2024, 21(3): 1044-1054
https://dx.doi.org/10.21123/bsj.2023.7310
P-ISSN: 2078-8665 - E-ISSN: 2411-7986

Baghdad Science Journal

Using the Elzaki decomposition method to solve nonlinear fractional differential equations with the Caputo-Fabrizio fractional operator

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Received 09/04/2022, Revised 24/02/2023, Accepted 26/02/2023, Published Online First 20/08/2023, Published 01/03/2024

Abstract

The techniques of fractional calculus are applied successfully in many branches of science and engineering, one of the techniques is the Elzaki Adomian decomposition method (EADM), which researchers did not study with the fractional derivative of Caputo Fabrizio. This work aims to study the Elzaki Adomian decomposition method (EADM) to solve fractional differential equations with the Caputo-Fabrizio derivative. We presented the algorithm of this method with the CF operator and discussed its convergence by using the method of the Cauchy series then, the method has applied to solve Burger, heat-like, and, couped Burger equations with the Caputo -Fabrizio operator. To conclude the method was convergent and effective for solving this type of fractional differential equations.

Keywords: Burger equation, Caputo-Fabrizio fractional operator, Ezaki decomposition method, Heat-like equation.

Introduction

Fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has grown in popularity and relevance over the last three decades, owing to its proven applications in a wide range of apparently disparate domains of science and engineering. It does, in fact, give some potentially valuable methods for solving differential and integral equations, as well as a variety of other problems requiring mathematical physics special functions, as well as their extensions and generalizations in one or more variables [1-4].

In the past decade, Caputo and Fabrizio introduced a new fractional differential operator many researchers studied this operator and researchers are still interested in this operator because of its importance, as some studies have applied methods of approximate solutions to equations that include this fractional operator [5,6]. The requirement for a model that represents the behavior of classic viscoelastic materials, thermal media, electromagnetic systems, and so on has piqued the attention of researchers in this unique technique. Plasticity, fatigue, and damage, as well as electromagnetic hysteresis, appear to benefit most from the original concept of fractional derivative. When these effects are missing, it appears that the new fractional derivative is more appropriate [7].

Despite the importance of integrative transformations in solving differential equations, most studies have focused on some of those transformations and neglected other transformations. Among these transformations is the Elzaki transformation presented by researcher Tariq Elzaki in recent years. Therefore, in this study, The Elzaki decomposition method will be presented for solving differential equations that include the fractional operator Caputo-Fabrizio [8-11].

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Fractional calculus preliminaries

Definition 1: [12] The fractional derivative with the Caputo-Fabrizio operator for $0 < a \le 1$ is defined as:

$$ {}^{\mathcal{CF}}\mathcal{D}_t^a u(t) = \frac{\varepsilon(a)}{1-a}\int_0^t \exp\left[-\frac{a(t-s)}{1-a}\right] u'(s)\mathrm{d}s, \qquad 1 $$

where $u \in \mathcal{H}^1(z_1, z_2), z_1 < z_2$, $u'(s)$ is the derivative of u, and $\varepsilon(a)$ is a normalization function such that $\varepsilon(0)=\varepsilon(1)=1$.

The operator's fundamental attributes are as follows [13,14]

1. ${}^{\mathcal{CF}}\mathcal{D}_t^a u(t) = u(t)$, where $a = 0$.
2. ${}^{\mathcal{CF}}\mathcal{D}_t^a [u(t) + v(t)] = {}^{\mathcal{CF}}\mathcal{D}_t^a u(t) + {}^{\mathcal{CF}}\mathcal{D}_t^a v(t)$, where $u, v \in \mathcal{H}^1(z_1, z_2)$
3. ${}^{\mathcal{CF}}\mathcal{D}_t^a (c) = 0$, c is constant.

Definition 2: [15] Over a set of functions $\mathcal{A}$, the Elzaki transform is defined,

$$ \mathcal{A} = \left\{ u(t) \middle| \begin{array}{c} \exists \mathcal{M}, \tau_1, \tau_2 > 0, |u(t)| < \mathcal{M} e^{\frac{|t|}{\tau_j}}, \\ if\ t \in (-1)^j \times [0, \infty) \end{array} \right\}, $$

by the following formula

$$ E\{u(t)\} = w \int_0^\infty u(t) e^{-\frac{t}{w}} dt, \quad w \in (\tau_1, \tau_2), \qquad 2 $$

where w is a parameter of Elzaki transform.

Elzaki transform for some functions [15-19]:

1. $E\{1\} = w^2$.
2. $E\{t\} = w^3$.
3. $E\{t^n\} = n!\, w^{n+2}$.
4. $E\{e^{at}\} = \frac{w^2}{1-aw}$.
5. $E\{\sin(at)\} = \frac{aw^3}{1+a^2w^2}$.
6. $E\{\cos(at)\} = \frac{w^2}{1+a^2w^2}$.

Lemma 1: The Elzaki transform for the Caputo-Fabrizio fractional operator is defined as follows if $0 < a \le 1$.

$$ E\{{}^{\mathcal{CF}}D_t^a u(t)\} = \frac{[E\{u(t)\} - w^2 u(0)]}{1-a+aw}, \qquad 3 $$

Proof:

$$ \begin{aligned} &E\{{}^{\mathcal{CF}}\mathcal{D}_t^a u(t)\} \\ &= E\left\{\frac{1}{1-a}\int_0^t u'(s)\ exp\left[-\frac{a(t-s)}{1-a}\right] ds \right\}, \\ &= \frac{1}{1-a} E\left\{\int_0^t u'(s)\ exp\left[-\frac{a(t-s)}{1-a}\right] ds \right\}, \\ &= \frac{w^{-1}}{1-a} E\{u'(t)\} E\left\{exp\left[-\frac{at}{1-a}\right]\right\}, \\ &= \frac{w^{-1}}{1-a}\left[\frac{1}{w}E\{u(t)\} - w\, u(0)\right]\left[\frac{w^2(1-a)}{1-a+aw}\right], \\ &= \frac{[E\{u(t)\} - w^2 u(0)]}{1-a+aw}. \end{aligned} $$

Analysis of EDM

In the Caputo-Fabrizio operator sense, consider the following nonlinear partial differential equation:

$$ {}_0^{\mathcal{CF}}\mathcal{D}_t^a u(x,t) + \mathcal{R}[u] + \mathcal{N}[u] = g(x,t), \qquad 4 $$

with initial condition $\quad u(x, 0) = u_0(x), \qquad 5$

where ${}_0^{\mathcal{CF}}\mathcal{D}_t^a u(x,t)$ is Caputo-Fabrizio operator of $u(x,t)$, a linear operator is $\mathcal{R}$, a nonlinear operator is $\mathcal{N}$ and a source term is g.

Applying the Elzaki transform to both sides of the Eq.4,

$$ \mathrm{E}\{{}_0^{\mathcal{CF}}\mathcal{D}_t^a u(x,t) + \mathcal{R}[u] + \mathcal{N}[u]\} = \mathrm{E}\{g\}, \qquad 6 $$

from Lemma 1 and Eq.5,

$$ \frac{[E\{u(t)\} - w^2 u(0)]}{1-a+aw} = \mathrm{E}\{g - \mathcal{R}[u] - \mathcal{N}[u]\}, \qquad 7 $$

or

$$ \begin{aligned} E\{u\} = {} & w^2 u_0(x) + (1-a+aw)\,\mathrm{E}\{g\} \\ & -(1-a+aw)\,\mathrm{E}\{\mathcal{R}[u]\} \\ & -(1-a+aw)\,\mathrm{E}\{\mathcal{N}[u]\}, \qquad 8 \end{aligned} $$

by using the Elzaki transform's inverse to Eq.8,

$$ \begin{aligned} u(x,t) = {} & E^{-1}\{w^2 u_0(x)\} \\ & + E^{-1}\{(1-a+aw)\,\mathrm{E}\{g\}\} \\ & - E^{-1}\{(1-a+aw)\mathrm{E}\{\mathcal{R}[u]\}\} \\ & - E^{-1}\{(1-a+aw)\mathrm{E}\{\mathcal{N}[u]\}\}. \qquad 9 \end{aligned} $$

Suppose that $u(x,t)$ is a solution of Eq.9, which it expressed as

$$ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \qquad 10 $$

the nonlinear term can be decomposed as

$$\mathcal{N}[u(x,t)] = \sum_{n=0}^{\infty} \mathcal{A}_n, \qquad 11$$

where,

$$\mathcal{A}_n = \frac{1}{n!}\frac{\partial^n}{\partial \zeta^n}\left[\mathcal{N}\left(\sum_{i=0}^{n} \zeta^i u_i(x,t)\right)\right]_{\zeta=0}, \qquad 12$$

where, $n = 0,1,2, \dots$

Substituting Eq.10 and Eq.11 into Eq.9 gives us the result that

$$\begin{aligned}\sum_{n=0}^{\infty} u_n = \; & E^{-1}\{w^2 u_0(x)\} + E^{-1}\{(1-a+aw)\mathrm{E}\{g\}\} \\ & -E^{-1}\{(1-a+aw)\,\mathrm{E}\{\mathcal{R}[\textstyle\sum_{n=0}^{\infty} u_n\,]\}\} \\ & -E^{-1}\{(1-a+aw)\,\mathrm{E}\{\textstyle\sum_{n=0}^{\infty} \mathcal{A}_n\,\}\}. \qquad 13\end{aligned}$$

When the left and right sides of Eq.13 are compared,

$$\begin{aligned} u_0 &= u_0(x) + E^{-1}\{(1-a+aw)\mathrm{E}\{g\}\}, \\ u_1 &= -E^{-1}\{(1-a+aw)\mathrm{E}\{\mathcal{R}[u_0\,]\}\} -E^{-1}\{(1-a+aw)\,\mathrm{E}\{\mathcal{A}_0\,\}\}, \\ u_2 &= -E^{-1}\{(1-a+aw)\,\mathrm{E}\{\mathcal{R}[u_1(x,t)\,]\}\} -E^{-1}\{(1-a+aw)\,\mathrm{E}\{\mathcal{A}_1\,\}\}. \qquad 14\end{aligned}$$

In its general form, a recursive relation is

$$\begin{aligned} u_0 &= u_0(x) + E^{-1}\{(1-a+aw)\mathrm{E}\{g\}\}, \\ u_{n+1} &= -E^{-1}\{(1-a+aw)\,\mathrm{E}\{\mathcal{R}[u_n\,]\}\} -E^{-1}\{(1-a+aw)\mathrm{E}\{\mathcal{A}_n\,\}\}). \qquad 15\end{aligned}$$

The approximate solution is given by

$$u(x,t) = u_0 + u_1 + u_2 + \cdots = \sum_{i=0}^{\infty} u_i. \qquad 16$$

Convergence analysis

This section discusses the convergence of Elzaki decomposition method to the exact solution of fractional differential equations as well as the estimated error resulting from the approximate solutions.

Theorem 1: Suppose that B is a Banach space, $\sum_{i=0}^{\infty} u_i$ in Eq.16 is convergence to $\delta \in B$ if $\exists(0 \le \propto < 1)$, s.t. $\forall \eta \in \mathbb{N} \;\Rightarrow \|u_\eta\| \le \propto \|u_{\eta-1}\|$.

Proof: The sequence of partial sums is defined as $\{\delta_\eta\}_{\eta=0}^{\infty}$

$$\begin{gathered} \delta_0 = u_0 \\ \delta_1 = u_0 + u_1 \\ \delta_2 = u_0 + u_1 + u_2 \\ \vdots \\ \delta_\eta = u_0 + u_1 + \cdots + u_\eta, \end{gathered}$$

now, it is necessary to prove that, $\{\delta_\eta\}_{\eta=0}^{\infty}$ is a Cauchy series in Banach space,

$$\|\delta_{\eta+1} - \delta_\eta\| = \left\|\sum_{i=0}^{\eta+1} u_i - \sum_{i=0}^{\eta} u_i\right\| = \|u_{\eta+1}\| \le \propto \|u_\eta\| \le \cdots \le \propto^{\eta+1} \|u_0\|, \qquad 17$$

for all $\eta, m \in \mathbb{N}$ $as\ \eta \ge m$,

$$\begin{aligned} \|\delta_\eta - \delta_m\| &= \|(\delta_\eta - \delta_{\eta-1}) + (\delta_{\eta-1} - \delta_{\eta-2}) + \cdots + (\delta_{m+1} - \delta_m)\| \\ &\le \|\delta_\eta - \delta_{\eta-1}\| + \|\delta_{\eta-1} - \delta_{\eta-2}\| + \cdots + \|\delta_{m+1} - \delta_m\| \\ &\le \propto^{\eta} \|u_0\| + \propto^{\eta-1} \|u_0\| + \cdots + \propto^{m+1} \|u_0\| \\ &= \propto^{m+1} \|u_0\|(\propto^{\eta-m-1} + \propto^{\eta-m-2} + \cdots + \propto) \\ &= \frac{1-\propto^{\eta-m}}{1-\propto} \propto^{m+1} \|u_0\|. \qquad 18 \end{aligned}$$

Since $(\propto^{\eta-m-1} + \propto^{\eta-m-2} + \cdots + \propto)$ is a geometric series and $0 \le \propto < 1$, then,

$$\lim_{\eta,m\to\infty} \|\delta_\eta - \delta_m\| = 0,$$

thus, $\{\delta_\eta\}_{\eta=0}^{\infty}$ is Cauchy sequence in Banach space B, therefore produces that the series solution $u = \sum_{i=0}^{\infty} u_i$, defined in Eq.16 converges.

Theorem 2: Suppose that the series solution $\sum_{i=0}^{\infty} u_i$ in Eq.16 is convergent to the solution $u(x,t)$ .If $\sum_{i=0}^{\infty} u_i$ is used as an approximation to the solution $u(x,t)$ of Eq.5 then the maximum error, $E_m(x,t)$ is estimated as

$$E_m(x,t) \le \frac{1}{1-\propto} \propto^{m+1} \|u_0\|.$$

Proof: From Theorem 1, inequality 18

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$$\left\|\delta_\eta - \delta_m\right\| \leq \frac{1-\propto^{\eta-m}}{1-\propto} \propto^{m+1} \|u_0\|,$$

for $\eta \geq m$, now, as $\eta \to \infty$ then $\delta_\eta \to u(x,t)$ so,

$$\|u(x,t) - \sum_{i=0}^{\infty} u_i\| \leq \frac{1-\propto^{\eta-m}}{1-\propto} \propto^{m+1} \|u_0\|. \quad 19$$

Also, since $0 \leq \propto < 1$ produces $(1-\propto^{\eta-m}) < 1$.

Therefor the above inequality 19 becomes

$$E_m(x,t) \leq \frac{1}{1-\propto} \propto^{m+1} \|u_0\|. \quad 20$$

Illustrative examples

Example 1: Consider the following, Burger's equation in the Caputo-Fabrizio sense is nonlinear.

$${}_{0}^{\mathcal{CF}}\mathcal{D}_t^a\, u + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \quad , \quad 0 < a \leq 1, \quad 21$$

subject to the initial condition $u(x,0) = x$. 22

By taking the Elzaki transform to both sides of Eq.21,

$$\mathrm{E}\left\{{}_{0}^{\mathcal{CF}}\mathcal{D}_t^a\, u\right\} = w^2 x + \mathrm{E}\left\{\frac{\partial^2 u}{\partial x^2}\right\} - \mathrm{E}\left\{u\frac{\partial u}{\partial x}\right\}, \quad 23$$

using the inverse Elzaki transform to both sides of Eq.23,

$$u = x + \mathrm{E}^{-1}\left\{(1-a+aw)\begin{bmatrix} \mathrm{E}\left\{\frac{\partial^2 \sum_{\mathrm{n}=0}^{\infty} u_{\mathrm{n}}}{\partial x^2}\right\} \\ -\mathrm{E}\{\sum_{\mathrm{n}=0}^{\infty} \mathcal{A}_{\mathrm{n}}\}\end{bmatrix}\right\}. \quad 24$$

As a result, the approximate solution may be derived using Eq.15

$$u_0(x,t) = x,$$

$$u_1(x,t) = \mathrm{E}^{-1}\left\{(1-a+aw)\begin{bmatrix} \mathrm{E}\{\mathcal{A}_0\} \\ -\mathrm{E}\left\{\frac{\partial^2 u_0}{\partial x^2}\right\}\end{bmatrix}\right\},$$

$$u_2(x,t) = \mathrm{E}^{-1}\left\{(1-a+aw)\begin{bmatrix} \mathrm{E}\{\mathcal{A}_1\} \\ -\mathrm{E}\left\{\frac{\partial^2 u_1}{\partial x^2}\right\}\end{bmatrix}\right\},$$

by the above algorithms and after simple steps,

$$u_0 = x,$$
$$u_1 = -x(1-a+at),$$
$$u_2 = x\begin{pmatrix} 2a^2 - 4a + a^2 t^2 \\ -4a^2 t + 4at + 2\end{pmatrix}, \quad 25$$
$$\vdots$$

and so on.

Therefore, the series solution $u(x,t)$ of Eq.21 is given by

$$u(x,t) = x - x(1-a+at) + x\begin{pmatrix} 2a^2 - 4a + a^2 t^2 \\ -4a^2 t + 4at + 2\end{pmatrix} - \cdots. \quad 26$$

If it was $a \to 1$ in Eq.26 then, the exact solution is

$$u(x,t) = x - xt + xt^2 - \cdots, = x\sum_{\mathrm{k}=0}^{\infty}(-t)^{\mathrm{k}} = \frac{x}{1+t}. \quad 27$$

Table 1 displays the values of the exact and approximate solutions of Eq.21 at the different a values, also includes the absolute error of the approximate solutions for the exact solutions. Fig. 1 shows graphs of the exact and approximate solutions of Eq.21 at the different a values between 0 and 1.

Table 1. The approximate and exact solution of Burger's equation with fractional operator CF, u_1 is app. sol. of u at $a = 0.9$, u_2 is app. sol. of u at $a = 1$, u_1 is exact sol. of u .

x	t	u_1	u_2	u_3	$\lvert u_1 - u_3\rvert$	$\lvert u_2 - u_3\rvert$
0.2500	0.2500	0.2089	0.2031	0.2000	0.0089	0.0031
0.7500	0.2500	0.6267	0.6094	0.6000	0.0267	0.0094
0.2500	0.5000	0.2131	0.1875	0.1667	0.0465	0.0208
0.5000	0.5000	0.4262	0.3750	0.3333	0.0929	0.0417
0.7500	0.5000	0.6394	0.5625	0.5000	0.1394	0.0625
0.2500	0.7500	0.2427	0.2031	0.1429	0.0998	0.0603
0.5000	0.7500	0.4853	0.4062	0.2857	0.1996	0.1205
0.7500	0.7500	0.7280	0.6094	0.4286	0.2994	0.1808

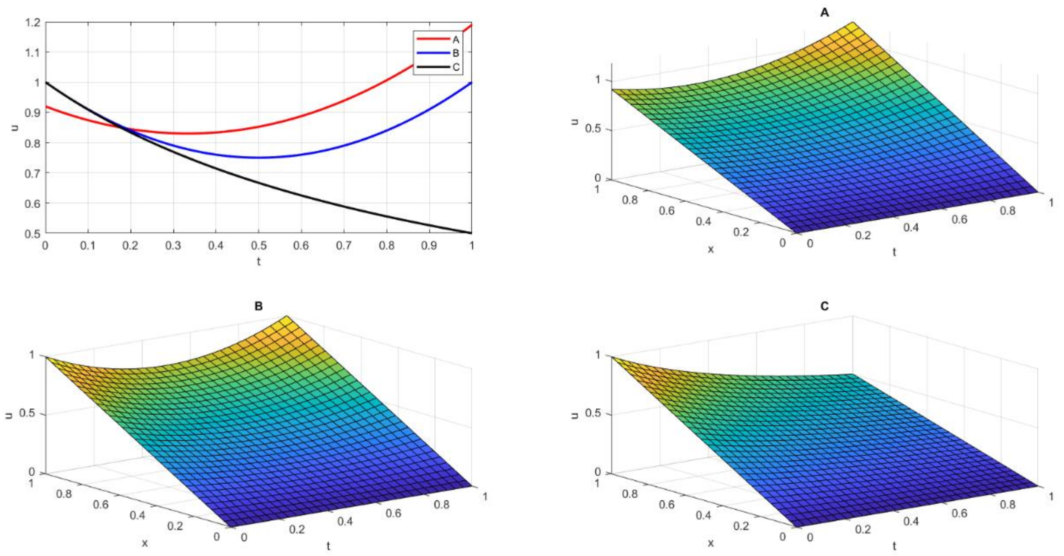

Figure 1. The graphs of the approximate and the exact solutions among different values of x and t in case (3D) and fixed x in case (2D) when a = 0.9,1 for nonlinear Burger's equation in the CF fractional operator,(A) is the app. sol. of u at $a = 0.9$, (B) is the app. sol. of u at $a = 1$, (C) is the exact sol. of u.

Example 2: Consider the following heat-like equation in the Caputo-Fabrizio sense

$$ {}^{\mathcal{CF}}_{0}\mathcal{D}_t^{a}\, u(x, y, t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad 28$$

where $0 \le x, y \le 2\pi\,, t > 0$, with the initial condition

$$u(x, y, 0) = \sin(x)\sin(y), \qquad 29$$

applying the Elzaki transform differentiation property on Eq.28

$$\mathrm{E}\{{}^{\mathcal{CF}}_{0}\mathcal{D}_t^{a}\, u\} = w^2 \sin(x)\sin(y) + (1 - a + aw)\begin{bmatrix} \mathrm{E}\left\{\frac{\partial^2 u}{\partial x^2}\right\} \\ +\mathrm{E}\left\{\frac{\partial^2 u}{\partial y^2}\right\}\end{bmatrix}, \qquad 30$$

by using the inverse Elzaki transform to both sides of Eq.30,

$$u = \sin(x)\sin(y) + \mathrm{E}^{-1}\left\{(1 - a + aw)\begin{bmatrix} \mathrm{E}\left\{\frac{\partial^2 u}{\partial x^2}\right\} \\ +\mathrm{E}\left\{\frac{\partial^2 u}{\partial y^2}\right\}\end{bmatrix}\right\}, \qquad 31$$

as a result, the approximate solution may be derived using Eq.15,

$u_0 = \sin(x)\sin(y)$,

$$u_n = \mathrm{E}^{-1}\left\{(1 - a + aw)\begin{bmatrix} \mathrm{E}\left\{\frac{\partial^2 u_{n-1}}{\partial x^2}\right\} \\ +\mathrm{E}\left\{\frac{\partial^2 u_{n-1}}{\partial y^2}\right\}\end{bmatrix}\right\}. \qquad 32$$

Hence, from Eq 29 and Eq 32, the components give as follows:

$u_0 = \sin(x)\sin(y)$,

$$u_1 = \mathrm{E}^{-1}\left\{(1 - a + aw)\begin{bmatrix} \mathrm{E}\left\{\frac{\partial^2 u_0}{\partial x^2}\right\} \\ +\mathrm{E}\left\{\frac{\partial^2 u_0}{\partial y^2}\right\}\end{bmatrix}\right\}, = -2\sin(x)\sin(y)\,(1 - a + at)$$

$$u_2 = \mathrm{E}^{-1}\left\{(1 - a + aw)\begin{bmatrix} \mathrm{E}\left\{\frac{\partial^2 u_0}{\partial x^2}\right\} \\ +\mathrm{E}\left\{\frac{\partial^2 u_0}{\partial y^2}\right\}\end{bmatrix}\right\}, = 4\sin(x)\sin(y)\left[(1 - 2a + a^2) + (2a - 2a^2)\mathrm{t} + \frac{1}{2}a^2t^2\right]. \qquad 33$$

and so on.

Therefore, the approximate solution of $u(x, y, t)$ of Eq.28 is given by

$$u = \sin(x)\sin(y) - 2\sin(x)\sin(y)\,(1 - a + at), + 4\sin(x)\sin(y)\begin{bmatrix} (1 - 2a + a^2) \\ +(2a - 2a^2)\mathrm{t} + \frac{1}{2}a^2t^2\end{bmatrix} + \cdots. \qquad 34$$

If it was $a \to 1$ in Eq.34 then, the exact solution is,

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https://dx.doi.org/10.21123/bsj.2023.7310
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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$$u = \sin(x)\sin(y)\left(1 - 2t + \frac{(2t)^2}{2!} - \cdots,\right.$$
$$= \sin(x)\sin(y)\, e^{-2t}. \qquad 35$$

Table 2 displays the values of the exact and approximate solutions of Eq.28 at the different a values, also includes the absolute error of the approximate solutions for the exact solutions. Fig. 2 shows graphs of the exact and approximate solutions of Eq.28 at the different a values between 0 and 1.

Table 2. The approximate and exact solution of heat-like equation with fractional operator CF, u_1 is app. sol. of u at $a = 0.9$, u_2 is app. sol. of u at $a = 1$, u_1 is exact sol. of u .

x	y	t	u_1	u_2	u_3	$\|u_1 - u_3\|$	$\|u_2 - u_3\|$
0.2500	0.2500	0.2500	0.0411	0.0383	0.0371	0.0040	0.0011
0.5000	0.2500	0.2500	0.0796	0.0741	0.0719	0.0077	0.0022
0.7500	0.2500	0.2500	0.1132	0.1054	0.1023	0.0109	0.0031
0.2500	0.5000	0.2500	0.0796	0.0741	0.0719	0.0077	0.0022
0.5000	0.5000	0.2500	0.1543	0.1437	0.1394	0.0149	0.0042
0.7500	0.5000	0.2500	0.2194	0.2042	0.1982	0.0211	0.0060
0.2500	0.7500	0.2500	0.1132	0.1054	0.1023	0.0109	0.0031
0.5000	0.7500	0.2500	0.2194	0.2042	0.1982	0.0211	0.0060
0.7500	0.7500	0.2500	0.3119	0.2904	0.2818	0.0301	0.0086
0.2500	0.2500	0.5000	0.0432	0.0306	0.0225	0.0206	0.0081
0.5000	0.2500	0.5000	0.0836	0.0593	0.0436	0.0400	0.0157
0.7500	0.2500	0.5000	0.1189	0.0843	0.0620	0.0569	0.0223
0.2500	0.5000	0.5000	0.0836	0.0593	0.0436	0.0400	0.0157
0.5000	0.5000	0.5000	0.1620	0.1149	0.0846	0.0775	0.0304
0.7500	0.5000	0.5000	0.2304	0.1634	0.1202	0.1102	0.0432
0.2500	0.7500	0.5000	0.1189	0.0843	0.0620	0.0569	0.0223
0.5000	0.7500	0.5000	0.2304	0.1634	0.1202	0.1102	0.0432
0.7500	0.7500	0.5000	0.3276	0.2323	0.1709	0.1566	0.0614
0.2500	0.2500	0.7500	0.0576	0.0383	0.0137	0.0440	0.0246
0.5000	0.2500	0.7500	0.1116	0.0741	0.0265	0.0852	0.0477
0.7500	0.2500	0.7500	0.1587	0.1054	0.0376	0.1211	0.0678
0.2500	0.5000	0.7500	0.1116	0.0741	0.0265	0.0852	0.0477
0.5000	0.5000	0.7500	0.2163	0.1437	0.0513	0.1651	0.0924
0.7500	0.5000	0.7500	0.3076	0.2042	0.0729	0.2347	0.1313
0.2500	0.7500	0.7500	0.1587	0.1054	0.0376	0.1211	0.0678
0.5000	0.7500	0.7500	0.3076	0.2042	0.0729	0.2347	0.1313
0.7500	0.7500	0.7500	0.4373	0.2904	0.1037	0.3337	0.1867

https://dx.doi.org/10.21123/bsj.2023.7310
P-ISSN: 2078-8665 - E-ISSN: 2411-7986

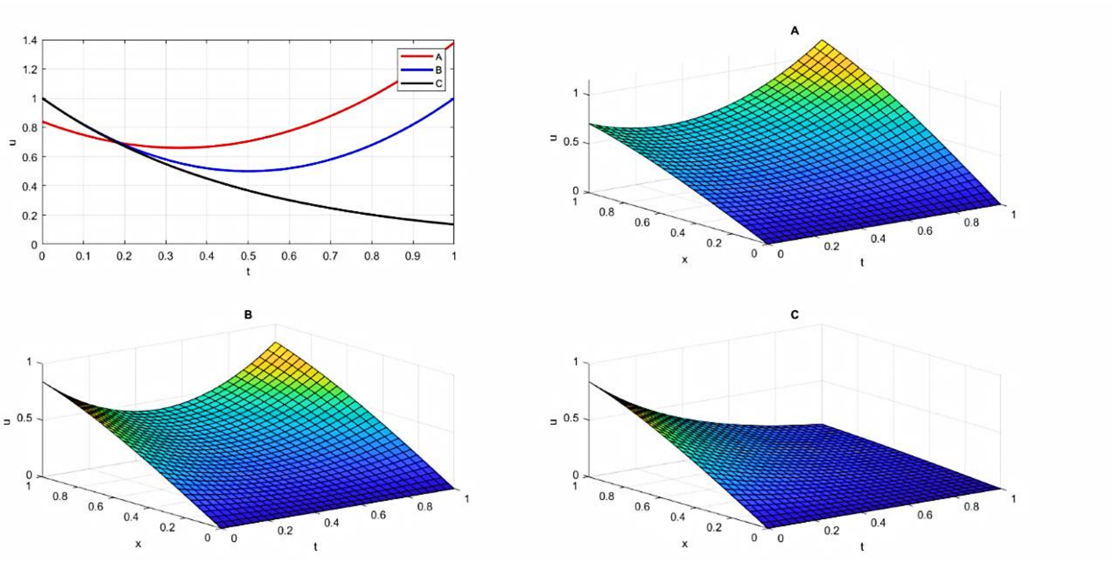

Figure 2. The graphs of the approximate and the exact solutions of heat-like equation among different values of x and t when y is fixed in case (3D) and x, y are fixed in case (2D) when a = 0.9,1 for nonlinear Burger's equation in the CF fractional operator ,(A) is the app. sol. of u at $a = 0.9$,(B) is the app. sol. of u at $a = 1$, (C) is the exact sol. of u.

Example 3: Consider the Caputo-Fabrizio operator's nonlinear system of time-fractional differential equations:

$$ {}^{CF}_{0}\mathcal{D}_t^{a} u(x,t) - u_{xx} - 2uu_x + (uv)_x = 0, $$

$$ {}^{CF}_{0}\mathcal{D}_t^{a} v(x,t) - v_{xx} - 2vv_x + (uv)_x = 0, \qquad 36 $$

where $0 < a, b \leq 1$ and the initial conditions are

$$ u(x,0) = \sin(x), $$
$$ v(x,0) = \sin(x). \qquad 37 $$

Taking the Elzaki transform on both sides of Eq.36,

$$ \mathrm{E}\{{}^{CF}_{0}\mathcal{D}_t^{a} u(x,t)\} = w^2 \sin(x) + (1 - a + aw)\mathrm{E}\left\{\frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(uv)\right\}, $$

$$ \mathrm{E}\{{}^{CF}_{0}\mathcal{D}_t^{b} v(x,t)\} = w^2 \sin(x) + (1 - a + aw)\mathrm{E}\left\{\frac{\partial^2 v}{\partial x^2} + 2v\frac{\partial v}{\partial x} - \frac{\partial}{\partial x}(uv)\right\}, \qquad 38 $$

when taking the inverse of the Elzaki transform to both sides of Eq.38,

$$ u(x,t) = \sin(x) + \mathrm{E}^{-1}\left\{(1 - a + aw)\mathrm{E}\left\{\begin{matrix}\frac{\partial^2 u}{\partial x^2}\\ +2u\frac{\partial u}{\partial x}\\ -\frac{\partial}{\partial x}(uv)\end{matrix}\right\}\right\}, $$

$$ v(x,t) = \sin(x) + \mathrm{E}^{-1}\left\{(1 - b + bw)\mathrm{E}\left\{\begin{matrix}\frac{\partial^2 v}{\partial x^2}\\ +2v\frac{\partial v}{\partial x}\\ -\frac{\partial}{\partial x}(uv)\end{matrix}\right\}\right\}, \qquad 39 $$

the solution is now represented as an infinite series as seen below

$$ u(x,t) = \sum_{\mathrm{n}=0}^{\infty} u_{\mathrm{n}}(x,t), $$
$$ v(x,t) = \sum_{\mathrm{n}=0}^{\infty} v_{\mathrm{n}}(x,t), \qquad 40 $$

and the nonlinear terms may be broken down into

$$ uu_x = \sum_{\mathrm{n}=0}^{\infty} \mathcal{A}_{\mathrm{n}}, $$
$$ vv_x = \sum_{\mathrm{n}=0}^{\infty} \mathcal{B}_{\mathrm{n}}, $$
$$ (uv)_x = \sum_{\mathrm{n}=0}^{\infty} \mathcal{C}_{\mathrm{n}}, \qquad 41 $$

where

$$ \mathcal{A}_0 = 2u_0\frac{\partial u_0}{\partial x}, $$
$$ \mathcal{A}_1 = 2u_0\frac{\partial u_1}{\partial x} + 2u_1\frac{\partial u_0}{\partial x} $$
$$ \mathcal{B}_0 = 2v_0\frac{\partial v_0}{\partial x}, $$
$$ \mathcal{B}_1 = 2v_0\frac{\partial v_1}{\partial x} + 2v_1\frac{\partial v_0}{\partial x} $$
$$ \mathcal{C}_0 = \frac{\partial}{\partial x}(u_0 v_0), $$
$$ \mathcal{C}_1 = \frac{\partial}{\partial x}(u_0 v_1 + u_1 v_0). \qquad 42 $$

Substituting Eq.41 and Eq.40 in Eq.39,

$$\sum_{N=0}^{\infty} u_n = \sin(x) + E^{-1}\left\{(1-a+aw)E\left\{\begin{matrix}\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} u_n\right)\\ +\sum_{n=0}^{\infty}\mathcal{A}_n\\ -\sum_{n=0}^{\infty}\mathcal{C}_n\end{matrix}\right\}\right\},$$

$$\sum_{n=0}^{\infty} v_n = \sin(x) + E^{-1}\left\{(1-b+bw)E\left\{\begin{matrix}\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} v_n\right)\\ +\sum_{n=0}^{\infty}\mathcal{B}_n\\ -\sum_{n=0}^{\infty}\mathcal{C}_n\end{matrix}\right\}\right\}. \quad 43$$

When both sides of the Eq.43 are compared,

$u_0 = \sin(x)$,

$v_0 = \sin(x)$,

$$u_1 = E^{-1}\left\{(1-a+aw)E\left\{\begin{matrix}\frac{\partial^2 u_0}{\partial x^2}\\ +\mathcal{A}_0\\ -\mathcal{C}_0\end{matrix}\right\}\right\},$$

$$= -\sin(x)\ (1-a+at),$$

$$v_1 = E^{-1}\left\{(1-b+bw)E\left\{\begin{matrix}\frac{\partial^2 v_0}{\partial x^2}\\ +\mathcal{B}_0\\ -\mathcal{C}_0\end{matrix}\right\}\right\},$$

$$= -\sin(x)\ (1-b+bt),$$

$$u_2 = E^{-1}\left\{(1-a+aw)E\left\{\frac{\partial^2 u_0}{\partial x^2}+\mathcal{A}_1-\mathcal{C}_1\right\}\right\},$$

$$= \sin(x)\left[(1-a)^2+(a-2a^2)t+\frac{1}{2}a^2t^2 + 2(a-\beta)\cos x\ \left[(1-a)+(2a-1)t - \frac{1}{2}a^2t^2\right]\right],$$

$$v_2 = E^{-1}\left\{(1-b+bw)E\left\{\begin{matrix}\frac{\partial^2 v_0}{\partial x^2}\\ +\mathcal{B}_1\\ -\mathcal{C}_1\end{matrix}\right\}\right\},$$

$$= \sin(x)\left[(1-b)^2+(2b-2b^2)t+\frac{1}{2}b^2t^2 + 2(b-a)\cos(x)\ \left[(1-b)+(2b-1) - \frac{1}{2}b^2t^2\right]\right].$$

Therefore, the approximate solution of Eq.36 is given by

$$u = \sin(x)\left[(1-a+a^2)+(a-2a^2)t+\frac{1}{2}a^2t^2 + 2(a-b)\cos(x)\left[(1-a)+(2a-1)t - \frac{1}{2}a^2t^2\right]+\cdots\right],$$

$$v = \sin(x)\left[(1-b+b^2)+(b-2b^2)t + \frac{1}{2}b^2t^2+2(b-a)\cos(x)\left[(1-b)+(2b-1)t-\frac{1}{2}b^2t^2\right]+\cdots\right]. \quad 44$$

If put $a \to 1$ and $b \to 1$ in Eq.44, the problem solution will be re-created as follows:

$$u(x,t) = \sin(x)\left(1-t+\frac{t^2}{2!}-\cdots\right),$$

$$v(x,t) = \sin(x)\left(1-t+\frac{t^2}{2!}-\cdots\right). \quad 45$$

This is the closed form equivalent of the precise solution:

$$u(x,t) = \sin(x)\,e^{-t},$$

$$v(x,t) = \sin(x)\,e^{-t}. \quad 46$$

Table 3 displays the values of the exact and approximate solutions of system 36 at the different a and b values, also includes the absolute error of the approximate solutions for the exact solutions. Fig. 3 shows graphs of the exact and approximate solutions of system 36 at the different a and b values between 0 and 1.

Table 3. The approximate and exact solution of system 36 with fractional operator CF, u_1 is app. sol. of u, v at $a = 0.9$, u_2 is app. sol. of u, v at $a = 1$, u_1 is exact sol. of u, v .

x	t	u_1	u_2	u_3	$\lvert u_1-u_3\rvert$	$\lvert u_2-u_3\rvert$
0.2500	0.2500	0.1869	0.1933	0.1927	0.0058	0.0006
0.5000	0.2500	0.3621	0.3746	0.3734	0.0113	0.0012
0.7500	0.2500	0.5149	0.5325	0.5309	0.0160	0.0017
0.2500	0.5000	0.1611	0.1546	0.1501	0.0111	0.0046
0.5000	0.5000	0.3122	0.2996	0.2908	0.0214	0.0089
0.7500	0.5000	0.4439	0.4260	0.4134	0.0305	0.0126
0.2500	0.7500	0.1479	0.1314	0.1169	0.0310	0.0146
0.5000	0.7500	0.2866	0.2547	0.2265	0.0601	0.0282
0.7500	0.7500	0.4075	0.3621	0.3220	0.0855	0.0401

2024, 21(3): 1044-1054
https://dx.doi.org/10.21123/bsj.2023.7310
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

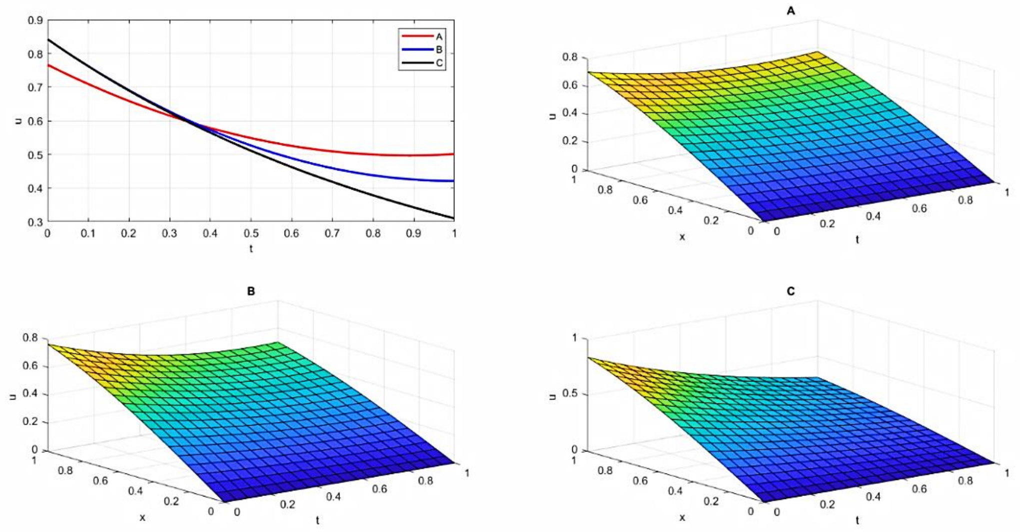

Figure 3. The graphs of the approximate and the exact solutions of nonlinear system 36 among different values of x and t in case (3D) and x is fixed in case (2D) when α = 0.9,1 for nonlinear system in the CF fractional operator, (A) is the app. sol. of u, v at $\alpha = 0.9$,(B) is the app. sol. of u, v at $\alpha = 1$, (C) is the exact sol. of u, v.

Remark. By comparing the results of this method with Yang decomposition method it appears that the results are similar to both methods [20].

Conclusion

In this article, the Elzaki decomposition method has been presented in terms of its derivation and convergence and its application to fractional differential equations (FDEs). The method was convergent and efficient to solve fractional differential equations with the Caputo-Fabrizio operator. The approximate solution of fractional differential equations (FDEs) with the derivative of Caputo-Fabrizio was convergent to the exact solution. Finally, this method can be adopted to solve fractional differential equations of this type.

Acknowledgment

We appreciate the efforts of everyone who contributed even a little to this work.

Authors' Declaration

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Furthermore, any Figures and images, that are not mine, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Thi-Qar.

References

1. Abd AL- Hussein WR, Fawzi RM. Solving Fractional Damped Burgers Equation Approximately by Using the Sumudu Transform (ST) Method. Baghdad Sci J. 2021; 18(1): 0803. https://doi.org/10.21123/bsj.2021.18.1(Suppl.).0803 .

2. Saad N, Muna S. Using Bernoulli Equation to Solve Burger's Equation. Baghdad Sci J. 2014; 11(2): 202-206. https://doi.org/10.21123/bsj.2014.11.2.202-206 .
3. Hussein M, Jassim H. Analysis of Fractional Differential Equations with Antagana-Baleanu Fractional Operator. Prog. Fract. Differ. Appl. 2023; 9(4): 1-6. http://dx.doi.org/10.18576/pfda/090411.
4. Sania Q, Rangaig A, Dumitru B. New numerical aspects of Caputo-Fabrizio fractional derivative operator. Mathematics. 2019; 7(4): 374. https://doi.org/10.3390/math7040374 .
5. Shaikh, A., Tassaddiq, A., Nisar, K.S. et al. Analysis of differential equations involving Caputo–Fabrizio fractional operator and its applications to reaction–diffusion equations. Adv Differ Equ 2019; 178 (2019). https://doi.org/10.1186/s13662-019-2115-3 .
6. Farooq M, Saqib K, Ahmad H. Flow of viscous fluid over an infinite plate with Caputo-Fabrizio derivatives. Open J Math Sci. 2019; 3(1): 115-120. https://doi.org/10.30538/oms2019.0054 .
7. Michele C, Mauro F. A new definition of fractional derivative without singular kernel. Progr Fract Differ Appl. 2015; 1(2): 73-85. http://dx.doi.org/10.12785/pfda/01020 .
8. Dineshkumar C, Nisar K S, Udhayakumar R, Vijayakumar V. A discussion on approximate controllability of Sobolev-type Hilfer neutral fractional stochastic differential inclusions. Asian J. Control. 2022; 24(5): 2378-2394. https://doi.org/10.1002/asjc.2650 .
9. Jassim H, Hussein M, and Ali M. An efficient homotopy permutation technique for solving fractional differential equations using Atangana-Baleanu-Caputo operator. AIP Conf Proc. 2023; 2845(1): 060008. https://doi.org/10.1063/5.0157148.
10. Jassim, Hassan Kamil, and Mohammed Abdulshareef Hussein. A New Approach for Solving Nonlinear Fractional Ordinary Differential Equations. Mathematics. 2023; 11(7): 1565. https://doi.org/10.3390/math11071565.
11. Varun Bose, C. S., and R. Udhayakumar. A note on the existence of Hilfer fractional differential inclusions with almost sectorial operators. Math Meth Appl Sci. 2022; 45(5):2530-2541. https://doi.org/10.1002/mma.7938 .
12. Jassim, H.K.; Hussein, M.A. A Novel Formulation of the Fractional Derivative with the Order $\alpha \geq 0$ and without the Singular Kernel. Mathematics 2022; 10(21):4123. https://doi.org/10.3390/math10214123 .
13. Hassan K, Mohammed A. On approximate solutions for fractional system of differential equations with Caputo-Fabrizio fractional operator. J Math Computer Sci. 2021; 23(1): 58-66. http://dx.doi.org/10.22436/jmcs.023.01.06 .
14. Muhammad A, Zakia H, Dumitru B. Modeling the dynamics of hepatitis E via the Caputo–Fabrizio derivative. Math Model Nat Phenom. 2019; 14(3): 311. https://doi.org/10.1051/mmnp/2018074 .
15. Tarig M. The new integral transform Elzaki Transform. Glob J Pure Appl Math. 2011; 7(1): 57-64.
16. Kalavathi A, Kohila T, Lalit M. On the degenerate Elzaki transform. Bull Pure Appl Sci Sect E Math Stat. 2021; 40: 99-107. https://doi.org/10.5958/2320-3226.2021.00010.2 .
17. Halil A. A Local Fractional Elzaki Transform Decomposition Method for the Nonlinear System of Local Fractional Partial Differential Equations. Fractal Fract. 2022; 6(3): 167. https://doi.org/10.3390/fractalfract6030167 .
18. Tarig M. Applications of new transform Elzaki transform to partial differential equations. Glob J Pure Appl Math. 2011; 7(1): 65-70.
19. Ziane D, Tarig M, Cherif M. Elzaki transform combined with variational iteration method for partial differential equations of fractional order. Fund J Math Appl. 2018; 1(1): 102-108. https://doi.org/10.33401/fujma.415892 .
20. Shah, Nehad Ali, Essam R. El-Zahar, and Jae Dong Chung. Fractional analysis of coupled burgers equations within yang caputo-fabrizio operator. J Funct Spaces. 2022; 2022. https://doi.org/10.1155/2022/6231921 .

2024, 21(3): 1044-1054
https://dx.doi.org/10.21123/bsj.2023.7310
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

استخدام طريقة تحليل الزاكي لحل المعادلات التفاضلية غير الخطية ذات الرتبة الكسرية مع المؤثر الكسري كابوتو-فابريزيو

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مركز البحث العلمي، جامعة العين، ذي قار، العراق

الخلاصة

يتم تطبيق تقنيات حساب التفاضل والتكامل الجزئي بنجاح في العديد من فروع العلوم والهندسة ، احدى هذه التقنيات طريقة تحليل الزكي وادوميان (EADM) ، والتي لم يدرسها الباحثون باستخدام المشتق الكسري لـ Caputo-Fabrizio. يهدف هذا العمل لدراسة طريقة تحليل الزكي وادوميان (EADM) لحل المعادلات التفاضلية الكسرية بمشتق Caputo-Fabrizio. قدمنا خوارزمية هذه الطريقة مع المؤثر CF وناقشنا تقاربها باستخدام طريقة سلسلة كوشي بعد ذلك ، تم تطبيق الطريقة لحل معادلات برجر، الشبية بالحرارة ، و برجر المزدوجة مع المؤثر Caputo -Fabrizio. في الختام أن الطريقة كانت متقاربة وفعالة لحل هذا النوع من المعادلات التفاضلية الكسرية.

الكلمات المفتاحية: معادلة برجر، المؤثر الكسري كابوتو-فابريزيو، طريقة تحليل الزكي، المعادلة الشبيه بالحرارة.