\lambda_1{\xi_1}^2\\+\lambda_2{\xi_2}^2-\frac{1}{2}t_1\xi_1-\frac{1}{2}t_2\xi_2\end{aligned}$$

Where
$\gamma_i = \gamma_i(\alpha, \beta), i = 1,2, \dots ,11$,
$\lambda_i = \lambda_i(\alpha, \beta, t), i = 1,2$
$\xi = (\xi_1, \xi_2), \delta = (\gamma_i, \lambda_i)$ such that λ, γ are parameters.

Proof:
To determine the key function of $V(w, \gamma, \psi)$ wall substituting Eq.27 in the functional

$$V(w, \gamma, \psi) = \frac{1}{2}\int_0^1 (-\alpha(w')^2 + \beta w^2 + w^3 + w(w') - w\psi)dx.$$

And after solving it get the function $W(\xi, \delta)$.

The geometry of the bifurcation of critical points and the principal asymptotic of the branches of bifurcating points for the function $W(\xi, \delta)$ are entirely determined by its principal part $U(\xi, \delta)$. The function $W(\xi, \delta)$ has all the topological and analytical properties of functional $V(w, \gamma, \psi)$. The spreading of the critical points of the function $W(\xi, \delta)$ depends on the change of parameter δ and will be discussed in this paper as follows:

The study of the discriminant set of function $W(\xi, \delta)$ it not easy to find so, we will use maple 16 to find the discriminant set of the above function $W(\xi, \delta)$, in particular, we will fix the values of $\lambda_1, \gamma_i, i = 1,2, . . ,11$. and then to find all sections of discriminant set in the $\lambda_2 t_1 t_2 -$ surfaces, so we have three cases.

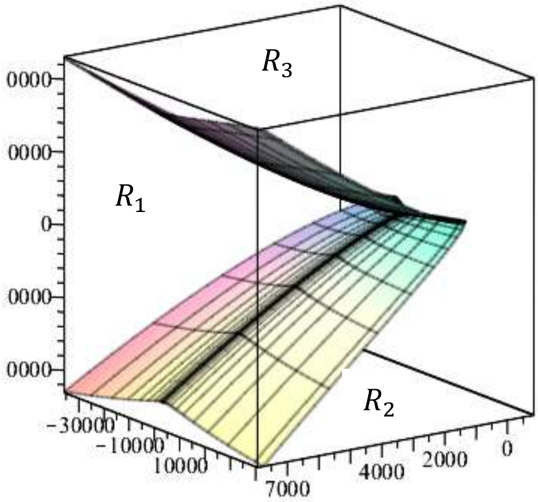

Figure 1. Describe Caustic when $\gamma_1 = \gamma_7 = 1, \gamma_2 = \gamma_8 = -2, \gamma_3 = \gamma_9 = 3, \gamma_4 = \gamma_{10} = 0.2, \gamma_5 = \gamma_{11} = 0.3, \gamma_6 = \lambda_1 = 0.4$

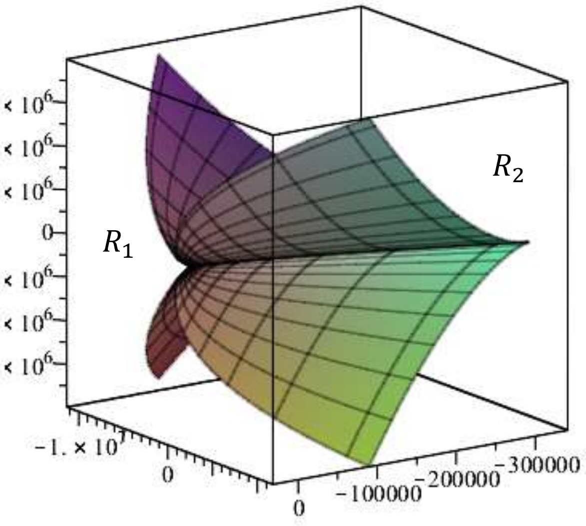

Figure 2. Describe Caustic when $\gamma_1 = 6, \gamma_2 = -5, \gamma_3 = \gamma_9 = 33, \gamma_4 = 0.56, \gamma_5 = 0.88, \gamma_6 = 0.77, \gamma_7 = 11, \gamma_8 = -22, \gamma_{10} = 0.92, \gamma_{11} = 0.93, \lambda_1 = 0.64$

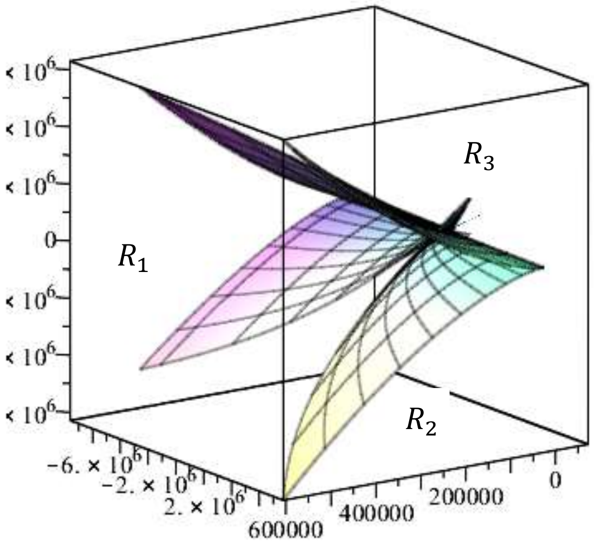

Figure 3. Describe Caustic when $\gamma_1 = -6, \gamma_2 = -5, \gamma_3 = -0.33, \gamma_4 = -56, \gamma_5 = -1, \gamma_6 = 77, \gamma_7 = 11, \gamma_8 = -22, = \gamma_9 = 33, \gamma_{10} = 92, \gamma_{11} = 22, \lambda_1 = 64$

The bifurcation propagation of the critical points to the function $W(\xi, \delta)$ is given as follows:

In Fig.1, the caustic (bifurcation set) of function $W(\xi, \delta)$ Split the space of parameters into regionsR_1, R_2, and R_3; in all regions, there is one real critical point (Saddle).

In Fig.2 the caustic (bifurcation set) of function $W(\xi, \delta)$ Split the space of parameters into regions R_1 and R_2; each region consists of a fixed number of three real critical points (Minimum, 2 Saddles).

In Fig.3, the caustic (bifurcation set) of function 21 Split the space of parameters into regions R_1, R_2 ,and R_3; each region consists of a fixed number of critical points so that the pervasion of the critical points is as follows: if the parameters (λ_1, t_1, t_2) belong to R_1, R_2, then have three real critical points (2 Maximum, Saddle), while haveing five real critical points (Minimum, 2 Saddles, 2 Maximum). When (λ_1, t_1, t_2) belong to R_3.

Nonlinear Ritz Approximation for the Benjamin-Bona-Mahony Equation (BBM)

In this section, we will give another example of our work in this paper. As in the above section, **MLSM** will be applied to the study of the existence of periodic solutions of the traveling wave in the form $u + v$ of the Benjamin-Bona-Mahony equation.

Consider the following nonlinear partial differential equation

$$u_t + \frac{3}{2}\frac{c_0}{h_0} u u_x + \int_{-\infty}^{\infty} K(x-\eta) u_\eta(\eta, t) d\eta = 0 \qquad 28$$

when $t, g, u(x,t), h_0$ are time, gravitational acceleration, and water wave velocity respectively while h_0 is the depth of the fluid such that $c_0 = \sqrt{g h_0}$, with a kernel $K(x)$, defined by

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k) e^{ikx} dx$$

By Taylor expansion, the partial deferential Eq. 28 reduces to the Korteweg-de Vries equation,

$$u_t + c_0 u_x + \frac{3}{2}\frac{c_0}{h_0} u u_x + \frac{1}{6} c_0 h_0^2 u_{xxx} = 0 \qquad 29$$

By assuming $u(x,t) = u(\eta), \eta = x - ct$, Eq.29 is transformed into the following ordinary differential equation for a variable $w(y)$,

$$\frac{d^2 w}{dx^2} + \alpha \mathrm{w} + w^2 = 0 \qquad 30$$

where α is a parameter.

In the present section, we will study Eq. 30 with the following boundary conditions which satisfy Eq. 30,

$$w(0) = w(2\pi) = 0,$$

where $w = w(x), x \in [0, 2\pi]$.

To obtain a nonlinear approximation for the Korteweg-de Vries equation, write Eq. 30 as a nonlinear Fredholm operator as follows:

$$g(w, \gamma) = w'' + \alpha w + w^2 \qquad 31$$

when $g\colon \mathrm{E} \longrightarrow M$ is the Fredholm operator which is nonlinear of index zero from Banach space E to Banach space M, where $E = C^2([0,2\pi], \mathbb{R})$ is the space of all continuous functions that have derivative of order at most two, $M = C([0,2\pi], \mathbb{R})$ is the space of every continuous function. The operator g own variational property, so there is a functional V defined by,

$$g(w, \alpha) = grad_H V(w, \alpha)$$

Where

$$\mathrm{V}(\mathrm{w}, \alpha, \psi) = \frac{1}{2\pi} \int_0^1 \left(\frac{(\mathrm{w}')^2}{2} + \alpha \frac{\mathrm{w}^2}{2} + \frac{\mathrm{w}^3}{3}\right) - \mathrm{w}\psi) dx$$

When $grad_H V$ denotes the gradient of V. Every solution of Eq. 30 is a solution of the operator equation,

$$g(w,\lambda)=\psi,\psi\in F \qquad 32$$

the Fréchet derivative at the point $(0,\alpha)$ of the nonlinear operator $g(w,\alpha)$ has the form,

$$dg(0,\alpha)\hbar=\hbar''+\alpha\hbar$$

And hence the linearized equation identical to Eq. 28 is defined by,

$$A\hbar=0,\hbar\in E$$

$$A=dg(0,\alpha)=\frac{\mathrm{d}^2}{dx^2}+\alpha,x\in[0,2\pi], \qquad 33$$

$$\hbar(0)=\hbar(2\pi)=0$$

Eq. 33 is called a linearized equation.

The solution of the linearized Eq. 33 verification of the boundary conditions is get by

$$e\ =\ a_1\,sin(x)+a_2\,cos(x), \qquad 34$$

As a result, (0,0) is a bifurcation point for Eq.28. And localized parameters for α gives by,

$$\hat{\alpha}=0+\Gamma$$

where Γ are parameters that small lead to the below modes over the bifurcation.

$$e_1=\sqrt{2}\sin(x),e_2=\sqrt{2}\,cos(x)$$

Where the norms of $e_1 and\, e_2$ are equal to one, and $a_1=a_2=\sqrt{2}$. This means that $e_1 and\, e_2$ are the orthonormal basis of null space $\ker(A)$.

Can separate the space E into subspace W and it's an orthogonal complement,

$$E=W\oplus\hat{E},\hat{E}=W^{\perp}\cap E=\{v\in E\colon v\perp W\}$$

Likewise, the space M separated to subspace N it's an orthogonal complement as follows

$$F=N\oplus\hat{F},\qquad \hat{F}=\mathrm{N}\cap F=\{v\in F\colon v\perp N\}$$

For that, there exist projections $j\colon E\to W$ & $I-j\colon E\to\hat{E}$ such that $jw=u$ and $(I-j)w=v$, so $\forall\, w\in E$ represented as $w=u+v,u=\sum_{i=1}^{2}\xi_i e_i\,,W\perp v\in\hat{E},\xi_i=\langle w,e_i\rangle$ by the same way there are projection $G\colon F\to N$ & $I-G\colon F\to\hat{F}$ in which

$$g(u,\gamma)=Gg(u,\gamma)+(I-G)g(u,\gamma)=\psi,\psi$$
$$=(w,t),t=(t_1,t_2)$$

Accordingly, Eq. 31 can be represented as follows,

$$Gg(u+v,\gamma)=\psi_1$$
$$(I-G)g(u+v,\gamma)=\psi_2$$

Such that $\psi_1=e_1t_1+e_2t_2\quad and$

$$\psi_2=a_1{t_1}^2+a_2t_1t_2+a_3{t_2}^2$$

From implicit function theory, obtain a map $\theta\colon W\to\hat{E}$ that is smooth satisfying,

$$W(\xi,\Gamma,\psi)=V(\theta(\xi,\alpha),\Gamma,\psi)$$

By finding the functions $v(x,\xi,\gamma)=O(\xi^2)$, $\mu(\xi)=O(\xi),\tilde{\mu}(\xi)=O(\xi),\xi=(\xi_1,\xi_2)$, can get the nonlinear Ritz approximation of $V(\theta(\xi,\alpha),\Gamma,\psi)$, when

$$\left.\begin{array}{c} q_1=\widetilde{q_1}+\mu(\xi_1,\xi_2),q_2=\widetilde{q_2}+\tilde{\mu}(\xi_1,\xi_2)\\ v(x,\xi,\gamma)=v_0(x,\lambda)\xi_1^2+v_1(x,\lambda)\xi_1\xi_2+v_2(x,\lambda)\xi_2^2+\cdots\\ \mu(\xi_1,\xi_2)=\mu_0\xi_1+\mu_1\xi_2\\ \tilde{\mu}(\xi_1,\xi_2)=\tilde{\mu}_0\xi_1++\tilde{\mu}_1\xi_2 \end{array}\right\}35$$

written Eq. 31 as follows

$$g(u,\alpha)=Au+Tu=\psi,$$

When $Aw=\frac{d^2w}{dx^2}+\alpha w$ represents a linear part while $Tw=w^2$ is a nonlinear part of Eq.30. Since

$$Qf(w,\lambda)=\sum_{i=1}^{2}\langle f(w,\lambda),e_i\rangle e_i=\psi_1,$$

obtaining

$$\sum_{i=1}^{2}\langle A(w)+T(w),e_i\rangle e_i=\sum_{i=1}^{2}\left(\int_0^{2\pi}(A(w)e_i+T(w)e_i)dx\right)e_i=\psi_1.$$

Thus,

$$\left(q_1\xi_1+\frac{1}{2\pi}\int_0^{2\pi}(\xi_1e_1+\xi_2e_2+v)^2e_1dx\right)e_1+\left(q_2\xi_2+\frac{1}{2\pi}\int_0^{2\pi}(\xi_1e_1+\xi_2e_2+v)^2e_2dx\right)e_2=t_1e_1+t_2e_2 \qquad 36$$

And

$$v''+\alpha v+(\xi_1e_1+\xi_2e_2+v)^2+\ q_1\xi_1e_1+q_2\xi_2e_2=a_1{t_1}^2+a_2t_1t_2+a_3{t_2}^2 \qquad 37$$

by substituting $q_1=\widetilde{q_1}+\mu(\xi_1,\xi_2)$ and $q_2=\widetilde{q_2}+\tilde{\mu}(\xi_1,\xi_2)$ in Eq.36 and Eq.37, obtaining

$$\left[\left(\widetilde{q_1}+\mu(\xi_1,\xi_2)\right)\xi_1+\frac{1}{2\pi}\xi_1^2\int_0^{2\pi}e_1^3dx+\frac{1}{\pi}\xi_1\xi_2\int_0^{2\pi}e_1^2e_2dx+\frac{1}{2\pi}\xi_2^2\int_0^{1}e_1e_2^2dx\right]e_1+\left[\left(\widetilde{q_2}+\tilde{\mu}(\xi_1,\xi_2)\right)\xi_2+\ \frac{1}{2\pi}\xi_1^2\int_0^{2\pi}e_1^2e_2dx+\frac{1}{\pi}\xi_1\xi_2\int_0^{2\pi}e_2^2e_1dx+\frac{1}{2\pi}\xi_2^2\int_0^{2\pi}e_2^3dx\right]e_2=t_1e_1+t_2e_2 \qquad 38$$

$$v''+\alpha v+\frac{1}{2\pi}(\xi_1^2e_1^2+2e_1e_2\xi_1\xi_2+\xi_2^2e_2^2+2ve_1\xi_1+2ve_2\xi_2+v^2))+\left(\widetilde{q_1}+\mu(\xi_1,\xi_2)\right)\xi_1e_1+\left(\widetilde{q_2}+\tilde{\mu}(\xi_1,\xi_2)\right)\xi_2e_2=a_1{t_1}^2+a_2t_1t_2+a_3{t_2}^2 \qquad 39$$

The functions $v(x,\xi,\lambda),\mu(\xi)\ and\ \ \tilde{\mu}(\xi)$ in Eq.35 determine by finding the coefficients $\mu_0,\mu_1,,\tilde{\mu}_0,\tilde{\mu}_1,,v_0,v_1,and\ v_2$ in Eq.38, 39, so have

$$\mu_0=\mu_1=\tilde{\mu}_0=\tilde{\mu}_1=\tilde{\mu}_1=0$$

$$v_0=\frac{1}{\alpha-4}coc(2x)-\frac{1}{\alpha}$$

$$v_1=\frac{-1}{\alpha-4}sin(2\,x)$$

$$v_2=-\frac{1}{\alpha-4}coc(2x)-\frac{1}{\alpha}$$

So, the nonlinear approximation for Eq.31 found by substituting the values of $\mu_0,\mu_1,,\tilde{\mu}_0,\tilde{\mu}_1,,v_0,v_1,and\ v_2$ in $U(x,\xi)$,

$$w(x,\xi)=\sqrt{2}\xi_1\sin(x)+\sqrt{2}\xi_2\cos(x)+\left[\frac{1}{\alpha-4}coc(2x)-\frac{1}{\alpha}\right]\xi_1^2+\left[\frac{-1}{\alpha-4}sin(2\,x)\right]\xi_1\xi_2+\left[-\frac{1}{\alpha-4}coc(2x)-\frac{1}{\alpha}\right]\xi_2^2 \qquad 40$$

$$q_1=\widetilde{q_1},$$
$$q_2=\widetilde{q_2}$$

Eq.40 is a solution of the functional $V(u,\alpha)$.which is represent the nonlinear Ritz approximation of V.

Now, will give the key function of functional $\mathrm{V}\ (\mathrm{w},\alpha,\psi)$.

Theorem 2. The functional

$$V\,(w,\alpha,\psi) = \frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{(w')^2}{2} + \alpha\frac{w^2}{2} + \frac{w^3}{3}\Big) - w\psi)dx.$$

has the key function of the form

$$\begin{aligned} W(\xi,\delta) = {} & \gamma_1{\xi_1}^6 + \gamma_2{\xi_2}^6 \\ & + \gamma_3{\xi_1}^4{\xi_2}^2 + \gamma_4{\xi_1}^2{\xi_2}^4 + \gamma_5{\xi_1}^4 \\ & + \gamma_6{\xi_2}^4 + \gamma_7{\xi_1}^2{\xi_2}^2 + \lambda_1{\xi_1}^2 \\ & + \lambda_2{\xi_2}^2 - t_1\xi_1 - t_2\xi_2 \end{aligned}$$

Such that

$\gamma_i = \gamma_i(\alpha), i = 1,2,\ldots,7$,

$\lambda_i = \lambda_i(\alpha,t), i = 1,2.$

Proof.

The proof is in the same manner as the proof of Theorem 2.

Conclusion:

The modified Lyapunov-Schmidt reduction for nonhomogeneous problems is used for finding the nonlinear Ritz approximation of nonlinear Fredholm functional when the dimension of the null space is equal to two. The method allowed us to get more information about the key function $W(\xi,\delta)$. The method can be used to find nonlinear Ritz approximation for Fredholm functional defined by the nonhomogeneous nonlinear differential equations like Camassa-Holm and Benjamin-Bona-Mahony equations. Nonlinear Ritz approximation solutions which have been obtained by **MLSR** experimented with in terms of thoroughness and convergence. Finding the caustic and discussing the bifurcation of critical points was difficult in previous studies, so the nonhomogeneous problems were studied to avoid this problem. In future work, we will study a new nonlinear equation using the modified Lyapunov-Schmidt method.

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Authors' contributions statement:

M. A. conceived of the presented idea**.** H. G**.** developed the theory and performed the computations. M. A. and H. G**.** verified the analytical methods. M. A. supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

References:

1. Sapronov YI. Finite Dimensional Reduction of Smooth Extremely Problems. Russ Math Surv. 51 1996; 51(1) : 97
2. Krasnoselskii MA. Topological Methods in the Theory of Nonlinear Equations, M. Gostehizdat, 1956. https://doi.org/10.1002/zamm.19640441041
3. Sapronov YI, Chemerzina EV. Direct parameterization of caustics of Fredholm functionals. J Math. 2007; 142(3): 2189-2197.
4. Sapronov YI, Darinskii BM. Discriminant sets and layerings of bifurcating solutions of fredholm equations J Math. 2005; 126(4): 1297-1311.
5. Abdul Hussain MA, Qaasim TH. On Bifurcation of Periodic Solutions of Nonlinear Fourth Order Ordinary Differential Equation Int J Nonlinear Anal Appl. 2018; 2018(1): 48-56.
6. Abdul Hussain MA. Lyapunov –Schmidt Reduction in the Study of Periodic Travelling Wave Solutions of Nonlinear Dispersive Long Wave Equation. TWMS. J App Eng Math. 2017; 7(2): 303-310.
7. Shawi ZA., Abdul Hussain MA. Bifurcation Solutions of Fourth Order Non-linear Differential Equation Using a Local Method of Lyapunov –Schmidt, Bas J Sci. 39(2) 2021, 221–233.
8. Abdul Hussain MM, Abdul Hussain MA. Bifurcation solutions of a fourth order Nonlinear Differential Equations system using "local method of Lyapunov –Schmidt". J Basrah Res. (Sci) 2020; 46(2): 163-174.
9. Abdul Hussain MA, Mizeal AA. Two-mode bifurcation in solution of a perturbed nonlinear fourth order differential equation. BRNO. Tomus. 2012; 48(1): 27-37.
10. Abdul Hussain MA. Nonlinear Ritz approximation for Fredholm functionals. Electron. J Differ Equ. 2015; 2015(294): 1–11.
11. Mohammed MJ. Lyapunov-Schmidt Reduction in the analysis of bifurcation solutions and caustic of nonlinear system of algebraic equation. Asian J Math.Comp Res. 2016; 14(4): 275-289. https://www.ikprress.org/index.php/AJOMCOR/article/view/751
12. Rosen A.H., Abdul Hussain M.A., On bifurcation solutions of nonlinear fourth order differential equation, Asian J Math.Comp Res. 2017;21(3): 145-155. https://www.ikppress.org/index.php/AJOMCOR/article/view/1151
13. Kadhim HK, Abdul Hussain MA. The analysis of bifurcation solutions of the Camassa–Holm equation by angular singularities. Probl Anal Issues Anal. 2020; 9(27) (1): 66–82.
14. Schmidt E. Zur Theorie der linearen und nichtlinearen Integral gleichungen. III. Teil: Über die Auflösung der nichtlinearen Integral gleichung und die Verzweigung ihrer Lösungen. Math Ann. 1908; 65(1908): 370-399.
15. Ouda EH. An Approximate Solution of some Variational Problems Using Boubaker Polynomials. Baghdad Sci J. 2018; 15(1): 106-109.

16. Zainab S. Madhi, Mudhir A. Abdul Hussain, Bifurcation Diagram of $W(u_j, \tau) -$ function with (p, q)-parameters, Iraqi J Sci., 63(2), 2022, 667-674.
17. Li J, Qiao Z. Bifurcations and Exact Traveling Wave Solutions for a Generalized CAMASSA-HOLM Equation. Int J Bifurcat Chaos. 2013; 23(3): 17 pages.
18. Hameed HH, Al-Saedi HM. Three-Dimensional Nonlinear Integral Operator with the Modelling of Majorant Function. Baghdad Sci J. 2021; 18(2): 296-305.

تقريب ريز غير الخطي للمعادلة كاماسا هولم باستخدام طريقة ليبنوف-شمدت المعدلة

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الخلاصة:

في هذا العمل، تم استخدام طريقة ليابونوف-شمدت المعدلة لايجاد تقريب ريتز غير الخطي لمؤثر فريدهولم المعرف بمعادلة كاماسا هولم غير المجانسة ومعادلة بنيامين بونا ماهوني. قدمنا طريقة ليابونوف- شمدت المعدلة في حالة المسائل غير المتجانسة عندما يكون بعد الفضاء الصفري مساو الى اثنان. أثبتنا ان تقريب ريتز غير الخطي لمعادلة كاماسا هولم يعطى بشكل دالة ذات بعد مرافق قيمته اربعة وعشرون.

الكلمات المفتاحية: حلول التفرع، معادلة بنيامين بونا ماهوني، معادلة كاماسا هولم، كاوستك، طريقة ليبنوف-شمدت المعدلة.