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Dynamics of Predator-prey Model under Fluctuation Rescue Effect

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Abstract:

This paper presents a novel idea as it investigates the rescue effect of the prey with fluctuation effect for the first time to propose a modified predator-prey model that forms a non-autonomous model. However, the approximation method is utilized to convert the non-autonomous model to an autonomous one by simplifying the mathematical analysis and following the dynamical behaviors. Some theoretical properties of the proposed autonomous model like the boundedness, stability, and Kolmogorov conditions are studied. This paper's analytical results demonstrate that the dynamic behaviors are globally stable and that the rescue effect improves the likelihood of coexistence compared to when there is no rescue impact. Furthermore, numerical simulations are carried out to demonstrate the impact of the fluctuation rescue effect on the dynamics of the non-autonomous model. The analytical and numerical results show a more coexisted model between prey and predator, which can help any extinction-threatened ecosystem.

Keywords: Coexistence, Kolmogorov analysis, Predator– prey model, Rescue effect, Stability analysis.

Introduction:

One of the causes for the evolution of the dynamics of living creatures is the change in population size; these dynamics take many forms[1-4]. The prey-predator model, which depicts population dynamics in many animals, is one of the most common types of this dynamics[5-8]. Many ecologists and mathematicians have expressed interest in this ecological model; recent researchers are mentioned, for instants[9-12].

Lotka[13] offered the essential pillar to study prey-predator dynamics for the first time. Moreover, Volterra[14]; represented the interactions between prey and predator; they devised a mathematical model based on nonlinear differential equations. Various researchers have been motivated by the Lotka-Volterra model, which has rendered them to develop it to replicate many biological interactions between any two competing species in life[15-18]. Further, many research papers with different ideas have discussed the interactions between more than two species[19-22].

The functional and numerical responses are the major components of the prey-predator paradigm[23]; they play an essential role in depicting the dynamic actions of prey and predator[24]. The functional response in the prey-predator model indicates the predator's rate of prey consumption, which is mainly determined by the density of prey in the environment in which they compete. The outcomes of this consumption also represent the numerical answer. For the first time, Solomon suggested the functional response in 1949[25]. Then, in 1959, Holling, a Canadian ecologist, conducted significant research on the functional response and classified it into three types: Holling type I, type II, and type III[26].

The prey-predator model has been examined and refined in the literature to make it more realistic by integrating various external forces that mirror real-life impacts[27-29]. In 1963, McLaren[30] investigated the effect of temperature on the prey-predator paradigm and the increase and reduction of population growth rates. In 1968, Glass [31] sought to construct a prey-predator model by including the effect of food restriction on the total use of resources in the ecosystem.

While Arditi and da Silva[32] in 1977, investigated the consequences of the predation delay on the prey-predator model and the effects of increased predator death rates and famine. In 1979, McArdle and Lawton[33,] the impact of prey age and its relationship with predation rates, as well as the effect of this element on the prey-predator model, were investigated. Furthermore, Yen[34], studied the effect of predator hunger and used the prey-predator model.

In 2004, Hethcote conducted a study on the predator-prey model's stability under the impact of the infected prey and their influence on the predator's growth rates[35]. In 2016, Alebraheem[36], the impacts of seasonality on the dynamical behavior of a predator-prey model were investigated. Al Basir, Tiwari, and Samanta[37] recently explored the stability and bifurcation in the diseased prey-predator model using the Holling type II functional response in 2021. During the same year, Alebraheem[38] aimed to investigate the dynamics of a predator-prey model with Crowley–Martin functional and numerical responses under the influence of prey migratory oscillations.

The impact of rescue is one of the crucial factors that help in keeping endangered species alive. It either adds new individuals to the environment in which they live or provides better living conditions for them in that environment to increase their life chances. This paper investigates the fluctuation of the rescue effect phenomenon on the predator-prey model with Holling type I, so a non-autonomous model is proposed. However, the non-autonomous model is converted to an autonomous model using the approximation method to analyze the dynamic behaviours. The boundedness, stability and Kolmogorov conditions are used to study the dynamic behaviours and verify the model's validity.

The remainder of this paper is organized as follows. Section two introduces the definition of the rescue effect and its types. Section three presents the mathematical formulation of the rescue effect of the prey in a prey-predator model. Section four explains the theoretical analyses of the autonomous proposed model. Section five shows the numerical simulations of the non-autonomous model. Finally, Section six presents the conclusion and discussion.

Rescue Effect

The rescue effect refers to the perception of providing support to endangered ecosystems at risk of extinction in ecological systems[39]. This support process is well shown in either adding new creatures to the ecosystem or providing those creatures with more appropriate living conditions in their environment. A prominent factor in the rescue effect is the process of immigration to decay ecosystems which helps to attain stability state in these ecosystems, which in turn sustain the continuity of the species survival[40].

In most ecosystems, two types of species coexist together, such as prey and predator systems, and the predator's survival is tightly linked to the prey's presence. When predation rates increase, the existence of this ecosystem, in general, will be threatened. Hence, the impact of rescue on prey is a major factor when addressing threatened ecosystems. Additionally, the rescue **e**ffect stability leads to the continuity of the presence of prey and predators in the ecosystem. That is, the immigration will be contrariwise symmetrical to the decay rates[41,42].

The Model

The rescue effect in our proposed model will depend on adding new individuals to the prey oscillating according to the capacity of the supporting ecosystems. The addition will be referred to as the "Immigration factor" since it symbolizes prey migration into the threatened environment. As a result, the sinusoid function may be used to characterize the immigration factor.

$$i\,(1+\varepsilon\sin(\omega t)) \qquad 1$$

where I, is the number of immigrants who prey on the ecosystem, angular frequency denotes the angular frequency of the fluctuations, fluctuation degree denotes the fluctuation degree 4e, and time denotes the duration.

The prey and predator model has been extensively studied in previous years, but this is the first time that it has been discussed under the influence of the rescue, so this research has contributed to displaying mathematically the benefits of the effect of rescue on the prey and predator model, which was done mathematically by including Eq. 1 in system 2 and showing how it increases the chances of coexistence between competing elements.

The non-autonomous predator–prey model is shown here, complete with Holling type I functional and numerical responses, as well as a fluctuation rescue effect factor is displayed:

$$\frac{dN_1}{dt} = zN_1\left(1-\frac{N_1}{k}\right) - \alpha N_1 N_2 + i\,(1+\varepsilon\sin(\omega t))$$
$$\frac{dN_2}{dt} = -uN_2 + e\alpha N_1 N_2 - e\alpha {N_2}^2 \qquad 2$$

For simplifying the mathematical operation, the fluctuation immigration component of the model will be approximated as follows,
since

$$-1 \le \sin(\omega t) \le 1 \qquad 3$$

Multiply 3 by ε,

$$-\varepsilon \leq \varepsilon \sin(\omega t) \leq \varepsilon \quad 4$$

Adding 1 to 4 we get,

$$1-\varepsilon \leq 1+\sin(\omega t) \leq 1+\varepsilon \quad 5$$

As a result, the volatility of the immigration factor with time may be estimated as $1 \pm \varepsilon$. This approximation reflects that there are temporal fluctuations, but positive or negative factors influence them. The system's autonomous system two is then explained as follows:

$$\frac{dN_1}{dt} = N_1\left(1-\frac{N_1}{k}\right) - \alpha N_1 N_2 + i\,(1 \pm \varepsilon)$$
$$\frac{dN_2}{dt} = -uN_2 + e\alpha N_1 N_2 - e\alpha {N_2}^2 \quad 6$$

where N_1 and N_2 denote prey-predator densities at time t, the rate of growth of prey $z = 1$, k is the carrying capacity, α the measured reliability of the predator searching and capture N_2, and u is the predator $N_2's$ rates of death of the predator N_2. αN_1 conveys the functional response, and $e\alpha N_1$ is the numerical responses of the predator N_2. i is the number of prey immigrants, and the degree of fluctuations will be denoted by ε. It is also worth mentioning that all parameters are more than zero.

Theoretical Analysis

In this section, we introduce the theoretical analysis of system 6, which represents the autonomous system.

The Boundedness of the Mathematical Model

Theorem 1: System 6 is bounded in R_+^2.

Proof: In the system 6, the prey equation is bounded through $\frac{dN_1}{dt} \leq N_1(1-\frac{N_1}{k})$.

The solution of $\frac{dN_1}{dt}$ from the logistic growth is

$$N_1(t) = \frac{k}{\left(\frac{k}{N_1(0)}-1\right)e^{-t}+1}. \quad 7$$

This means, $N_1(t) \leq k + i\,(1 \pm \varepsilon),\ \forall\, t > 0$

Consider, $D(t) = N_1(t) + N_2(t)$. By finding the derivative of D, it has:

$$\frac{dD}{dt} = \frac{dN_1}{dt} + \frac{dN_2}{dt} \quad 8$$

By plugging the predator and prey equations in Eq. 8

$$\frac{dD}{dt} = N_1(1-\frac{N_1}{k}) - \alpha N_1 N_2 + i\,(1 \pm \varepsilon) + (-uN_2 + e\alpha N_1 N_2 - e\alpha {N_2}^2) \quad 9$$

since the immigration factor is a constant, we can sitting $i(1 \pm \varepsilon) = I$.

As, all parameters are positive and the solution is nonnegative in R^2, can be supposed

$$\frac{dD}{dt} \leq N_1\left(1-\frac{N_1}{k}\right) - \alpha N_1 N_2 + I + \left(-uN_2 + e\alpha N_1 N_2 - e\alpha {N_2}^2\right) \quad 10$$

$max\left(N_1(1-\frac{N_1}{k})\right) = \frac{k}{4}$. By plugging it in Eq. 10 and adding $D(t)$ to both sides, one gets,

$$\frac{dD}{dt} + D(t) \leq \frac{k}{4} + N_1 + I + (-uN_2 + e\alpha N_1 N_2 - e\alpha {N_2}^2) \quad 11$$

Also, we have $N_1 \leq k$,

$$\frac{dD}{dt} + D(t) \leq \frac{k}{4} + k + I + \left(-uN_2 + e\alpha k N_2 - e\alpha {N_2}^2\right) \quad 12$$

But, $max(-uN_2 + e\alpha k N_2 - e\alpha {N_2}^2) = \frac{(-u+e\alpha k)^2}{4e\alpha}$.

So Eq. 12 become

$$\frac{dD}{dt} + D(t) \leq Q, \text{ where } Q = \frac{k}{4} + k + I + \frac{(-u+e\alpha k)^2}{4e\alpha}. \quad 13$$

Using the separation of variables gives $D(t) \leq Q - e^{-t}$ (where $t \to \infty$). That means $D(t) \leq Q$. Consequently, we get that system 6 is bounded.

Equilibrium Points

By sitting $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$, system 6 has four equilibrium points. Which obtained as follow: $E_1(N_1 = 0, N_2 = 0)$, $E_2(N_1 \neq 0, N_2 = 0)$, $E_3(N_1 = 0, N_2 \neq 0)$, and $E_4(N_1 \neq 0, N_2 \neq 0)$. Only the nonnegative equilibria are taken into consideration since they are related to a biological significance. While the negative equilibrium points are ignored. To get unique equilibrium points and conditions, we take the positive side of ε. The following lists are the equilibrium points.

1. $E_1(N_1, N_2)$ is (0,0), which is resulted from assuming $N_1 = N_2 = 0$ in system 6.
2. $E_2(N_1, N_2)$ is obtained by sitting $N_2 = 0$ in system 6 and getting, $N_1 = \frac{k+\sqrt{k^2+4ki(1+\varepsilon)}}{2}$. That means the second point is $\left(\frac{k+\sqrt{k^2+4ki(1+\varepsilon)}}{2}, 0\right)$. Where the positive root of N_1 is positive without any conditions on its parameters.
3. $E_3(N_1, N_2)$ is obtained by assuming $N_1 = 0$ in system 6 to give $N_2 = \frac{-u}{e\alpha}$. That means the third point is $\left(0, \frac{-u}{e\alpha}\right)$. Which is neglected since it is negative.
4. $E_4(N_1, N_2)$ is obtained by determining N_2 from the predator equation and substituted it in the prey equation as follows:

Let $N_1\left(1-\frac{N_1}{k}\right) - \alpha N_1 N_2 + i(1+\varepsilon) = 0$ then, $N_2 = \frac{1}{\alpha} - \frac{N_1}{\alpha k} + \frac{i(1+\varepsilon)}{\alpha N_1} = \overline{N_2}$, plugging N_2 in predator equation gives:

$$-u + e\alpha N_1 - e + \frac{eN_1}{k} - \frac{ei(1+\varepsilon)}{N_1} = 0 \quad 14$$

By multiplying Eq. 14 by N_1, it results

$$-uN_1 + e\alpha {N_1}^2 - eN_1 + \frac{e}{k}{N_1}^2 - ei(1+\varepsilon) = 0 \quad 15$$

$\left(e\alpha+\frac{e}{k}\right)\mathrm{N}_1{}^2+(-u-e)\mathrm{N}_1+\left(-ei(1+\varepsilon)\right)=0$
16

Therefore N_1 is equal to:

$$\mathrm{N}_1=\frac{u+e+\sqrt{(-u-e)^2-4\left(e\alpha+\frac{e}{k}\right)\left(-ei(1+\varepsilon)\right)}}{2\left(e\alpha+\frac{e}{k}\right)} \quad 17$$

$$\mathrm{N}_1=\frac{ku+ke+\sqrt{(ku+ke)^2+4eki(1+\varepsilon)(ke\alpha+e)}}{2(ke\alpha+e)}=\overline{\mathrm{N}_1} \quad 18$$

The positive root of $\overline{\mathrm{N}_1}$ is always higher than 0 in this case, regardless of the condition.

As a result, by swapping $\overline{\mathrm{N}_1}$ in $\overline{\mathrm{N}_2}$, one gets

$$\overline{\mathrm{N}_2}=\frac{-2u-ku\alpha+ke\alpha+\alpha\sqrt{(ku+ke)^2+4eki(1+\varepsilon)(ke\alpha+e)}}{2\alpha(ke\alpha+e)} \quad 19$$

The positive root of $\overline{\mathrm{N}_2}>0$ if $2u+ku\alpha<ke\alpha+\alpha\sqrt{(ku+ke)^2+4eki(1+\varepsilon)(ke\alpha+e)}$.

Thus, $E_4(\overline{\mathrm{N}_1},\overline{\mathrm{N}_2})$ is positive**.**

Local Stability

There are several approaches for studying the system's localized stability at its equilibrium points. In this part, eigenvalues are employed, as well as trace and determinant approaches[43,44].

The Jacobian matrix of system 6 is

$$J(\mathrm{N}_1,\mathrm{N}_2)=\begin{bmatrix}1-\frac{2\mathrm{N}_1}{k}-\alpha\mathrm{N}_2 & -\alpha\mathrm{N}_1\\ e\alpha\mathrm{N}_2 & -u+e\alpha\mathrm{N}_1-2e\alpha\mathrm{N}_2\end{bmatrix} \quad 20$$

Theorem 2: $E_1(0,0)$ is a saddle point.

Proof: As a result of substituting $E_1(0,0)$ in Eq. 4, we get $J(0,0)=\begin{bmatrix}1 & 0\\ 0 & -u\end{bmatrix}$. By solving the characteristic equation of $J(0,0)$ we get that the eigenvalues are opposite signs ($\lambda_1=1$, $\lambda_2=-u$). So $E_1(0,0)$ is a saddle point

Theorem 3: Under the following conditions $E_2\left(\frac{k+\sqrt{k^2+4ki(1+\varepsilon)}}{2},0\right)$ is locally stable:

- $e\alpha\Upsilon<u$ 21
- $k<2\Upsilon$ 22

Proof: Assume that $\Upsilon=\frac{k+\sqrt{k^2+4ki(1+\varepsilon)}}{2}>0$

The Jacobian matrix of system 6 after substituting Υ in Eq. 4 is

$$J(A,0)=\begin{bmatrix}1-\dfrac{2\Upsilon}{k} & -\alpha\Upsilon\\ 0 & -u+e\alpha\Upsilon\end{bmatrix}$$

The eigenvalues are ($\lambda_1=1-\frac{2\Upsilon}{k}$, $\lambda_2=-u+e\alpha\Upsilon$). That means the point is locally stable if $k<2\Upsilon$ and $e\alpha\Upsilon<u$.

Theorem 4: In system 6 $E_4(\overline{\mathrm{N}_1},\overline{\mathrm{N}_2})$ is locally stable under the following conditions:

- $2\overline{\mathrm{N}_2}>\overline{\mathrm{N}_1}$ 23
- $\alpha\overline{\mathrm{N}_2}>1$ 24
- $2\overline{\mathrm{N}_1}>k$ 25

Proof: since $\overline{\mathrm{N}_1},\overline{\mathrm{N}_2}>0$. The jacobian of $E_4(\overline{\mathrm{N}_1},\overline{\mathrm{N}_2})$ is,

$$J(\overline{\mathrm{N}_1},\overline{\mathrm{N}_2})=\begin{bmatrix}1-\dfrac{2}{k}\overline{\mathrm{N}_1}-\alpha\overline{\mathrm{N}_2} & -\alpha\overline{\mathrm{N}_1}\\ e\alpha\overline{\mathrm{N}_2} & -u+e\alpha\overline{\mathrm{N}_1}-2e\alpha\overline{\mathrm{N}_2}\end{bmatrix}$$

Then, by solving the determinant of this matrix, we get

$$|J(\overline{\mathrm{N}_1},\overline{\mathrm{N}_2})|=u(\alpha\overline{\mathrm{N}_2}-1)+2e\alpha(\alpha\overline{\mathrm{N}_2}-1)+e\alpha\overline{\mathrm{N}_1}+\frac{2\overline{\mathrm{N}_1}u}{k}+\frac{2\overline{\mathrm{N}_1}e\alpha}{k}(2\overline{\mathrm{N}_2}-\overline{\mathrm{N}_1})$$

Where it is positive if we take $(2\overline{\mathrm{N}_2}>\overline{\mathrm{N}_1})$ and also $(\alpha\overline{\mathrm{N}_2}>1)$.

Also, by finding the trace of $J(\overline{\mathrm{N}_1},\overline{\mathrm{N}_2})$ we obtain,

$tr\left(J(\overline{\mathrm{N}_1},\overline{\mathrm{N}_2})\right)=-\left(\frac{2}{k}\overline{\mathrm{N}_1}-1\right)-\alpha\overline{\mathrm{N}_2}-u-e\alpha(2\overline{\mathrm{N}_2}-\overline{\mathrm{N}_1})$

then, by sitting $(2\overline{\mathrm{N}_1}>k)$ and $(2\overline{\mathrm{N}_2}>\overline{\mathrm{N}_1})$ the trace will be negative. Therefore, $E_4(\overline{\mathrm{N}_1},\overline{\mathrm{N}_2})$ is stable under conditions 23- 25.

Periodic Dynamic and Global Stability

The next theorem will establish the existence of periodic dynamics for the coexistence point (i.e. the non-trivial equilibrium point).

Theorem 5: In R_+^2, system 6 owns no periodic solution.

Proof: let the Dulac function $H(\mathrm{N}_1,\mathrm{N}_2)=\frac{1}{\mathrm{N}_1\mathrm{N}_2}$, which is smooth in R_+^2

Let $M(\mathrm{N}_1,\mathrm{N}_2)=\frac{d\mathrm{N}_1}{dt}$ and $(\mathrm{N}_1,\mathrm{N}_2)=\frac{d\mathrm{N}_2}{dt}$.

A Dulac function multiplied by the prey equation yields the following result:

$$HM=\frac{1}{\mathrm{N}_2}-\frac{\mathrm{N}_1}{k\mathrm{N}_2}-\alpha+\frac{1}{\mathrm{N}_2}i(1+\varepsilon)\mathrm{N}_1{}^{-1} \quad 26$$

To both sides of Eq. 26, take the partial derivative with respect to N_1,

$$\frac{\partial HM}{\partial\mathrm{N}_1}=-\frac{1}{k\mathrm{N}_2}-\frac{i(1+\varepsilon)}{\mathrm{N}_1{}^2\mathrm{N}_2} \quad 27$$

Also, when multiplying the Dulac function by the predator equation, the following result appears:

$$HN=\frac{-u}{\mathrm{N}_1}+e\alpha-\frac{e\alpha\mathrm{N}_2}{\mathrm{N}_1} \quad 28$$

To both sides of Eq. 28, take the partial derivative with respect to N_2,

$$\frac{\partial HN}{\partial\mathrm{N}_2}=-\frac{e\alpha}{\mathrm{N}_1}$$

Thus, $\Delta\,(HM,HN)=\frac{\partial HM}{\partial\mathrm{N}_1}+\frac{\partial HN}{\partial\mathrm{N}_2}=\frac{-1}{k\mathrm{N}_2}-\frac{i(1+\varepsilon)}{\mathrm{N}_1{}^2\mathrm{N}_2}-\frac{e\alpha}{\mathrm{N}_1}$. It is clear that $\Delta\,(HM,HN)$ does not change sign also, dose not identically to zero in R_+^2. Therefore, system 6 has no periodic solution by the Bendixson-Dulac criterion.

Corollary 6: The non-trivial $E_4(\overline{N_1}, \overline{N_2})$ equilibrium point of system 6 is globally asymptotically stable.

Proof: since $E_4(\overline{N_1}, \overline{N_2})$ has no periodic solution in R_+^2. So, $E_4(\overline{N_1}, \overline{N_2})$ is globally asymptotically stable by the Poincare-Bendixon theorem.

It is observed from Theorem 5 and Corollary 6 that the dynamic behavior is globally stable without conditions, and this implies it is locally stable without conditions.

Kolmogorov Analyses

Kolmogorov criteria were employed to establish the biological viability of the suggested non-dimensional autonomous prey-predator model with Holling type I functional and numerical responses and to get the conditions of coexistence and extinction. It is worth mentioning that[45] it goes into great depth about these circumstances.

System 6 is shown as follows:

$$XM(N_1, N_2) = N_1\left(1 - \frac{N_1}{k} - \alpha N_2 + \frac{i\,(1+\varepsilon)}{N_1}\right) \qquad 29$$
$$YN(N_1, N_2) = N_2(-u + e\alpha N_1 - e\alpha N_2)$$

where $M(N_1, N_2)$ and $N(N_1, N_2)$ are the growth rate of the species to the current density of prey N_1 and predator N_2, respectively.

1. According to the first condition of Kolmogorov $\frac{\partial M}{\partial N_2} < 0$ and $\frac{\partial N}{\partial N_2} < 0$, the growth rate of prey and predators decreases when the number of prey is constant and the number of predators increases.

By applying the first condition of Kolmogorov on system 29, one gets

$$\frac{\partial M}{\partial N_2} = -\alpha < 0 \qquad 30$$
$$\frac{\partial N}{\partial N_2} = -e\alpha < 0 \qquad 31$$

Eq. 30 and Eq. 31 since e and α are both positive factors, and they are always negative. This instance will therefore result in a drop in both of their growth rates.

2. System 3 has a carrying capacity that is $\frac{k+k\sqrt{1+i\,(1+\varepsilon)}}{2}$. According to the second condition of Kolmogorov $\exists P > 0 \ni M(P, 0) = 0$, then P is the carrying capacity. Using this condition on system 29, it has

$$M(P, 0) = 1 - \frac{P}{k} - \alpha(0) + \frac{i\,(1+\varepsilon)}{P} = 0 \qquad 32$$
$$P - \frac{1}{k}P^2 - i\,(1+\varepsilon) = 0 \qquad 33$$

By solving this quadratic equation, the positive root of $P = \frac{k+k\sqrt{1+i\,(1+\varepsilon)}}{2}$ is obtained, which represents the carrying capacity of the system 29.

3. The least number of prey that maintains even when the predator population is small in the system 29 is $\boldsymbol{\tau} = \frac{\boldsymbol{u}}{\boldsymbol{e\alpha}}$. According to the third condition of Kolmogorov $\exists\tau > 0 \ni U(\tau, 0) = 0$, then τ is the least number of prey. This condition is applied to system 29 to obtain

$$U(\tau, 0) = -u + e\alpha\tau = 0 \qquad 34$$

Therefore, $\tau = \frac{u}{e\alpha}$ is the least number of prey.

4. The prey coexists with predators if $P > \tau$. So, this gives

$$\frac{k+k\sqrt{1+i\,(1+\varepsilon)}}{2} > \frac{u}{e\alpha} \qquad 35$$

It is well established that any ecosystem may continue to exist if the carrying capacity exceeds the quantity of prey. As a result, there will be more prey that will boost the predator's carrying capacity.

It is concluded the following theorem from the Kolmogorov Analyses:

Theorem 7: The preys coexist with predators if the condition 35 is satisfied.

The rescue effect, as seen in condition 35, enhances the likelihood of coexistence compared to when there is no rescue effect, which can help any ecosystem on the verge of extinction. Theorem 7 leads to the following consequence:

Corollary 8: The rescue effect raises the possibility of the prey and predators' coexistence.

Numerical Simulations

The present section describes several numerical simulations that were carried out using the MATHEMATICA program to demonstrate the fluctuation of the rescue influence on the dynamics of the non-autonomous system (i.e. model 2). Two sorts of figures are used to depict dynamic behaviors: time series and phase plane figures. The parameter of immigration is changed in each scenario, while the remaining parameters and beginning circumstances are maintained as follows:

$k = 4$, $\alpha = 1.5$, $e = 0.5$, $u = 0.5$, $\varepsilon = 0.5$, $\omega = 0.75$, $N_1(0) = 2$, $N_2(0) = 1$.

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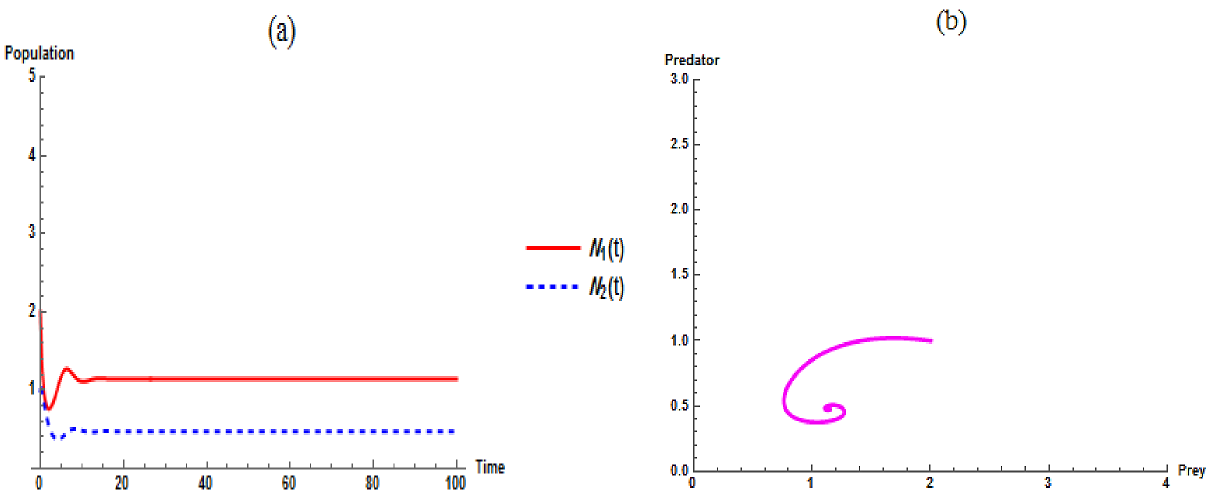

Figure 1. Dynamic behaviour of system 1 when ($i = 0$): (a) time series; (b) phase plane.

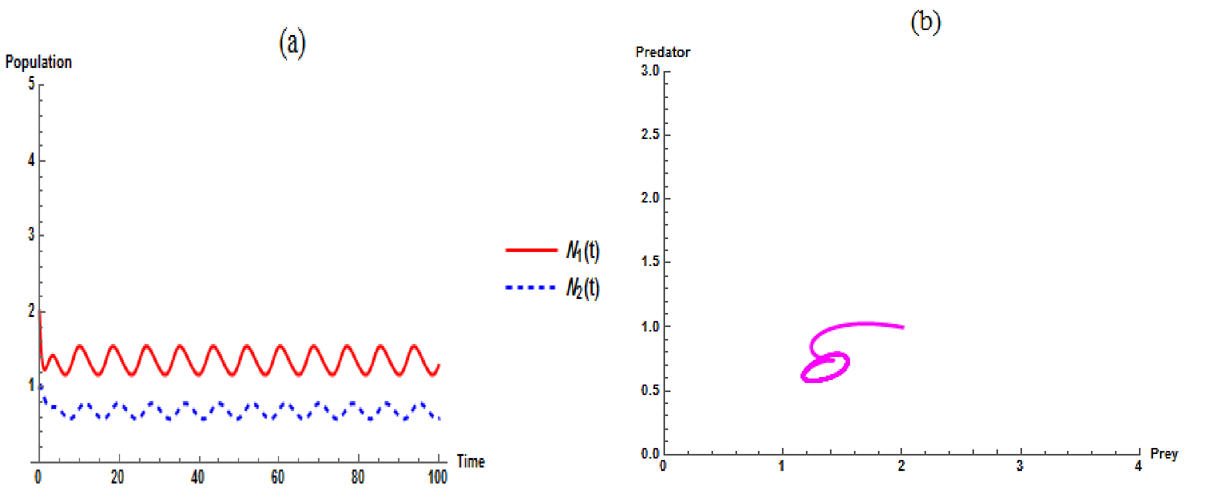

Figure 2. Dynamic behaviour of system 1 when ($i = 0.5$): (a) time series; (b) phase plane.

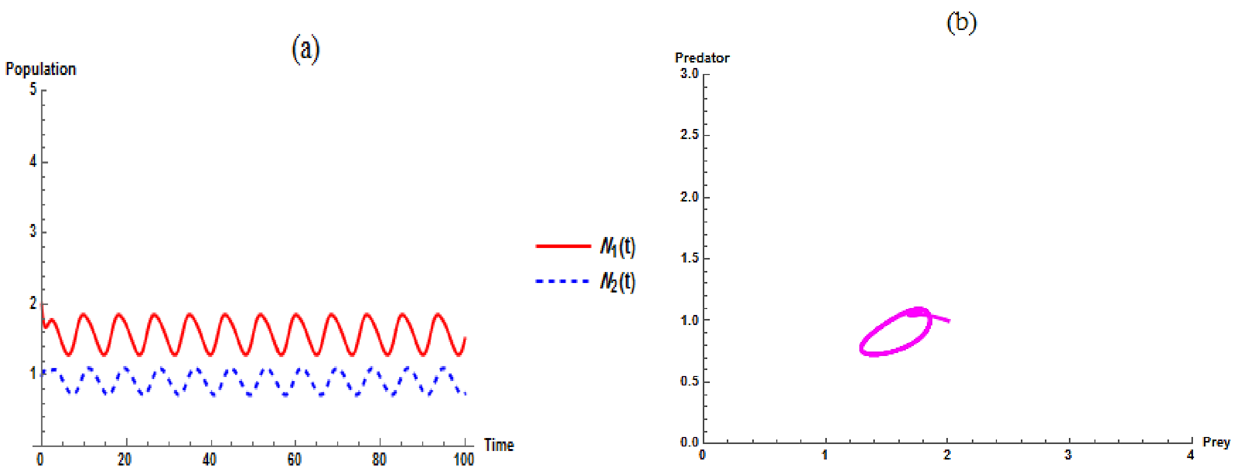

Figure 3. Dynamic behavior of system 1 when ($i = 1.25$): (a) time series; (b) phase plane.

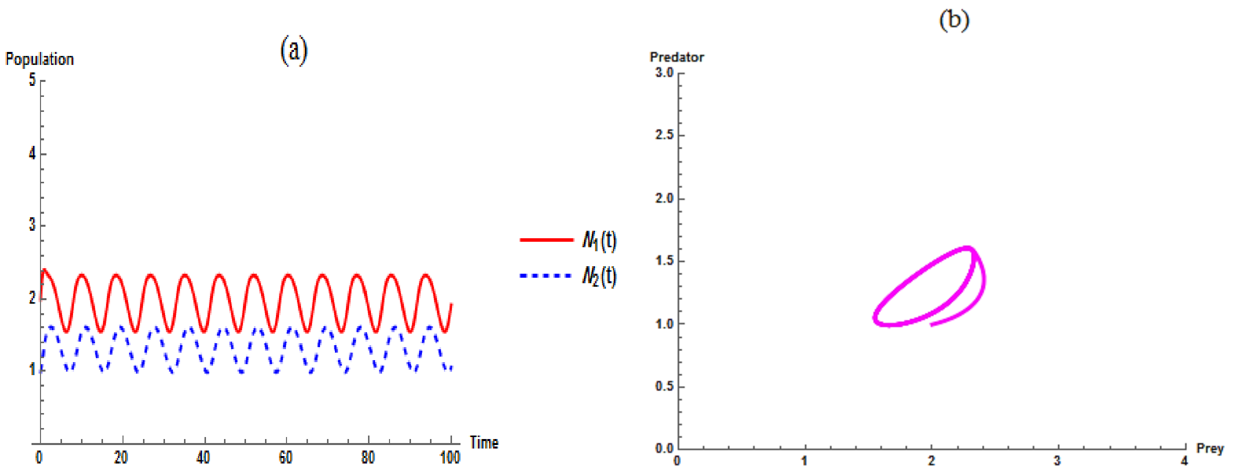

Figure 4. Dynamic behaviour of system 1 when ($i = 3.00$): (a) time series; (b) phase plane.

Fig. 1 shows that the dynamic behavior has coexisted steadily, but when the immigration factor I increases, the dynamic behaviors oscillate and become more oscillated, as seen in Figs. (2 – 4), respectively, implying that there is an oscillation component. In general, the dynamics of theoretical results (Theorem 5 and corollary 6) with fixed effect of rescue phenomena (i.e. system 6) is stable, but the periodic dynamics arise with fluctuation immigration involved (i.e. system 2). Although the dynamic behaviors grow more oscillated as the immigration factor increases, the predator and prey numbers increase, improving the likelihood of the prey-predator model coexisting.

Discussion and Conclusions:

In this paper, the prey-predator model was investigated under the impact of the fluctuation rescue effect.

Holling type I functional and numerical responses with the fluctuation rescue effect factor are used to form the non-dimensional non-autonomous predator-prey model, then approximated into an autonomous model to aid mathematical study. The boundedness of the system was studied in order to study its stability. It is concluded in theorems 2 – 4 that the equilibrium points of system 2 are locally stable under specific conditions. Since system 6 does not contain periodic solutions, as shown in theorem 5, it turns out that the system is globally stable Corollary 6.

Kolmogorov conditions showed that the addition of the fluctuation rescue effect increases the chances of coexistence between prey and predator. In turn, this feature can be helpful in saving any system in life that is threatened with extinction. Simulations of the system 2 confirmed that the fluctuation rescue effect increases the probability of coexistence between prey and predator and the periodic dynamics appear in contrast with the fixed rescue effect that shows stable dynamics as proved theoretically.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in Majmaah University.

Authors' contributions statement:

Gh. E. A., J. A., and W. B. Y. contributed to the suggestion of the subject of the study, to the creation and simulation of the proposed model, to the analysis of the results and to the writing of the manuscript.

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ديناميكيات نموذج المفترس والفريسة تحت تأثير الانقاذ المتذبذب

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الخلاصة:
تهدف هذه الورقة إلى دراسة فكرة جديدة تتضمن التحقيق في تأثير إنقاذ الفريسة مع تأثير التذبذب لاقتراح نموذج معدّل من مفترس - فريسة يشكل نموذجًا غير مستقل .ومع ذلك، يتم تحويل النموذج غير المستقل إلى نموذج مستقل بطريقة التقريب لتبسيط التحليل الرياضي واتباع السلوكيات الديناميكية .تمت دراسة بعض الخصائص النظرية للنموذج المستقل المقترح مثل المحدودية والاستقرار وظروف كولموغوروف . أظهرت النتائج التحليلية في هذا البحث أن السلوكيات الديناميكية مستقرة عموميا وأن تأثير الإنقاذ يزيد من إمكانية التعايش مقارنة بغير تأثير الإنقاذ .بالإضافة إلى ذلك، يتم تنفيذ عمليات المحاكاة العددية لإظهار تأثير الإنقاذ المتذبذب على ديناميكيات النموذج غير المستقل .تقدم النتائج التحليلية والرقمية نموذجًا أكثر تعايشا بين الفريسة والمفترس، وهذا يمكن أن يدعم أي نظام بيئي مهدد بالانقراض.

الكلمات المفتاحية: نموذج مفترس فريسة، تأثير الإنقاذ، تحليل الاستقرار، التعايش، تحليل كولموغوروف.