Solution of Variable Delay Integral Equations Using Variational Approach

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Abstract:

The main objective of this paper is to discuss and solve delay integral equations using the subject of calculus of variation in which the delay term appears in the integral equation is not a constant, but a function of the independent variable. Also, in this paper we considered Volterra delay integral equation of the second kind, as atest problem.

1. Introduction:

Many real life problems could not be solved using obvious methods in differential equations or integral equations. But reformulating these problems as a functional differentiation or integral equations is used in solving such type of problems,[Thekra,2001]

The importance of such type of delay or functional integral equations is of its ability to give information about the solution not only at the present time, but also for an interval of time step, which has many applications in real life problems, such as population growth model, [Myer,1990],[Ali,2000], [Gorecki,1989].

Many numerical and approximate methods could be applied to solve such type of delay integral equations, and among these methods is the variational methods which is of minimizing certain functional.

2. Preliminaries [El'sgolts, 1977]:

Variable Delay Volterra integral equations has the form:

$$f(t) = g(t) + \int_a^t K(t,s)f(s-d(h(s)))ds$$

where a is aconstant, g(t) is acontinous function called the derived term, k(t,s) called the kernel and f(t) are given function, and h(s) is acontinous function.

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is solved by using the method of variational technique.

The main aim is to find a functional F(f) defined on the domain of the linear operator L, whose critical points are the solutions of the given equation

Lf = g. The variational formulation of variable delay Volterra integral equation, when will be found in which its Euler-Lagrange equation is given by Lf = g, which is the desired delay Volterra integral equation.

The first step is to define an operator L related to the delay integral equation which is linear and non-degenerate (u, v) = <u, v>.

The next theorem given by Magri in 1974 [Magri, 1974] gives the equivalence between the variational formulation and the integral equation, i.e., between the critical points of the variational formulation and the solution of the integral equation.

Theorem (1) [Magri, 1974]:

The given linear operator L is symmetric with respect to the chosen non-degenerate bilinear form <u v>, then u is the solution of the equation Lu = f if and only if u is the critical points of the functional:

$$F(f) = \frac{1}{2} <Lu, u> - <f, u>$$

The bilinear form considered in formulating the integral equation in the variational formulation is given by, [Burden, 1985], [Stroud, 1966], [Bunday, 1984]:

$$<u, v> = \int_0^T u(t)\,v(t)\,dt\,, \quad T > 0$$

3. The Main Results:

3.1 Variable Delay Volterra Integral Equations :

The variational formulation related to the variable delay Volterra integral equation, is as follows:

Consider the following variable delay Volterra integral equation:

$$f(t) = g(t) + \int_a^x K(t,s) f(s - d(s, h(s)))\,ds \ldots (1)$$

and define the linear operator L by:

$$L = I - \int_a^t K(t,s) D\,ds$$

where I stands for the identity operator and D is defined by:

$$Df(t) = f(t - d(t, h(t)))$$

L is the linear operator related to equation (1), since it is satisfy Lf = g, i.e.,

$$\left(I - \int_a^t K(t,s) D\,ds \right) f(t) = g(t)$$

or equivalently:

$$f(t)-\int_a^x K(x,s)f(s-d(s,h(s)))ds = g(t)$$

which is the same considered variable delay Volterra integral equation.

3.2 Linearity :

It easily shown that L is a linear operator, since for all $u_1(t)$, $u_2(t)$ $\in U$, $\alpha_1, \alpha_2 \in R$, we have:

$L(\alpha_1 u_1 + \alpha_2 u_2) =$

$$\left(I-\int_a^t k(t,s)\,D\,ds\right)(\alpha_1 u_1 + \alpha_2 u_2)$$

$$= \alpha_1 u_1 + \alpha_2 u_2 -$$

$$\int_a^t K(t,s)\,D\,(\alpha_1 u_1 + \alpha_2 u_2)\,ds$$

$$= \alpha_1 u_1 + \alpha_2 u_2 -$$

$$\alpha_1 \int_a^t K(t,s)\,Du_1(s)ds -$$

$$\alpha_2 \int_a^t K(t,s)Du_2(s)ds$$

$$= \alpha_1\left(u_1(t)-\int_a^t K(t,s)\,u_1(s-d(s,h(s)))ds\right)+$$

$$\alpha_2\left(u_2(t)-\int_a^t K(t,s)\,u_2(s-d(s,h(s)))ds\right)$$

$$= \alpha_1\left(u_1(t)-\int_a^t K(t,s)\,Du_1(s)ds\right)+$$

$$\alpha_2\left(u_2(t)-\int_a^t K(t,s)\,Du_2(s)ds\right)$$

$$= \alpha_1\left(I-\int_a^t K(t,s)Dds\right)u_1(t)+$$

$$\alpha_2\left(I-\int_a^t K(t,s)Dds\right)u_2(t)$$

$$= \alpha_1 L(u_1) + \alpha_2 L(u_2)$$

3.3 Symmetry:

the linear operator L is symmetric with respect to the choosen bilinear form $(u, v) = \langle u, Lv\rangle$, since:

$$(Lu_1, u_2) = \langle Lu_1, Lu_2\rangle =$$

$$\int_0^T Lu_1(t)Lu_2(t)dt$$

$$= \int_0^T\left[\left(I-\int_a^t K(t,s)Dds\right)u_1(t)\right]$$

$$\left[\left(I-\int_a^t K(t,s)Dds\right)u_2(t)\right]dt$$

$$= \int_0^T\left[u_1(t)-\int_a^t K(t,s)u_1(s-d(s,h(s)))ds\right]\Bigg]$$

$$u_2(t)-\int_a^t K(t,s)u_2(s-d(s,h(s)))ds\Bigg]dt$$

$$= \int_0^T\Bigg[u_2(t)-\int_a^t K(t,s)u_2(s-d(s,h(s)))ds\Bigg]\Bigg]$$

$$\left. u_1(t) - \int_a^t K(t,s)u_1(s-d(s,h(s)))ds \right] dt$$

$$= \int_0^T \left[\left(u_2(t) - \int_a^t K(t,s)u_2(s)ds \right) \left(u_1(t) - \int_a^t K(t,s)u_1(s)ds \right) \right] dt$$

$$= \int_0^T \left[\left(I - \int_a^t K(t,s)D\,ds \right) u_2(t) \right] \left[\left(I - \int_a^t K(t,s)D\,ds \right) u_1(t) \right] dt$$

$$= \int_0^T Lu_2(t)Lu_1(t)\,dt$$

$= <Lu_2, Lu_1> = (Lu_2, Lu_1)$

3.4 Variational Formulation of variable Delay Volterra Integral Equations:

Since the linear operator L is symmetric with respect to the choosen bilinear form (u, v) = <u, Lv> and using theorem (1), the solution of equation (1) is the critical points of the functional:

$$F(f) = \frac{1}{2}\langle Lf, Lf \rangle - \langle g, Lf \rangle$$

$$= \frac{1}{2}\int_0^T \left[I - \int_a^t K(t,s)D\,ds \right] f(t) \left[I - \int_a^t K(t,s)D\,ds \right] f(t)\,dt$$

$$- \int_0^T g(t)\left(I - \int_a^t K(t,s)D\,ds \right) f(t)\,dt$$

$$= \int_0^T \left[\frac{1}{2}\left[f(t) - \int_a^t K(t,s)Df(s)ds \right]^2 - \left[g(t)\left(f(t) - \int_a^t K(t,s)Df(s)ds \right) \right] \right] dt$$

$$= \int_0^T \left[\frac{1}{2}\left[f(t) - \int_a^t K(t,s)f(s,h(s))ds \right]^2 - \left[g(t)\left(f(t) - \int_a^t K(t,s)f(s - d(s,h(s)))ds \right) \right] \right] dt \quad ...(2)$$

The functional F(f) is the variational formulation of the variable delay Volterra integral equation of the second kind.

Also, there exists an equivalence between the solution of delay ordinary differential equations and delay integral equations.[Corduneanu,1991], [Bushra,2005]

This is explained in the following theorem:

Theorem (2) [Earl, 1961]:

A function ϕ is a solution to the initial value problem:

$y' = f(t, y(t, h(t)), y(t_0) = y_0$, on the interval I

if and only if it is a solution to the delay integral equation

$$y = y_0 + \int_0^t f(s, y(s-h(s)))ds$$

The illustration of the above criteria will be given in the next example:

4. Illustrative Example:

Consider the following delay Volterra integral equation:

$$y(t) = t - \int_0^t \left[y\left(e^{1-\frac{1}{s}} \right) \right] ds$$

which is a non-homogeneous variable delay Volterra integral equation of the second kind with non-constant delay.

The analytical solution for this problem is given by $y(t) = \ln(t)$. Now, the linear operator is taken to be:

$$L = I - \int_{t_0}^t K(t,s) D\, ds$$

Where I is the identity operator and D is defined by: $Dy(t) = y(t - d(t, h(t))$

Then equation (2) leads to:

$F(y) =$

$$\int_0^T \left[\frac{1}{2}\left[y(t) - \int_{t_0}^t K(t,s) y(s - d(s,h(s)))ds \right]^2 - \left[g(t) \left(y(t) - \int_{t_0}^t K(t,s) y(s - d(s,h(s)))ds \right) \right] \right] dt$$

$$= \int_0^1 \left[\frac{1}{2}\left[y(t) - \int_0^t y(s - d(s,h(s)))ds \right]^2 - \left[t \left(y(t) - \int_0^t y(s-d(s,h(s)))ds \right) \right] \right] dt \quad \ldots(3)$$

hence, letting:

$y(t) = a_0 + a_1 t + a_2 t^2)$ ……………..(4)

Therefore:

$$y\left(e^{1-\frac{1}{t}} \right) = a_0 + a_1 \left(e^{1-\frac{1}{t}} \right) + a_2 \left(e^{1-\frac{1}{t}} \right)^2 \quad \ldots\ldots(5)$$

and substituting equations (4), (5) back into the functional (3), yields:

$F(y) =$

$$\int_0^1 \left[\frac{1}{2}\left[a_0 + a_1 t + a_2 t^2 - \int_0^t y\left(e^{1-\frac{1}{s}} \right) ds \right]^2 - \left[t \left(a_0 + a_1 t + a_2 t^2 - \int_0^t y\left(e^{1-\frac{1}{s}} \right) ds \right) \right] \right] dt$$

$$= \int_0^1 \left[\frac{1}{2} \left[a_0 + a_1 t + a_2 t^2 - \int_0^t \left\{ a_0 + a_1 \left(e^{1-\frac{1}{s}} \right) + a_2 \left(e^{1-\frac{1}{s}} \right)^2 \right\} ds \right]^2 - \left[t \left(a_0 + a_1 t + a_2 t^2 - \int_0^t \left\{ a_0 + a_1 \left(e^{1-\frac{1}{s}} \right) + a_2 \left(e^{1-\frac{1}{s}} \right)^2 \right\} ds \right) \right] \right] dt \quad \ldots(6)$$

and hence the problem is reduced now to minimize the functional (6), [Bundy, 1984]. The results obtained using computer programs are given by:

$a_0 = 0.015$, $a_1 = 0.33$, $a_2 = 0.403$.

and hence the solution is given by:

$$y(t) = 0.015 + 0.33t + 0.403t^2$$

Remark:

The numerical method used in integration is the Gaussian quadratur integration method of degree 7, [Stroud, 1966], [Burden, 1885].

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حل المعادلات التكاملية التباطؤية باستخدام طرق التغاير

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المستخلص:

الهدف الرئيسي من هذا البحث هو أستخدام طرق حسبان التغاير لحل المعادلات التكاملية التباطؤية متى ماكان التباطؤ هو دالة للزمن t. حيث أن المعادلة التكاملية التباطؤية المستخدمة في هذا البحث هي من نوع فولتيرا.