The Coincidence Lefschetz Number For Self – Maps of Lie groups

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Abstract

Let $f, h : G \to G$ be any two self maps of a compact connected oriented Lie group G. In this paper, for each positive integer k, we associate an integer with f^k, h^k. We relate this number with Lefschetz coincidence number. We deduce that for any two differentiable maps $f, h : G \to G$, there exists a positive integer k such that $k \leq \lambda+1$, and there is a point $x \in G$ such that $f^k(x) = h^k(x)$, where λ is the rank of G.

Introduction

Let G be an n-dimensional compact connected Lie group with multiplication μ (i.e., $\mu : G \times G \to G$ such that $\mu(x, y) = x.y$) and unit e. Let $[G, G]$ be the set of homotopy classes of maps $G \to G$. Given two maps $f, f' : G \to G$, following [3], we write $f.f'$ to denote the map $G \to G$ defined by $(f.f')(g) = \mu(f(g), f'(g)) = f(g).f'(g)$, $g \in G$.

Given a point $g \in G$ and a differentiable map $F: G \to G$, write G_g to denote the tangent space of G at g [4,p.10], and denote by d_gF the linear map $d_gF : T_gG \to T_{F(g)}G$ induced by F, it is called the differential of F at g [4,p.22]. Let $L_g, R_g : G \to G$ be respectively the left translation $L_g(g') = \mu(g, g')$, and the right translation $R_g(g') = \mu(g', g)$. Then there is a natural homomorphism Ad, the adjoin representation, from G to $GL(G_e)$, (the group of nonsingular linear transformations of G_e) defined as follows:-

$$Ad(g) = d_gR_{g^{-1}} \circ d_eL_g .$$

Note that $d_gR_{g^{-1}} \circ d_eL_g = d(R_{g^{-1}}(L_g(e))) \circ d_eL_g = d_e(R_{g^{-1}} \circ L_g) = d_e(L_g \circ R_{g^{-1}}) = d(L_g(R_{g^{-1}}(e))) \circ d_eR_{g^{-1}} = d_{g^{-1}}L_g \circ d_eR_{g^{-1}}$. Since G is connected, the image of Ad belongs to the connected component of $GL(G_e)$ containing the identity, i.e. for each $g \in G$, $\det Ad(g) > 0$. By Exercise A1

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[4,p. 147] we have

***Lemma* 1 :-**

1) If T is the map $G \to G$ is defined by $T(g) = g^{-1}$, then

$d_g T = -d_e L_{g^{-1}} \circ d_g R_{g^{-1}} = -d_e R_{g^{-1}} \circ d_g L_{g^{-1}}$,

2) If μ is the mapping $(g_1, g_2) \to g_1 g_2$ of $G \times G$ into G, then if $X \in G_{g_1}, Y \in G_{g_2}$.

$d_{(g_1,g_2)}\mu(X,Y) = d_{g_2}L_{g_1}(Y) + d_{g_1}R_{g_2}(X), (X,Y) \in G_{g_1} \times G_{g_2}$.

In [3] , the author shows that if $f:G \to G$ is a differentiable map,then there exists a positive integer $k \le \lambda+1$ and a point $x \in G$ such that $f^k(x)=x$, where λ is the rank of G, i .e . the dimension of G of any maximal torus in G.

THE MAIN RESULTS

Let $f, h : G \to G$ be differentiable maps of compact connected oriented Lie group G. A point $g \in G$ is called a coincidence point if $f(g) = h(g)$, following [7],[8]. Assume f and h have isolated coincidence points then by compactness of G, f and h have only finitely many coincidence points .Also if g is an isolated coincidence point of f and h then $d_g f - d_g h$ has no nonzero fixed point , i.e. $\det(d_g f - d_g h) \neq 0$.In [6] the author defines the Lefschetz coincidence number as

$$Le(f,h) = \sum_{f(g)=h(g)} sign(\det(d_g f - d_g h)).$$

If f is the identity map then $Le(h)$ is the Lefschetz number,for more details see[2].

Let $Le:[G,G] \to Z$ be the function that sends each element in $[G,G]$ to its Lefschetz coincidence number . Then the Lefschetz coincidence point theorem states that "if f, $h : G \to G$ are maps with $Le(f,h) \neq 0$, then f, h have a coincidence point " . Now we define another function $B(f,h) : [G,G] \to Z$ by setting $B(f,h)$= degree$(f.h^{-1})$ .Since $B(f,h) \neq 0$ implies $f.h^{-1}$ is surjective ,i.e.,$e \in \operatorname{Im}(f \circ h^{-1})$. this function also possesses the property that " if $B(f,h) \neq 0$, f and h have a coincidence point,i.e.,if $B(f,h) \neq 0$ then $e \in \operatorname{Im}(fh^{-1})$ which means there is a point $g \in G$ such that $f(g)h^{-1}(g) = e$,i.e., $f(g)=h(g)$".

***Theorem* (2) :-**

The two functions Le , $(-1)^n B$: $[G,G] \to Z$ coincide , where n is the dimension of G.

proof :-

The left translation L_g of G onto itself is an analytic diffeomorphism then $d_e L_g$ is an isomorphism.So for each $g \in G$,we identify G_g with G_e by the differential of left translation for any L_g at e.

For any two maps in $[G,G]$, we take the representations $f,h: G \to G$ that satisfy the following :

(1) f and h are differentiable;

(2) f and h have only finitely many coincidence points $g_1,\dots,g_k$, i.e.,have isolated coincidence points $\{g_1,\dots,g_k\}$.

(3) $\det(d_{g_i} h - d_{g_i} f) \neq 0$.

Then $Le(f,h) = \sum_1^k sign \det(d_{g_i} h - d_{g_i} f)$.

$(f.h^{-1})^{-1}(e)=\{g_1,...,g_k\}$, and for each i the differential of $(f.h^{-1})\circ L_{g_i}$ at e is

$G_e \xrightarrow{dl_{g_i}} G_{g_i} \xrightarrow{d\Delta} G_{g_i}\times G_{g_i} \xrightarrow{df\times dh^{-1}}$

$G_{f(g_i)}\times G_{h^{-1}(g_i)} \xrightarrow{d\mu} G_e$,

where $:G\to G\times G$ is the diagonal map and we use Lemma(2) for $d_{(f(g_i),h^{-1}(g_i))}\mu$.

It follows from Lemma 2 that the above homomorphism is just the same as

$Ad(g^{-1})(d_e(L_{f^{-1}(g)}\circ(fL_g))-d_e(L_{h^{-1}(g)}\circ(hL_g)):G_e\to G_e$.

Denote this map by A_i. Then by the assumption (3) $\det(A_i)\neq 0$

$(-1)^n sign\det A_i = sign\det(d_{g_i}h-d_{g_i}f)$.

So we see that e is a regular value of fh^{-1},(a point y in Y is regular value for a smooth map of manifolds $f:X\to Y$, is called a regular value for f if $d_xf:T_xX\to T_yY$ is surjective at every point $x\in X$ such that $f(x)=y$,[2]) and

$$B(f,h)=\sum_{1}^{k} sign\det A_i=(-1)^n Le(f,h).$$

This completes the proof. □

To give an application of the above theorem, recall that $H^*(G,Q)$ is an exterior algebra $\wedge(x_1,...x_\lambda)$ generated by primitive elements x_i of odd degree [9,p.155] with $\lambda=$ rank G . Also from [9 , p149] we have

Lemma (3) :-

If f, $f':G\to G$ are two maps, and if $x\in H^*(G,Q)$ is primitive ,then

$(f.f')^*(x)=f^*(x)+f'^*(x)$.

By Lemma (3) $(fh^{-1})^*(x_i)=f^*(x_i)-h^*(x_i)$. Let $\cap$ be the cup product in $H^*(G;Q)$ then Lemma (3) also implies

$$\begin{aligned}B(f,h)^*(x_1\cup...\cup x_\lambda)&=(fh^{-1})^*(x_1\cup...\cup x_\lambda)\\&=(fh^{-1})^*x_1\cup...\cup(fh^{-1})^*x_\lambda\\&=(f^*(x_1)-h^*(x_1))\cup...\\&\cup(f^*(x_\lambda)-(h^*(x_\lambda))\end{aligned}$$

Recall that a maximal torus is a maximal abelian compact subgroup of $G$and any two maximal tori are conjugate in particular,all maximal tori have the same dimension λ, the integer λ is called the rank of G. It is known that $n\equiv\lambda$ mod 2 . We can rewrite theorem 2 as follows : -

***Theorem* (4) : -**

$$Le(f,h)x_1\cup...\cup x_\lambda=(h^*(x_1)-f^*(x_1))\cup...\cup(h^*(x_\lambda)-f^*(x_\lambda)).$$

Given a map $f:G\to G$ and integer $k>0$,let ${}^kf$ be the $k-$fold product of f defined inductively ${}^1f=f$, ${}^kf=f.{}^{k-1}f$.

Theorem (5) : -

For any integer k, $Le({}^kf,{}^kh)(x_1\cup...\cup x_\lambda)=x^\lambda Le(f,h)(x_1\cup...\cup x_\lambda)$.

Proof :-

$$\begin{aligned}Le({}^kf,{}^kh)(x_1\cup...\cup x_\lambda)&=(({}^kh)^*(x_1)-({}^kf)^*(x_1))\\&\cup...\cup(({}^kh)^*(x_\lambda)-({}^kf)^*(x_\lambda))\\&=(kh^*(x_1)-kf^*(x_1))\\&\cup...\cup(kh^*(x_\lambda)-kf^*(x_\lambda))\\&=k(h^*(x_1)-f^*(x_1))\\&\cup...\cup k(h^*(x_\lambda)-f^*(x_\lambda))\end{aligned}$$

$$= k^{\lambda}((h^*(x_1) - f^*(x_1)) \cup .. \cup (h^*(x_\lambda) - f^*(x_\lambda)));$$
$$= k^{\lambda} L(f,h)(x_1 \cup .. \cup x_\lambda)$$

Corollary (6) :-

For any differentiable maps f,h: G→G, there is an integer k with 0 <k≤ λ + 1 that such $Le({}^k f, {}^k h) \neq 0$.

Proof :-

Given maps $f_1, h_1 : G \to G$, , regard the expression , $(f_1 - h_1)(x_1) \cup ... \cup (f_1 - h_1)(x_\lambda)$, as a formal polynomial in the $(f_1 - h_1)^*(x_i)$. For each integer t with $0 \le t \le \lambda$, there exists an element sum of $A_t(f_1, h_1)$ in Q such that $A_t(f_1, h_1)x_1 \cup ... \cup x_\lambda$ = sum of the monomials appearing in the above polynomial , and containing just t elements. $(f_1 - h_1)^*(x_i)$. Then Lemma (3) and theorem (4) imply that

$$Le({}^k f, {}^k h) = \sum_0^{\lambda} k^t A_t(f,h) \quad \text{for any } k > 0.$$

So, if $H = (a_{st})$ is the $(\lambda+1) \times (\lambda+1)$ Vandermonde matrix [5] , defined by $a_{st} = t^{s-1}, 1 \le s, t \le \lambda + 1$ then

$$(Le(f,h), Le({}^2 f, {}^2 h), ..., Le({}^{\lambda+1} f, {}^{\lambda+1} h)) = (A_0(f,h), A_1(f,h), ..., A_\lambda(f,h))H.$$

Since *det* $(H) \neq 0$ and $A_0(f,h) = 1 \neq 0$ then there is an integer k, $0 < k \le \lambda+1$ such that $Le({}^k f, {}^k h) \neq 0$. □

Corollary (7) :-

For any differentiable maps f,h: G → G, there is an integer k with 0 < k ≤ λ+1 such that $f^k(x) = h^k(x)$.

Proof :-

By corollary (6)there is an integer k with $0<k\le\lambda+1$ such that $Le({}^k f, {}^k h) \neq 0$. Therefore , by Lefschetz coincidence point theorem f^k and h^k have a coincidence point . □

Suppose $f,h: G \to G$ are homomorphisms. Then for any primitive element $x \in H^*(G;Q)$, $(f-h)^*(x)$ is also primitive. Since the primitive elements form a submodule of $H^*(G;Q)$ with basis $\{x_1, ..., x_\lambda\}$,there exists a $\lambda \times \lambda$ matrix M_{f-h} over Q such that

$$(f^*(x_1) - h^*(x_1), ..., f^*(x_\lambda) - h^*(x_\lambda)) = (x_1, ... x_\lambda) M_{f-h}.$$
$$((fh^{-1})^*(x_1), ..(fh^{-1})^*(x_\lambda)) = (x_1, ... x_\lambda) M_{f-h}.$$
$$((fh^{-1})^*(x_1, ... x_\lambda)) = (x_1, ... x_\lambda) M_{f-h}.$$
$$B(f,h)^*(x_1, ... x_\lambda)) = (x_1, ... x_\lambda) M_{f-h}.$$

By theorem 4 we have

$Le(f,h) = \det(M_{f-h})$.The above discussion proves the following corollary :

Corollary (8) :-

$$Le(f,h) = \det(M_{f-h}).$$

Corollary (9) :-

$$Le({}^k Id) = (1-k)^\lambda, Le({}^{k-1}(f^{-1})) = (1+k)^\lambda,$$

where Id is identity map on G.

Proof :-

See [1] . □

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عدد لبشز للتطابق لدوال من زمرة لي الى نفسها

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المستخلص

لتكن G زمرة لي المرصوصه المترابطه ، لتكن $f, h : G \to G$ دالتين مستمرتين معرفتين على G وليكن t عدد صحيح موجب ، التكرار t لـ f و h يرمز له h', f' على التوالي .

في هذا البحث لكل عدد صحيح موجب t ، يرافق عدد صحيح مع h', f' ويربط هذا العدد مع عدد لبشز للتطابق . وبالنتيجه لأي دالتين $f, h : G \to G$ قابلتين للأشتقاق يوجد عدد صحيح موجب k بحيث أن $k \leq \lambda+1$ ويوجد نقطة $x \in G$ بحيث أن $f^k(x) = h^k(x)$ حيث λ رتبة G .