About the Construction of Fuzzy Inner Product Spaces

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Abstract:

In this paper, a new inner product function is given and defined by using the extension principle of Zadeh, which will be called the fuzzy inner product, this inner product together with the space of all fuzzy subsets of the universal inner product space X form the so called fuzzy inner product space or pre – fuzzy Hilbert space which will be denoted by ($\tilde{\rho}$, <., .>˜)).

1- Introduction:

Zadeh in 1965, introduced the notion of fuzzy set provided a convenient point of departure for the construction of a conceptual frame work which parallels in many respects, the frame used in the case of ordinary system , but in more general than the later and, rotationally, many prove to nave a much reader scope of applicability, particularly in the field of pattern classification and information processing. Essentially, such a frame work provides a natural way of dealing with problems in which the source of imprecision of classical membership rather than the presence. of random variables, [4].

Other attempts are given by other researchers, and the followings are some of them. Rosenfeld in 1971, used the concept to develop the theory of fuzzy groups and found that many basic properties in group theory are successfully carried out on fuzzy groups. Since that, these ideas have been applied to other algebraic structures, such as semigroups, rings, ideals, modules, etc.. [2].

Also, Kramosil I. in 1975 [2] studied fuzzy metric spaces and its connection with statistical metric spaces. Systematic treatment of abstract fuzzy dynamical systems was presented in 1973 by Nazaroff and further investigation was done by Warren [5]. But in 1980 fuzzy topology has been used to define neighborhood structure of fuzzy by Ming P. P. and Min L. Y.

Tracking their steps, many mathematical researchers used the subject in other fields of mathematics, applied and pure, such as linear algebra, optimization theory and so on. But, still there is a wide gap in using the subject in studying fuzzy inner product spaces.

Among the most important notions in fuzzy set theory, which are necessary for extending ordinary inner product spaces to fuzzy inner product spaces is the extension principle

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described by L. A. Zadeh [5] provides a natural way for extending the domain of a mapping defined on a set X to a fuzzy subset of X, [3] or in other words it provides a general method for extending non fuzzy mathematical concepts onto fuzzy set theory [1].

Now we will summarize some basic definitions and concepts related to abstract theory of fuzzy sets.

Definition 1 [5]:

Let X be any set of elements. A *fuzzy set* A in X is characterized by a membership function, $\mu_A : X \longrightarrow I$, where I is the closed unit interval [0, 1]. Then we can write a fuzzy set A by the set of points:

$A = \{(x_i, \mu_A(x_i)) \mid x_i \in X, i = 1, 2, \dots, n, 0 \leq \mu_A(x_i) \leq 1\}$

The collection of all fuzzy sets in X will be denoted by I^X or $\tilde{\rho}(X)$, i.e.,

$I^X = \{A : A \text{ is fuzzy subset of } X\}$.

Remarks 1 [2], [5]:

Following, we lists some concepts related to the basic operations of fuzzy subsets of X:

Let A and B be fuzzy sets in X with membership functions μ_A and μ_B respectively, then for all $x \in X$:

1- $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$.

2- $A = B$ if and only if $\mu_A(x) = \mu_B(x)$.

3- $E = X - A$ if and only if $\mu_E(x) = 1 - \mu_A(x)$.

4- $A = \varnothing$ if and only if $\mu_A(x) = 0$.

5- $C = A \cap B$ if and only if $\mu_C(x) = \min\{\mu_A(x), \mu_B(x)\}$.

6- $D = A \cup B$ if and only if $\mu_D(x) = \max\{\mu_A(x), \mu_B(x)\}$.

Definition 2 [4]:

Let $A_1, A_2, \dots, A_n$ be n – fuzzy subsets of universes $X_1, X_2, \dots, X_n$, respectively, then the ***Cartesian product*** of $A_1, \dots, A_n$ is denoted by $A_1 \times \dots \times A_n$ and is defined as a fuzzy subset of $X_1 \times X_2 \times \dots \times X_n$, whose membership function is defined by:

$$\mu_{A_1 \times A_2 \times \dots \times A_n}(x_1, x_2, \dots, x_n) = \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \dots, \mu_{A_n}(x_n)\}$$

Let f be a mapping from $X_1 \times X_2 \times \dots \times X_n$ to a universe Y such that $Y = f(x_1, x_2, \dots, x_n)$, the extension principle allows us to induce from n fuzzy sets A_i, a fuzzy set B on Y through f, such that:

$$\mu_B(y) = \begin{cases} \sup\limits_{(x_1,\dots,x_n) \in f^{-1}(y)} \min\{\mu_{A_1}(x_1), \mu_{A_2}, \dots, \mu_{A_n}(x_n)\}, & \text{if } f^{-1}(y) \neq \varnothing \\ 0, & \text{if } f^{-1}(y) = \varnothing \end{cases} \quad \dots\dots(1)$$

where $f^{-1}(y)$ denotes inverse image of y

2 - Fuzzy Inner Product Between Fuzzy Sets:

The subject of Fuzzy inner product spaces and in a special case the inner product between Fuzzy subsets of a universe product space could be considered as a problem which is not established clearly by other researchers. Therefore, we have seen that it is satisfactory to use the extension principle of Zadeh to define and construct fuzzy inner product spaces.

Definition 3:

Let $(X, \langle\cdot,\cdot\rangle)$ be an ordinary inner product space, and $A, B \in \tilde{\rho}(X)$ (the fuzzy vector space over X). Define the fuzzy function $<.,.>^{\sim} : \tilde{\rho}(X)\times \tilde{\rho}(X) \longrightarrow \tilde{\rho}(K)$ (where $\tilde{\rho}(K)$ is the set of all fuzzy subsets of K) by the extension principle for Zadeh as follows:

$$\mu_{<A,B>^{\sim}}(w) = \begin{cases} \sup\limits_{w=<x,y>} \min\{\mu_A(x), \mu_B(y)\}, & \text{if } w=<x,y> \\ 0, & \text{if } w \neq <x,y> \end{cases} \quad .....(2)$$

Now, we have to show that the function $<.,.>^{\sim}$ defined by (2) indeed is an inner product and therefore ($<A\ B>^{\sim}$, $<.,.>^{\sim}$) is a fuzzy inner product.

From the extension principle, it is well known that if $A_1, A_2, \ldots, A_n$ are n–fuzzy subsets of universes $X_1, X_2, \ldots, X_n$, respectively, then $A_1+A_2+\ldots+A_n$ is a fuzzy subsets, with membership function:

$$\mu_{A_1+\ldots+A_n}(y) = \underset{w=x_1+\ldots+x_n}{\text{aup}} \min\{\mu_{A_1}(x_1),\ldots,\mu_{A_n}(x_n)\}$$

and for $\lambda \in$ □ and $\tilde{B}$ a fuzzy set in X, where $X = X_1\times X_2\times\ldots\times X_n$, we define λB as in the following lemma:

Lemma:

a) For $\lambda \neq 0$,

$$\mu_{\lambda B}(w) = \mu_B\left(\frac{1}{\lambda}w\right), \forall w \in x.$$

b) For $\lambda = 0$,

$$\mu_{\lambda B}(x) = \begin{cases} 0 & \text{if } w \neq 0 \\ \sup\limits_{x} \mu_B(x) & \text{if } w = 0 \end{cases}$$

Proof:

For the first part, we have using the extension principle:

$$\mu_{\lambda B}(w) = \underset{\substack{w=\lambda x \\ \frac{1}{\lambda}w=x}}{\text{Sup}} \mu_{\lambda B}(x)$$

$$\geq \mu_B(x) = \mu_B\left(\frac{1}{\lambda}w\right)$$

Hence:

$$\mu_{\lambda\tilde{B}}(w) \geq \mu_{\tilde{B}}\left(\frac{1}{\lambda}w\right) \ldots\ldots(3)$$

On the other hand

$$\mu_B\left(\frac{1}{\lambda}w\right) = \mu_{\frac{1}{\lambda}g(B)}\left(\frac{1}{\lambda}w\right)$$

$$= \underset{\frac{1}{\lambda}w=\frac{1}{\lambda}g(x)}{\text{Sup}} \mu_{g(B)}(w)$$

$$\geq \mu_{g(B)}(w) = \mu_{\lambda B}(w)$$

and therefore:

$$\mu_B\left(\frac{1}{\lambda}w\right) \geq \mu_{\lambda B}(w) \ldots(4)$$

As a result, from (3) and (4), we have:

$$\mu_{\lambda B}(w) = \mu_{\tilde{B}}\left(\frac{1}{\lambda}w\right)$$

For the second part, if $\lambda = 0$ and $w \neq 0$ $\mu_{\lambda B}(w) = 0$, and since $\mu_{\lambda B}(w) = \sup\limits_{x} \mu_{B}(x)$ and since $w \neq \lambda x$, because if $w = \lambda x$ then $w = 0x = 0$ but $w \neq 0$, i.e., the inverse image of x is empty by definition of the extension principle. If $\lambda = 0$ and $w = 0$, then:

$$\mu_{\lambda B}(w) = \underset{\substack{x \\ w=0=0x}}{\sup} \mu_B(x). \quad \blacksquare$$

Now, we can define the following:

$(\langle A, C\rangle^{\sim} + \langle B, C\rangle^{\sim})(w) =$

$$\sup_{\substack{x,y,z \\ w=\langle x+y,z\rangle}} \min\left(\mu_{\langle A,C\rangle^{\sim}}(\langle x,z\rangle), \mu_{\langle B,C\rangle^{\sim}}(\langle y,z\rangle)\right)$$

$= \langle x,y\rangle + \langle y,z\rangle$ …(5)

where A, B and C are fuzzy subsets of X, and:

$\mu_{\lambda\langle A, B\rangle^{\sim}}(w) =$

$$\begin{cases} \mu_{\langle A,B\rangle^{\sim}}\left(\frac{1}{\lambda}w\right) & \text{if } \lambda\neq 0 \\ 0 & \text{if } \lambda=0, w\neq 0 \\ \sup_{\langle x,y\rangle} \mu_{\langle A,B\rangle^{\sim}}(\langle x,y\rangle) & \text{if } \lambda=0, w=0 \end{cases}$$

…..(6)

Theorem:

$(\tilde{\rho}(X), \langle .,.\rangle^{\sim})$ is a fuzzy inner product space.

Proof:

We have to prove that the following four conditions are satisfied for all A, B and C $\in \tilde{\rho}(X)$ and for all $\lambda \in K$.

1- $\langle A, B\rangle^{\sim} = \overline{\langle A,B\rangle^{\sim}}$.
2- $\langle A + B, C\rangle^{\sim} = \langle A, C\rangle^{\sim} + \langle B, C\rangle^{\sim}$.
3- $\langle \lambda A, B\rangle^{\sim} = \lambda\langle A, B\rangle^{\sim}$.
4- $\langle A, A\rangle^{\sim} \geq 0$ and $\langle A, A\rangle^{\sim} = 0$ if and only if $A = 0$

For the first condition.

Let $S = \{w = \langle x,y\rangle \mid \langle x,y\rangle = \overline{\langle y,x\rangle}, x,y \in X, w \in K\}$

We have to show that

$$\mu_{\langle A,B\rangle^{\sim}}(w) = \mu_{\overline{\langle B,A\rangle^{\sim}}}(w).$$

Assume first that $w \notin \delta$, then:

$$\mu_{\langle A,B\rangle^{\sim}}(w) = 0 = \mu_{\overline{\langle A,B\rangle^{\sim}}}(w)$$

Now, assume next that $w \in S$, then:

$$\mu_{\langle A,B\rangle^{\sim}}(w) = \sup_{\substack{x,y \\ w=\langle x,y\rangle}} \min\{\mu_A(x), \mu_B(y)\}$$

$$= \underline{\sup}_{w=\overline{\langle y,x\rangle}} \min\{\mu_B(y), \mu_A(x)\}$$

$$= \mu_{\overline{\langle B,A\rangle^{\sim}}}(w).$$

Hence, for all $w \in K$,

we have $\langle A, B\rangle^{\sim} = \overline{\langle A,B\rangle^{\sim}}$.

For the second condition.

We have to show that

$$\mu_{\langle A+B,C\rangle^{\sim}}(w) = \mu_{\langle A,C\rangle^{\sim}+\langle B,C\rangle^{\sim}}(w)$$

, for all $w \in A$.

If $w \notin S$, then

$$\mu_{\langle A+B,C\rangle^{\sim}}(w) = 0 = \mu_{\langle A,C\rangle^{\sim}+\langle B,C\rangle^{\sim}}(w),$$

while if $w \in S$, then given any $\varepsilon > 0$, there exists x, y and z $\in X$ with $\langle x + y, z\rangle = w$, such that:

$$\mu_{\langle A+B,C\rangle^{\sim}}(w) = \sup_{w=\langle x+y,z\rangle} \min\{\mu_{A+B}(x+y), \mu_C(z)\}$$

$$= \sup_{w=\langle x+y,z\rangle} \left\{ \sup_{x,y} \min\{ \mu_A(x), \mu_B(y), \mu_C(z)\} \right\}$$

(from the definition of the sum of two fuzzy sets using the extension principle)

$$\geq \sup_{w=<x+y,z>} \min\{\min\{\mu_A(x),\mu_B(y)),\mu_C(z)\}\}$$

$$= \sup_{w=<x+y,z>} \min\{\mu_A(x),\mu_B(y)),\mu_C(z)\}$$

$$\geq \min\{\mu_A(x),\mu_B(y)),\mu_C(z)\}$$

Hence:

$$\mu_{<A+B,C>^{\sim}}(w) \geq \min\{\mu_A(x),\mu_B(y)),\mu_C(z)\}$$

Now, suppose there is a real number ε (small enough), such that:

$$\min\{\mu_A(x),\mu_B(y)),\mu_C(z)\}$$

$$> \mu_{<A+B,C>^{\sim}}(w) - \varepsilon \quad ...(7)$$

then, we have:

$$\mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w) =$$

$$\mu_{<A,C>^{\sim}+<B,C>^{\sim}}(<x+y,z>)$$

$$= \mu_{<A,C>^{\sim}+<B,C>^{\sim}}(<x,z>+<y,z>)$$

$$= \sup_{\substack{w=<x+y,z> \\ =<x,z>+<y,z> \\ =\delta_1+\delta_2}} \min\{\mu_{<A,C>^{\sim}}(<x,z>),\mu_{<B,C>^{\sim}}(<y,z>)\} \text{ (from (5))}$$

$$= \sup_{\substack{w=<x+y,z> \\ =<x,z>+<y,z> \\ =\delta_1+\delta_2}} \min\left\{ \sup_{\substack{x,z \\ \delta_1=<x,z>}} \min\{\mu_A(x),\mu_C(z)\}, \sup_{\substack{y,z \\ \delta_2=<y,z>}} \min\{\mu_B(y),\mu_C(z)\} \right\}$$

$$\geq \sup_{w=<x+y,z>} \min\{\min\{\mu_A(x),\mu_C(z)\},\min\{\mu_B(y),\mu_C(z)\}\}$$

$$= \sup_{w=<x+y,z>} \min\{\mu_A(x),\mu_B(y)),\mu_C(z)\}$$

$$\geq \min\{\mu_A(x),\mu_B(y)),\mu_C(z)\}$$

therefore from (7), we have:

$$\mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w)$$

$$\geq \min\{\mu_A(x),\mu_B(y)),\mu_C(z)\}$$

$$> \mu_{<A+B,C>^{\sim}}(w) - \varepsilon$$

and as $\varepsilon \longrightarrow 0$, we have:

$$\mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w)$$

$$\geq \mu_{<A+B,C>^{\sim}}(w) \quad ...(8)$$

On the other hand, given any $\varepsilon > 0$, then there exist $\delta_1, \delta_2 \in K$ with $\delta_1 + \delta_2 = w$, such that:

$$\mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w)$$

$$= \mu_{<A,C>^{\sim}+<B,C>^{\sim}}(\delta_1+\delta_2)$$

$$= \sup_{w=\delta_1+\delta_2} \min\{\mu_{<A,C>^{\sim}}(\delta_1),\mu_{<B,C>^{\sim}}(\delta_2)\}$$

$$\geq \min\{\mu_{<A,C>^{\sim}}(\delta_1),\mu_{<B,C>^{\sim}}(\delta_2)\}$$

Thus $\mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w) \geq$

$$\min\{\mu_{<A,C>^{\sim}}(\delta_1),\mu_{<B,C>^{\sim}}(\delta_2)\}.$$

It follows that:

$$\min\{\mu_{<A,C>^{\sim}}(\delta_1),\mu_{<B,C>^{\sim}}(\delta_2)\} >$$

$$\mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w) - \varepsilon$$

taking $\varepsilon < \mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w)$

(if $\mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w) = 0$

then $\mu_{<A+B,C>^{\sim}}(w) = 0$

and we have nothing to prove).

If $\delta_1, \delta_2 \in S$, therefore, there exists x, y and z in X, with $<x, z> = \delta_1$ and $<y, z> = \delta_2$, such that:

$$\mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w) \geq \min\left\{\mu_{<A,C>^{\sim}}(\delta_1), \mu_{<B,C>^{\sim}}(\delta_2)\right\}$$

$$= \min\left\{\sup_{\delta_1=<x,z>}\{\mu_A(x), \mu_C(z)\}, \sup_{\delta_2=<y,z>}\{\mu_B(x), \mu_C(z)\}\right\}$$

$$\geq \min\{\mu_A(x), \mu_B(y)), \mu_C(z)\}$$

hence $\mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w) \geq$

$$\min\{\mu_A(x), \mu_B(y)), \mu_C(z)\}$$

It follow that

$$\min\{\mu_A(x), \mu_B(y)), \mu_C(z)\} > \mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w) - \varepsilon$$

Since $<x + y, z> = <x, z> + <y, z> = w$, therefore:

$$\mu_{<A+B,C>^{\sim}}(w) = \sup_{\substack{x,y,z \\ w=<x+y,z>}} \min\{\mu_{A+B}(x+y), \mu_C(z)\}$$

$$= \sup_{\substack{x,y,z \\ w=<x+y,z>}} \min\left\{\sup_{x,y,z} \min\{\mu_A(x), \mu_B(y)\}, \mu_C(z)\right\}$$

$$\geq \sup_{x,y,z} \min\left\{\min\{\mu_A(x), \mu_B(y)\}, \mu_C(z)\right\}$$

$$\geq \min\{\mu_A(x), \mu_B(y)), \mu_C(z)\}$$

$$> \mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w) - \varepsilon$$

Thus $\mu_{<A+B,C>^{\sim}}(w) >$

$$\mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w) - \varepsilon,$$

and hence as $\varepsilon \longrightarrow 0$, we have:

$$\mu_{<A+B,C>^{\sim}}(w) \geq \mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w) \ldots (9)$$

Now, from (8) and (9), we have:

$$\mu_{<A+B,C>^{\sim}}(w) = \mu_{<A,C>^{\sim}+<B,C>^{\sim}}(w)$$

which completes the proof of part (2).

For the third condition, and considering also the set S and if $w \notin S$, then $\mu_{<\lambda A,B>^{\sim}}(w) = 0$ and hence from the extension principle $\mu_{\lambda<A,B>^{\sim}}(w) = 0$, and therefore $\mu_{<\lambda A,B>^{\sim}}(w) = \mu_{\lambda<A,B>^{\sim}}(w)$

Suppose that $w \in S$, if $\lambda \neq 0$, then:

$$\mu_{\lambda<A,B>^{\sim}}(w) = \mu_{<A,C>^{\sim}}\left(\frac{1}{\lambda}w\right)$$

$$= \mu_{<A,B>^{\sim}}\left(\frac{1}{\lambda} < x, y >\right)$$

$$= \mu_{<A,B>^{\sim}}\left(< \frac{1}{\lambda}x, y >\right)$$

$$= \sup_{w=<x,y>} \min\left\{\mu_A\left(\frac{1}{\lambda}x\right), \mu_B(y)\right\}$$

$$= \sup_{w=<x,y>} \min\{\mu_{\lambda A}(x), \mu_B(y)\}$$

$$= \mu_{\lambda<A,B>^{\sim}}(w)$$

Next assume that $\lambda = 0$, if $w \neq 0$, then from (5) $\mu_{\lambda<A,B>^{\sim}}(w) = 0$. Also:

$$\mu_{<\lambda A,B>^{\sim}}(w) = \mu_{<\lambda A,B>^{\sim}}(<x,y>)$$

$$= \sup_{w=<x,y>} \min\{\mu_{\lambda A}(x), \mu_B(y)\}$$

$$= \sup_{w=<x,y>} \min\{\mu_{0A}(x), \mu_B(y)\}$$

(since $w \neq 0$, then x and $y \neq 0$, where if x or y = 0, then $w = <x, y> = 0$, which is a contradiction.

$$= \sup_{w=<x,y>} \min\{0, \mu_B(y)\} = 0$$

and if $w = 0$, then given any $\varepsilon > 0$, there exists x, y in X with $w = <0, y> = <0x, y> = 0<x, y>$, such that:

$$\mu_{<0A,B>^{\sim}}(w) = \mu_{<0A,B>^{\sim}}(0)$$

$$= \sup_{w=<0x,y>} \min\{\mu_{0A}(0x), \mu_B(y)\}$$

$$= \sup_{w=<0x,y>} \min\left\{\sup_x \mu_A(x), \mu_B(y)\right\}$$

$$\geq \sup_{w=<0x,y>} \min\{\mu_A(x), \mu_B(y)\} \geq$$

$\min\{\mu_A(x), \mu_B(y)\}$

Therefore, $\mu_{<0A,B>^{\sim}}(w) \geq \min\{\mu_A(x), \mu_B(y)\}$. It follows that:

$\min\{\mu_A(x), \mu_B(y)\} >$

$\mu_{<0A,B>^{\sim}}(w) - \varepsilon$

We have also:

$$\mu_{0<A,B>^{\sim}}(w) = \mu_{0<A,B>^{\sim}}(<0x,y>)$$

$$= \mu_{0<A,B>^{\sim}}(0<x,y>)$$

$$= \sup_{w=<0x,y>} \mu_{<A,B>^{\sim}}(<x,y>) \text{(by (5))}$$

$$= \sup_{w=<x,y>} \min\{\mu_A(x), \mu_B(y)\} \geq$$

$\min\{\mu_A(x), \mu_B(y)\}$

then $\mu_{0<A,B>^{\sim}}(w) \geq$

$\min\{\mu_A(x), \mu_B(y)\} \geq$

$\mu_{<0A,B>^{\sim}}(w) - \varepsilon$, and as $\varepsilon \longrightarrow 0$,

we get $\mu_{0<A,B>^{\sim}}(w) \geq$

$\mu_{<0A,B>^{\sim}}(w)$.

On the other hand, given any $\varepsilon > 0$, then there exist δ_1 in K with $\lambda\delta_1 = w$, such that:

$$\mu_{\lambda<A,B>^{\sim}}(w) = \mu_{\lambda<A,B>^{\sim}}(\lambda\delta_1)$$

$$= \mu_{0<A,B>^{\sim}}(0\delta_1) =$$

$$\sup_{\delta_1} \mu_{<A,B>^{\sim}}(\delta_1) \text{ (using (5))}$$

hence $\mu_{\lambda<A,B>^{\sim}}(w) \geq \mu_{<A,B>^{\sim}}(w)$.

It follows that $\mu_{<A,B>^{\sim}}(w) >$

$\mu_{\lambda<A,B>^{\sim}}(w) - \varepsilon$,

and upon taking $\varepsilon < \mu_{\lambda<A,B>^{\sim}}(w)$

(if $\mu_{\lambda<A,B>^{\sim}}(w) = 0$, then

$\mu_{<\lambda A,B>^{\sim}}(w) = 0$

and we have noting to prove) we have that $\delta_1 \in S$. Therefore, there exists x, y $\in$ X, with $<x, y> = \delta_1$, such that:

$$\mu_{\lambda<A,B>^{\sim}}(w) = \mu_{<A,B>^{\sim}}(\delta_1)$$

$$= \sup_{\delta_1=<x,y>} \min\{\mu_A(x), \mu_B(y)\} \geq$$

min $\{\mu_A(x), \mu_B(y)\}$

It follow that min

$\{\mu_A(x), \mu_B(y)\} > \mu_{\lambda<A,B>^{\sim}}(w) - \varepsilon.$

Now, since <0x, y> = 0<x, y>, we get

$\mu_{<\lambda A,B>^{\sim}}(w) =$

$\mu_{<0A,B>^{\sim}}(0<x,y>)$

$= \mu_{<0A,B>^{\sim}}(<0x,y>)$

$$= \sup_{\substack{w=0<x,y> \\ =<0x,y>}} \min\{\mu_{0A}(0x), \mu_B(y)\}$$

$$= \sup_{w=0x\ ,y>} \min\left\{\sup_x \mu_A(x), \mu_B(y)\right\}$$

(from lemma (2.1))

$$\geq \sup_{w=0<x,y>} \min\{\mu_A(x), \mu_B(y)\}$$

$\geq$ min $\{\mu_A(x), \mu_B(y)\} >$

$\mu_{\lambda<A,B>^{\sim}}(w) - \varepsilon$

and as $\varepsilon \longrightarrow 0$, we have

$\mu_{<\lambda A,B>^{\sim}}(w) \geq \mu_{\lambda<A,B>^{\sim}}(w),$

which completes the proof (3).

this prove (3)

Finally, to prove the fourth condition, i.e., to prove that $<A, A>^{\sim} \geq 0$ and $<A, A>^{\sim}$ = if and only if A = 0.

It is clear that $<A, A>^{\sim} \geq 0, \forall A$

(since $\sup_{<A,B>^{\sim}} \min\{\mu_A(x), \mu_B(y)\}$

since $\mu_A(x) \in [0, 1]$ and $\mu_A(y) \in [0, 1]$.

Hence $\mu_{<A,A>^{\sim}}(w) \geq 0, \forall$ w.

Now, we have to show that $<A, A>^{\sim} = 0$ if and only if $A = \tilde{0}$ suppose that $A = \tilde{0}$ therefore $\mu_A(x) = \mu_0(x) = 0, \forall x \in X$, then:

$\mu_{<A,A>^{\sim}}(w) =$

$$\sup_{<A,B>^{\sim}} \min\{\mu_A(x), \mu_B(y)\}$$

(by def. of Fuzzy inner product)

$= 0$ (since A = 0, then $\mu_A(x) = 0, \forall x \in X$)

it follow that

$\mu_{<A,A>^{\sim}}(w) = 0$ if w = <x, y>

$= 0$ if w $\neq$ <x, y>

Hence $<A, A>^{\sim} = 0, \forall A$

On the other hand, suppose that $<A, A>^{\sim} = 0$ but $A \neq 0$, therefore, there exist x in X such that $\mu_A(x) \neq 0$, we have:

$0 < \mu_A(x) = \min\{\mu_A(x), \mu_A(x)\}$

$$= \sup_{w=<x,x>} \min\{\mu_A(x), \mu_B(x)\}$$

$= \mu_{<A,A>^{\sim}}(w)$

which contradicts the fact that

$<A, A>^{\sim} = 0, \forall A$ and hence A = 0.

Hence $(\tilde{\rho}(X), <.,.>^{\sim}$

is a fuzzy inner product space. ■

As a consequence to the above theorem, the set of all fuzzy subsets $\tilde{\rho}(X)$ of the inner space X form an inner product space which is called fuzzy inner product space or pre-fuzzy Hilbert space which is denoted by

We give the following example of fuzzy inner product space.

Example:

Let R^n be a inner product space with the inner product $(\tilde{\rho}(X), <.,.>^{\sim})$.

$<x, y> = x_1y_1 + \ldots + x_ny_n$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and hence $(R^n, <.,.>)$ is the inner product space

Let A and B be two fuzzy sets defined on X, then the membership function of the fuzzy set D = <A, B>$^{\sim}$ $\in \tilde{\rho}$ (R) can be defined using the extension principle, as follows:

$$\mu_{D=<A,B>^{\sim}}(w) = \sup_{\substack{x,y \\ w=<x,x>}} \min\{\mu_A(x), \mu_B(y)\}$$

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حول إنشاء فضاءات الضرب الداخلي في المجموعات الضبابية

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المستخلص:

في هذا البحث، تم أعطاء تعريف دالة ضرب داخلي باستخدام مبدأ التوسيع في المجموعات الضبابية والذي أقترح من قبل العالم زادة في عام 1965، والذي سوف يطلق عليه الضرب الداخلي الضبابي، هذا الضرب الداخلي الضبابي مع فضاء كل المجموعات الضبابية من المجموعة الشاملة X والتي هي أيضاً فضاء ضرب داخلي، تسمى فضاء الضرب الداخلي الضبابي والذي سيرمز له بالرمز $(\tilde{\rho}(x), <.,.>^{\sim})$.