Effective Computational Methods for Solving the Jeffery-Hamel Flow Problem

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Abstract

In this paper, the effective computational method (ECM) based on the standard monomial polynomial has been implemented to solve the nonlinear Jeffery-Hamel flow problem. Moreover, novel effective computational methods have been developed and suggested in this study by suitable base functions, namely Chebyshev, Bernstein, Legendre, and Hermite polynomials. The utilization of the base functions converts the nonlinear problem to a nonlinear algebraic system of equations, which is then resolved using the Mathematica®12 program. The development of effective computational methods (D-ECM) has been applied to solve the nonlinear Jeffery-Hamel flow problem, then a comparison between the methods has been shown. Furthermore, the maximum error remainder ($MER_n$) has been calculated to exhibit the reliability of the suggested methods. The results persuasively prove that ECM and D-ECM are accurate, effective, and reliable in getting approximate solutions to the problem.

Keywords: Approximate solution, Bernstein polynomials, Chebyshev polynomials, Hermite polynomials, Legendre polynomials.

Introduction:

In several fields of engineering and applied sciences, nonlinear ordinary differential equations (NODE) play a significant role in simulating many real-life issues. Many phenomena, including engineering, fluid mechanics, physics, chemical matters, biology, and electrostatics, have been mathematically formulated using these types of equations. The exact solution for nonlinear problems is difficult or sometimes cannot be obtainable. Therefore authors want to develop efficient either numerical or approximate methods to solve these types of problems.1-4

Several analytical and approximate methods have been proposed by researchers to solve nonlinear differential equations, such as the Adomian decomposition method (ADM) and Direct Homotopy Analysis Method (DHAM) 5, the Bernoulli collocation method 6, the Hermite polynomial method 7, the Taylor collocation method 8, and the Gegenbauer wavelet method 9. In particular, Singh 10 has used the Jacobi collocation method to solve the fractional advection-dispersion equation. Ganji et al. 11 have used the fifth-kind Chebyshev polynomials to solve differential equations with multiple variable orders and non-local and non-singular kernels. Also, Singha et al. 12 used Boubaker polynomials to solve a class of fractional optimal control problems. Yuttanan et al. 13 solved the non-linear distributed fractional differential equations using the Legendre wavelets method and some other approximation methods, see 14-16.

One of the most important applications in fluid mechanics and biomechanical engineering is the flow between two nonparallel plates 17, Jeffery 18 and Hamel 19 introduced incompressible viscous fluid movement in convergent and divergent channels, and this is known as Jeffery-Hamel flow.

Many researchers have attempted to develop analytical approximations methods to solve the Jeffery-Hamel flow: such as optimal iterative perturbation technique 20, Bernstein collocation method (BCM) 21, modified Adomian decomposition method (MADM) 22,23, Homotopy analysis method (HAM) 24, Homotopy perturbation method (HPM) 25, Bernoulli collocation method 26, Hermite wavelet method 27, differential transform method (DTM) 28. More recently, AL-Jawary et al. 29, has implemented three semi-analytical iterative
methods namely the Daftardar-Jafari method (DJM), Temimi-Ansari method (TAM), and Banach contraction method (BCM) to obtain the solution for this problem. In addition, AL-Jawary et al. [30], has employed two operational matrices techniques (OMM) based on Bernstein and Chebyshev polynomials to solve a similar problem.

More recently, the Turkyilmazoglu has proposed an analytic approximate method namely the effective computational method (ECM), and implemented it to solve various types of problems. For example, Lane-Emden-Fowler singular nonlinear equations [31], high-order Fredholm integro-differential equations [32], high-order Volterra-Fredholm-Hammerstein integro-differential equations [33], heat transfer of fin problems [34], and initial and boundary value problems for linear differential equations of any order [35]. The approach was based on well-chosen general-type basis functions, such as classical polynomials, and that exact solution is obtained under particular conditions. A nonlinear equation’s solution is also converted into a nonlinear algebraic equations system that can be solved numerically.

Recently, orthogonal functions and polynomials have received a lot of attention from researchers since they are very useful tools and techniques in dealing with many different problems in approximation theory as well as numerical analysis [30]. On the other hand, these techniques are mainly characterized by simplifying the required solution effectively by transforming the problem into a system of algebraic equations, where it can be solved simply by using any computational program [36-39]. Accordingly, the problems are simplified substantially and the unknown function is approximated using a series of powers of polynomials. Thus, all integrals and differentials are eliminated by using the operational matrices procedure. Furthermore, the literature is full of the applications that have been discussed by OMM of orthogonal polynomials, for instance, see [40-43].

The motivation for this research work is our great interest in finding the approximate solutions of the nonlinear ordinary differential equations, in particular the Jeffrey-Hamel flow problem, which is one of the most important applications in fluid mechanics and biomechanics. Moreover, this study aims to implement the ECM based on the standard polynomial to solve the Jeffrey-Hamel problem, and another aim is to develop and suggest a novel ECM based on various orthogonal polynomials such as Chebyshev, Bernstein, Legendre, and Hermite polynomials, and then D-ECM has been applied to solve the Jeffrey-Hamel flow problem.

This paper is organized as follows: The mathematical description of the Jeffrey-Hamel flow problem is presented in section two. Section three explains the basic concepts of the proposed methods. Solving the Jeffrey-Hamel flow problem by the proposed methods will be given in section four. In section five, the numerical results will be displayed and explained. Finally, in section six, a conclusion will be presented.

The Mathematical Formulation of Jeffrey Hamel’s Flow Problem

The Jeffrey-Hamel flow problem represented by the NODE is the steady flow of a viscous, conductive, incompressible fluid in two dimensions at the intersection of two plane rigid and non-parallel walls that get together at an angle 2α. It is assumed that the flow is perfectly radial and symmetric. Therefore, the velocity field is only along the radial direction and depends on r and θ, so it can be given by V(𝑢(𝑟, θ), 0), as illustrated in (Fig. 1) [30].

Figure 1. Jeffrey-Hamel flow’s geometry [30].

The continuity equations and the Navier-Stokes equations can be expressed in polar coordinates as follows:

\[ \rho \frac{\partial}{\partial r} (r u(r, \theta)) = 0, \]  
\[ u(r, \theta) \frac{\partial u(r, \theta)}{\partial r} = - \frac{1}{\rho} \frac{\partial P}{\partial r} + v \left[ \frac{\partial^2 u(r, \theta)}{\partial r^2} + \frac{1}{r} \frac{\partial u(r, \theta)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u(r, \theta)}{\partial \theta^2} - \frac{u(r, \theta)}{r^2} \right] - \frac{\sigma B_0^2}{\rho r^2} u(r, \theta), \]  
\[ - \frac{1}{\rho} \frac{\partial P}{\partial \theta} + 2v \frac{\partial u(r, \theta)}{\partial \theta} = 0, \]  
where \( u(r, \theta) \) is the radial velocity, \( B_0 \) is denoted by the electromagnetic induction and \( \sigma \) is a fluid’s conductivity, \( P \) is the pressure of the fluid, \( \rho \) is the fluid density constant, and \( v \) is the kinematic viscosity parameter.

Eq.1 can be written as:

\[ g(\theta) = ru(r, \theta), \]  
By using dimensionless parameters [39], so

\[ w(x) = \frac{g(\theta)}{g_{max}}, \text{ where, } x = \frac{\theta}{\alpha}. \]
By eliminating $P$ term from Eq.2 and Eq.3, and using the formulas given in Eq.4 and Eq.5, a nonlinear third-order ODE is obtained:

$$w''''(x) + 2a \text{Re} w'(x) w''(x) + (4-Ha) \alpha^2 w'(x) = 0,$$

with the boundary conditions as follows:

$$w(0) = 1, \quad w'(0) = 0, \quad w(1) = 0, \quad w''(1) = 0,$$

where, $\text{Re} = \frac{a \mu_{\text{max}}}{\nu}$, and $Ha^2 = \frac{\sigma B^2}{\nu \rho}$, are the Reynolds number and the Hartmann number's square, respectively.

The Basic Concepts of the Proposed Methods

A description of the suggested methods will be presented in this section. Also, orthogonal polynomials and the operational matrices will be offered, which are used in the development of the ECM algorithm to get the approximate solution to the problem.

The Basic Concepts of ECM

Consider $m^{th}$-order non-linear ODE as follows

$$f(x, y, y', y'', ..., y^{(m)}) = h(x), \quad \alpha \leq x \leq \beta,$$

with either the I.C:

$$y^{(i)}(\alpha) = \omega_i, \quad 0 \leq i \leq m - 1,$$

or the following B.C:

$$y^{(i)}(\alpha) = \mu_i, \quad y^{(i)}(\beta) = \delta_i, \quad 0 \leq i \leq \frac{m}{2} - 1,$$

where $h(x)$ is a function that is known and $\omega_i, \mu_i, \delta_i$, are constants. The essential assumption is that Eq.8 has a unique solution with the initial or boundary conditions given in Eq.9 or Eq.10. Moreover, a function $y(x) \in L^2[0,1]$ can be expressed by a linear combination of $m^{th}$-order function series based on the classical standard monomial polynomials as:

$$y(x) = \sum_{i=0}^{m} c_i \varphi_i(x),$$

where $c_i$ are the coefficients whose values will be found by giving the following definitions

$$X = [\varphi_0 \varphi_1 \varphi_2 \cdots \varphi_m], \quad C = [c_0 c_1 c_2 \cdots c_m]^T$$

where $\varphi_m$ represents the base functions from the classical polynomials. By using the dot product, the $m^{th}$ order approximation of the series solution provided in Eq.11 is as follows:

$$y(x) = \sum_{i=0}^{m} c_i \varphi_i(x) = X C,$$

Assume that the derivative of vector $X$ will be defined as below

$$D[X] = X B,$$

where $B_{(m+1)\times(m+1)}$ is the operational auxiliary matrix with the given entries in classical monomials:

$$B = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots m \\
0 & 0 & 0 & 0
\end{bmatrix}_{(m+1)\times(m+1)}$$

Also, the higher derivatives can be written as,

$$D^m[X] = X B^m \quad \text{where} \quad m = 1, 2, \ldots$$

Therefore, Eq.13 can be used to write the derivatives in the following format:

$$y^{(m)}(x) = X B^m C \quad m \geq 1.$$}

Now, substituting the Eqs.12, and 14 in Eqs.8-10, the matrix equation with the restrictions $3^1$, can be obtained:

$$f(x, X, C, X B, X B^2 C, \ldots, X B^m C) = h(x), \quad m = 1, 2, \ldots$$

and

$$X(0) B^i C = \omega_i, \quad 0 \leq i \leq m - 1, \quad i = 0, \ldots, m - 1,$$

Consider the Hilbert space $H = L^2[0,1]$, which has the inner product as follows:

$$\langle f_1, f_2 \rangle = \int_0^1 f_1(x) f_2(x) dx, \quad \text{Assume a set of functions that are linearly independent in } H$$

$$\psi = \{\psi_0, \psi_1, \ldots, \psi_m\},$$

where $\psi_m$ be the base function of a standard monomial polynomials $x^i, \forall i = 0, 1, 2, \ldots, m$ or any other type of polynomial $3^{1,2}$. Then, by applying the inner product given in Eq.17 with the elements of $\psi$ defined in Eq.18, the following matrix equation $3^3$ will be shown:

$$G = E,$$

The $i^{th}$ row of $G$ and $E$, respectively, is made up of:

$$\langle \psi_i, f(x, X, C, X B C, X B^2 C, \ldots, X B^m C) \rangle, \quad \langle \psi_i, h(x) \rangle, \quad 0 \leq i \leq m.$$

In addition, by applying the initial or boundary conditions in Eqs.15, and 16, some entries of Eq.19 are modified from the left-hand side $G$ and the corresponding right-hand side $E$ $3^{5}$. Thus, a system of $(m + 1)$ nonlinear algebraic equations for unknown $C$ will be obtained. By solving the resulting system numerically or sometimes analytically, unique values can be obtained for unknown elements $c_0, c_1, c_2, \ldots, c_m$, this will be substituted in Eq.12 to obtain an approximate solution to Eq.8.
**First Kind Chebyshev Polynomials**

The first kind of Chebyshev polynomials \( T_i(x) \) of degree \( i \) is defined by:

\[
T_i(x) = \sum_{j=0}^{i} (-1)^{i-j} 2^j \frac{(i+j-1)!}{(i-j)! (2j)!} (x-1)^j
\]

The unknown function \( y(x) \) can be represented as:

\[
y(x) = \sum_{i=0}^{\infty} c_i T_i(x),
\]

where,

\[
c_i = \langle y, T_i \rangle = (2i+1) \int_0^1 y(x) P_i(x) dx; \ i \geq 0.
\]

In general, only the first \( (m+1) \) terms of the Chebyshev polynomials have been expressed \(^{30}\), so

\[
y(x) = \sum_{i=0}^{m} c_i T_i(x) = C^T \tilde{\phi}(x), \tag{22}
\]

where, \( C^T = [c_0, c_1, c_2, ..., c_m] \) and \( \tilde{\phi}(x) = [T_0(x), T_1(x), ..., T_m(x)]^T \). Moreover, the derivatives of \( \phi(x) \) can be considered as:

\[
D[\phi(x)] = D_T \phi(x), \quad D^2[\phi(x)] = D_T^2 \phi(x), \ldots, D^m[\phi(x)] = D_T^m \phi(x), \tag{23}
\]

where \( D_T \) is the operational matrix of the provided derivative, which is defined as follows:

\[
D_T = (di,j) = \begin{cases} \frac{2i}{\rho_j}, & \text{for } i = j - k, \\ 0, & \text{otherwise}, \end{cases}
\]

where, \( k = 1, 3, 5, ..., m - 1 \) if \( m \) is even, or \( k = 1, 3, 5, ..., m \) if \( m \) is odd, \( \rho_0 = 2 \), and \( \rho_k = 1 \) for all \( k \geq 1 \).

For example, if \( m \) is even then the \( D_T \) is expressed as follows:

\[
D_T = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
3 & 0 & 6 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
5 & 0 & 8 & 0 & 8 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
(m-1) & 0 & 2(m-1) & 0 & 2(m-1) & \cdots & 2(m-1) & 0 & 0 \\
0 & 2m & 0 & 2m & 0 & \cdots & 0 & 2m & 0
\end{pmatrix}, \tag{24}
\]

In addition, if \( m \) is odd then the matrix \( D_T \) is defined as follows:

\[
D_T = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
3 & 0 & 6 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 8 & 0 & 8 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 2(m-1) & 0 & 2(m-1) & \cdots & 2(m-1) & 0 & 0 \\
0 & 2m & 0 & 2m & 0 & \cdots & 0 & 2m & 0
\end{pmatrix}, \tag{25}
\]

Hence, the derivatives can be written by using Eq.23 in the following form:

\[
\frac{d^m y}{dx^m} = C^T D_T^m \phi(x), \quad \frac{d^m y}{dx^m} = C^T D_T^m \phi(x), \tag{26}
\]

**Bernstein Polynomials**

The degree \( n \) Bernstein polynomials in \([0, 1]\) are defined by \(^{44}\):

\[
B_{j,n}(x) = \binom{n}{j} x^j (1-x)^{n-j}, \quad 0 \leq j \leq n \tag{27}
\]

There is \((n+1)\) degree of the Bernstein Polynomials. Also, these polynomials have two most significant properties \(^{30}\):

i) Property of unity partition, \( \sum_{j=0}^{n} B_{j,n}(x) = 1, \quad 0 \leq x \leq 1 \)

ii) Positivity property, \( B_{j,n}(x) \geq 0, \quad \text{for} \quad 0 \leq j \leq n \) and \( B_{j,n}(x) = 0 \) if \( j < 0 \) or \( n < j \).

In general, the \( y(x) \) can be approximated by the linear combination of Bernstein polynomial shown in the following formula below:

\[
y(x) = \sum_{j=0}^{n} c_j B_{j,n}(x) = C^T \phi(x), \tag{28}
\]

where \( C^T = [c_0, c_1, c_2, ..., c_n] \), and \( \phi(x) = [B_{0,n}, B_{1,n}, B_{2,n}, ..., B_{n,n}]^T \).

Moreover, the vector \( \phi(x) \) can be decomposed as a square matrix multiplication \( A_{(n+1) \times (n+1)} \) and a vector \( X_{(n+1) \times 1} \) as:

\[
\phi(x) = AX, \quad X = [1, x, x^2, ..., x^n]^T,
\]

Define the vector \( A_{j+1} \) as \(^{30}\):
\[ A_{j+1} = \begin{bmatrix} \binom{n}{j}, \binom{n-j}{j}, \binom{n-j}{j}, \cdots, \binom{n-j}{j} \end{bmatrix}, \text{ for } 0 \leq j \leq n. \]

Also, if \( A_{(n+1)\times (n+1)} \) such that \( A = [A_1, A_2, \ldots, A_{n+1}]^T \) the following matrix will be exposed:

\[
A = \begin{bmatrix}
(-1)^0 \binom{n}{0} & (-1)^1 \binom{n}{0} & \cdots & (-1)^n \binom{n}{n-1} \\
0 & (-1)^0 \binom{n}{1} & \cdots & (-1)^n \binom{n}{n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (-1)^n \binom{n}{n-1}
\end{bmatrix}
\]

Therefore, the derivatives of \( \phi(x) \) can be defined by:

\[
D^n[\phi(x)] = D_B^n \phi(x), \quad n = 1, 2, \ldots.
\]

Therefore, the derivatives can be expressed as follows:

\[
d^n y \over dx^n = C^T D_B^n \phi(x) = C^T (A U B^*)^n \phi(x) \quad \text{where } n = 1, 2, \ldots
\]

**Legendre Polynomials**

The Legendre polynomials, \( P_m(x) \), on \([-1,1]\) of \( m^{th} \) order are defined as:

\[
P_0(x) = 1, \quad P_1(x) = x, \quad P_{m+1}(x) = \frac{2m+1}{m+1} x P_m(x) - \frac{m}{m+1} P_{m-1}(x),
\]

Also, the Legendre polynomials \( P_m(x) \) can be obtained in the analytical formula by the following:

\[
D_p = \begin{cases} (2j-1), & j = i - k, \quad (k = 1, 3, \ldots, m, \quad \text{if } m \text{ odd}, \\
0 & \text{Otherwise} \end{cases}
\]

Therefore, the derivatives can be expressed as follows:

\[
d^m y \over dx^m = C^T D_p^m \phi(x), \quad \text{where, } m \geq 1
\]

**Hermite Polynomials**

The Hermite polynomials, \( H_m(x) \), on \((-\infty, \infty)\) of \( m^{th} \) order are defined as:

\[
H_m(x) = \sum_{j=0}^{K} \frac{(-1)^j}{j! (m - 2j)} (2x)^{m-2j}.
\]

Where \( K = \frac{m-1}{2} \) if \( m \) is odd and \( K = \frac{m}{2} \) if \( m \) is even. Also, the Hermite polynomials \( H_m(x) \) can be written as follows:

\[
H_m(x) = \sum_{j=0}^{K} \frac{(-1)^j}{j!} m(m-1) \cdots (m-2j + 1)(2x)^{m-2j}
\]

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The function \( y(x) \) is defined by a truncated Hermite polynomials \( \mathbf{H}^n(x) \), as:

\[
y(x) = \sum_{j=0}^{K} c_j H_j(x) = \phi(x) \mathbf{C},
\]

where, \( \phi(x) = [H_0(x), H_1(x), ..., H_K(x)] \) and, \( \mathbf{C} = [c_0, c_1, c_2, ..., c_K]^T \). On the other hand, Hermite polynomials \( \mathbf{H}^n(x) \) and the powers \( x^m \) are related to the following relation \( 45 \),

\[
x^{2m} = \frac{(2m)!}{2^m} \sum_{m=0}^{s} \frac{H_{2m}(x)}{(s-m)!(2m)!}, \quad 0 \leq x
\]

and, \( D_M = \begin{pmatrix} 1 & 0 & 0 & 0 & ... & 0 \\ 0 & \frac{1}{2} & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{k!}{2^{(K-1)}!} & 0 & \frac{k!}{2^{(K-2)}!} & \ldots & \frac{k!}{2^{K-2}!} \\ \end{pmatrix} \)

and for even \( K \), the matrix \( D_M \) is defined as follows \( 45 \):

\[
D_M = \begin{pmatrix} 1 & 0 & 0 & 0 & ... & 0 \\ 0 & \frac{1}{2} & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{k!}{2^{(K-1)}!} & 0 & \frac{k!}{2^{(K-2)}!} & \ldots & \frac{k!}{2^{K-2}!} \\ \end{pmatrix}
\]

From above, the expression of \( \phi(x) \) will be written as follows:

\[
\phi(x) = X(x)(D_M)^{-1}T
\]

and, \( \phi(x)^{(n)} = X^{(n)}(x)(D_M)^{-1}T \quad n = 1, 2, ..., \) Furthermore, the below relation can be applied to obtain the \( X^{(n)}(x) \) by using terms of the \( X(x) \) \( 36 \):

\[
X^{(1)}(x) = X(x)G, \quad X^{(2)}(x) = X(x)G^2,
\]

\[
X^{(n)}(x) = X(x)G^n
\]

where \( G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & ... \\ 0 & 0 & 0 & ... \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & ... \end{pmatrix} \) \( (K+1) \times (K+1) \)

Similarly, the derivatives \( y^{(n)}(x) \) can be expressed as:

\[
d^n y \over dx^n = \phi(x)^{(n)} = X(x)G^n \quad \text{where} \quad n = 1, 2, ..., \)

**Solving the Jeffery-Hamel Flow Problem by the ECM and D-ECM**

The proposed methods from section three will be implemented in this section to provide accurate approximation solutions to the Jeffery-Hamel flow problem.

The D-ECM depends on the base functions of different polynomials such as Chebyshev, Bernstein, Legendre, and Hermite polynomials that are given in the Eqs.21, 27, 31, 35, respectively, and applying the operational matrices corresponding to these polynomials represented on Eqs.24, 25, 29, 33, 39, 40, respectively. To increase the accuracy and efficiency of ECM, these polynomials are used
By implementing the processes as presented in the matrix equation, which are given in Eq.19.

By substituting the initial or boundary conditions in Eqs.15, and 16, some entries of Eq.19 are modified. Thereafter, $(m + 1)$ nonlinear algebraic equations for unknown $C$ can be obtained by solving this system numerically by Mathematica®12, where unique values are given for unknown elements $c_0, c_1, c_2, ..., c_m$, to achieve the approximate solution to the problem.

The ECM and D-ECM procedures can be used to solve Eq.6 with boundary conditions Eq.7, by using Eqs.12, 14, replacing unknown function $w(x)$ with its derivatives as matrices, for ECM:

$$X B^2 C + 2a Re (X C)(X B C)
+ (4 - H a) a^2 (X B C) = 0,
(X C)(0) = 1, (X B C)(0) = 0, (X C)(1)
= 0$$

Then, the process has been used as presented in Eqs.19, 20. so:

$$\langle x^i, X B^2 C + 2a Re (X C)(X B C)
+ (4 - H a) a^2 (X B C) = \langle x^i, 0 \rangle,
\forall i = 0, 1, 2, ..., m$$

Applying Eqs.22, 26 for D-ECM based on the first kind of Chebyshev polynomials, it follows:

$$C^T D_T^3 \varphi(x)
+ 2a Re (C^T \varphi(x))(C^T D_T \varphi(x))
+ (4 - H a) a^2 (C^T D_T \varphi(x)) = 0,$n
$C^T \varphi(0) = 1, C^T D_T \varphi(0) = 0, C^T \varphi(1)
= 0$$

Using the procedures as given in the Eqs.19, and 20, hence:

$$\langle T_i(x), C^T D_T^3 \varphi(x)
+ 2a Re (C^T \varphi(x))(C^T D_T \varphi(x))
+ (4 - H a) a^2 (C^T D_T \varphi(x)) \rangle
= \langle T_i(x), 0 \rangle,
\forall 0 \leq i \leq m$$

By setting the Eqs.28, and 30 for D-ECM based on the Bernstein polynomials, the following is obtained:

$$C^T D_B^3 \varphi(x) + 2a Re (C^T \varphi(x))(C^T D_B \varphi(x))
+ (4 - H a) a^2 (C^T D_B \varphi(x)) = 0,$n
$C^T \varphi(0) = 1, C^T D_B \varphi(0) = 0, C^T \varphi(1)
= 0$$

By implementing the processes as presented in Eqs.19, 20, Eq.47 will be shown

$$\langle B_{j,n}(x), C^T D_B^3 \varphi(x)
+ 2a Re (C^T \varphi(x))(C^T D_B \varphi(x))
+ (4 - H a) a^2 (C^T D_B \varphi(x)) \rangle = \langle B_{j,n}(x), 0 \rangle,
\forall j = 0, 1, 2, ..., n.$$
\( w(x) \approx 1 - 1.12597 x^2 + 8.79819 \times 10^{-8} x^3 + 0.166622 x^4 + 0.000024296 x^5 - 0.046882 x^6 + 0.000494395 x^7 + 0.00618549 x^8 + 0.00177057 x^9 - 0.00325326 x^{10} + 0.00117476 x^{11} - 0.000162363 x^{12}. \)

- By using D-ECM based on the Bernstein polynomials,
  \( w(x) \approx 1 - 1.12597 x^2 + 5.81122 \times 10^{-7} x^3 + 0.166618 x^4 + 0.0000498819 x^5 - 0.0469669 x^6 + 0.000676843 x^7 + 0.00592638 x^8 + 0.00201225 x^9 - 0.00339574 x^{10} + 0.00122292 x^{11} - 0.000169476 x^{12}. \)

- By using D-ECM based on the Legendre polynomials,
  \( w(x) \approx 1 - 1.12597 x^2 + 8.93943 \times 10^{-8} x^3 + 0.166622 x^4 + 0.0000245082 x^5 - 0.0468829 x^6 + 0.000496724 x^7 + 0.00618166 x^8 + 0.0017746 x^9 - 0.0032559 x^{10} + 0.00117574 x^{11} - 0.000162521 x^{12}. \)

- By using D-ECM based on the Hermite polynomials,
  \( w(x) \approx 1 - 1.12597 x^2 + 6.8438 \times 10^{-8} x^3 \)

\( w(x) \approx 1 - 1.12597 x^2 + 5.81122 \times 10^{-7} x^3 + 0.166622 x^4 + 0.0000498819 x^5 - 0.0468829 x^6 + 0.000676843 x^7 + 0.00592638 x^8 + 0.00201225 x^9 - 0.00339574 x^{10} + 0.00122292 x^{11} - 0.000169476 x^{12}. \)

**The Numerical Results and Discussion:**

In this section, an example is presented when the value of \( n = 3, \alpha = 5^\circ, Re = 10, \) and \( Ha = 0, \) to illustrate the approach of the proposed methods to solve the Jeffery-Hamel flow problem.

To explain the technique of ECM, by using the Eqs.12, and 14, it follows:

\[ w(x) = X \mathbf{C} = [\varphi_0 \varphi_1 \varphi_2 \varphi_3] \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}^T, \]

and,

\[ w''(x) = X \mathbf{B} \mathbf{C} = [\varphi_0 \varphi_1 \varphi_2 \varphi_3] \begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}^T. \]

Now, substituting the \( w'(x), w''(x) \) in Eqs.6, 7, and applying the inner product to solve the left and right sides of the matrix equation given in Eq.43, with boundary conditions Eq.42, four nonlinear algebraic equations for unknown \( c_0, c_1, c_2, c_3, \) can be obtained as:

\[
\frac{100}{3} \pi^2 c_1 + \frac{100}{3} \pi c_0 c_1 + 25 \pi c_1 c_2 + 50 \pi^2 c_2 \\
+ 50 \pi c_0 c_2 + 60 \pi c_1 c_2 + \frac{5}{27} \pi^2 c_2^2 \\
+ 2 c_3 + 60 \pi^2 c_3 + 60 \pi c_0 c_3 \\
+ \frac{25}{27} \pi c_1 c_3 + \frac{63}{24} \pi c_2 c_3 \\
+ \frac{24}{27} \pi c_3^2 = 0, \\
c_0 = 1, \\
c_1 = 0,
\]
c_0 + c_1 + c_2 + c_3 = 0.
By solving this system numerically by Mathematica®12, unique values of c_0, c_1, c_2, c_3 are given as follows:
\[ c_0 = 1, \ c_1 = 0, \ c_2 = -1.14626, \ c_3 = 0.146262. \]

Hence, the values of c_0, c_1, c_2, c_3, will be substituted in Eq.52 to obtain an approximate solution to the Eq.6, as:
\[ w(x) \approx 1 - 1.14626 x^2 + 0.146262 x^3. \]

\[ w'(x) = C^T D_T \varphi(x) = [c_0 \ c_1 \ c_2 \ c_3] \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 3 & 0 & 6 & 0 \end{pmatrix} [T_0(x), T_1(x), T_2(x), T_3(x)]^T, \]

Also,
\[ w''(x) = C^T D_T^3 \varphi(x) = [c_0 \ c_1 \ c_2 \ c_3] \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 \end{pmatrix} [T_0(x), T_1(x), T_2(x), T_3(x)]^T. \]

Therefore, substituting the w'(x), w''(x) in Eqs.6, 7, and employing the inner product of the matrix equation given in Eq.45, with boundary conditions Eq.44, four nonlinear algebraic equations for unknown c_0, c_1, c_2, c_3, are achieved as:
\[ \begin{align*}
-100 & c_1 - \frac{100}{3} c_0 c_1 + \frac{220}{3} c_1 c_2 - 200 c_2^2 \\
&+ \frac{40}{27} \pi c_2^2 - 8 c_3 + 180 c_3 c_2 \\
&+ 180 c_0 c_3 - 400 c_1 c_3 \\
&+ \frac{80}{27} \pi c_1 c_3 - 1100 c_2 c_3 \\
&+ \frac{400}{63} \pi c_2 c_3 + 1200 c_3 c_2 \\
&- \frac{20}{3} \pi c_3^2 = 0,
\end{align*} \]
\[ \begin{align*}
c_0 - c_2 & = 1, \\
c_1 - 3 c_3 & = 0, \\
c_0 + c_1 + c_2 + c_3 & = 0.
\end{align*} \]

Solving this system numerically by Mathematica®12, the following unique values of c_0, c_1, c_2, c_3 will be obtained:
\[ c_0 = 0.419839, \ c_1 = 0.120242, \ c_2 = -0.580161, \ c_3 = 0.0400806. \]

Hence, the values of c_0, c_1, c_2, c_3, will be substituted in Eq.53 to obtain an approximate solution to the Eq.6, as:
\[ w(x) \approx 1 - 1.16032 x^2 + 0.160322 x^3. \]

By implementing the Eqs.28, 30, for D-ECM based on the Bernstein polynomials, the following is obtained:
\[ w(x) = C^T \varphi(x) = c_0 B_{0,3} + c_1 B_{1,3} + c_2 B_{2,3} + c_3 B_{3,3}, \]
where, \( B_{0,3} = 1 - 3x + 3x^2 - x^3, \ B_{1,3} = 3x - 6x^2 + 3x^3, \ B_{2,3} = 3x^2 - 3x^3, \ B_{3,3} = x^3, \) as matrices, the derivatives \( w'(x), w''(x) \) may be written as:
\[ w'(x) = C^T D_B \varphi(x) = [c_0 \ c_1 \ c_2 \ c_3] \begin{pmatrix} -3 & -1 & 0 & 0 \\ 3 & 1 & -2 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 3 \end{pmatrix} [B_{0,3}, B_{1,3}, B_{2,3}, B_{3,3}]^T, \]

and,
\[ w''(x) = C^T D_B^3 \varphi(x) = [c_0 \ c_1 \ c_2 \ c_3] \begin{pmatrix} -6 & -6 & -6 & -6 \\ -18 & 18 & 18 & 18 \\ -18 & -18 & -18 & -18 \\ 18 & 18 & 18 & 18 \end{pmatrix} [B_{0,3}, B_{1,3}, B_{2,3}, B_{3,3}]^T. \]

Thus, if the \( w'(x), w''(x) \) substituting into Eqs.6, 7, and using the inner product of the matrix equation from Eq.47 with the boundary conditions from Eq.46, four nonlinear algebraic equations for unknown c_0, c_1, c_2, c_3, are attained as follows:
In addition, four nonlinear algebraic equations with unknowns $c_0, c_1, c_2, c_3$, are obtained by substituting $w'(x)$ and $w''(x)$ in Eqs. 6, 7, and applying the inner product of the matrix equation from Eq.49 with the boundary conditions from Eq.48:

$$
\begin{align*}
\frac{25}{2} c_1^2 + \frac{75}{2} c_2^2 + \frac{75}{2} c_0 c_2 + 60 c_1 c_2 \\
+ \frac{75}{2} c_2^2 + 100 c_2^2 + 100 c_0 c_3 \\
+ \frac{520}{7} c_1 c_3 \\
+ \frac{575}{16} c_3^2 = 0,
\end{align*}
$$

$$
c_0 - \frac{c_2}{2} = 0,
$$

$$
c_1 - \frac{3 c_3}{2} = 0,
$$

where $c_0 + c_1 + c_2 + c_3 = 0$.

The following unique values of $c_0, c_1, c_2, c_3$ will be found by numerically solving this system with Mathematica.:

$$
c_0 = 1, \quad c_1 = 1, \quad c_2 = 0.60497, \quad c_3 = 0.
$$

To achieve an approximate solution to Eq.6, the values of $c_0, c_1, c_2, c_3$ will be substituted in Eq.54, as follows:

$$
w(x) \approx 1.18509 x^2 + 0.185091 x^3
$$

By applying the Eqs.32, and 34, for D-ECM based on the Legendre polynomials, the following is achieved:

$$
w(x) = C^T \phi(x)
$$

$$
= c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x)
$$

$$
+ c_3 P_3(x),
$$

where, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = -\frac{1}{2} + \frac{3 x^2}{2}$, $P_3(x) = -\frac{3 x}{2} + \frac{5 x^3}{2}$, and the derivatives of $w(x)$, can be written as matrices:

$$
w'(x) = C^T D_p \phi(x) = [c_0 c_1 c_2 c_3] \begin{bmatrix} 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
1 & 0 & 5 & 0 \end{bmatrix} [P_0(x), P_1(x), P_2(x), P_3(x)]^T,
$$

and,

$$
w''(x) = C^T D_p^3 \phi(x) = [c_0 c_1 c_2 c_3] \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
15 & 0 & 0 & 0 \end{bmatrix} [P_0(x), P_1(x), P_2(x), P_3(x)]^T.
$$

Moreover, by using the Eqs. 36, 41, a description of the D-ECM based on the Hermite polynomials procedure follows:

$$
w(x) = \phi(x) C
$$

$$
= H_0(x) c_0 + H_1(x) c_1 + H_2(x) c_2
$$

$$
+ H_3(x) c_3,
$$

where $H_0(x) = 1$, $H_1(x) = 2 x$, $H_2(x) = -2 + 4x^2$, $H_3(x) = -12 x + 8x^3$, and the derivatives $w'(x), w''(x)$ as matrices can be obtained as:

$$
w'(x) = [H_0(x) H_1(x) H_2(x) H_3(x)] \begin{bmatrix} 0 & 2 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 \end{bmatrix} [c_0 c_1 c_2 c_3]^T,
$$

As a result, the values $c_0, c_1, c_2, c_3$, will be substituted in Eq.55 to give an approximate solution to Eq.6, as follows:

$$
w(x) \approx 1.105406 x^2 + 0.0540639 x^3
$$

Then, using Mathematica, to solve this system numerically, the following unique values of $c_0, c_1, c_2, c_3$, will be obtained:

$$
c_0 = 0.648645, \quad c_1 = 0.0324384, \quad c_2 = -0.702709, \quad c_3 = 0.0216256.
$$
Substituting $w'(x)$, $w'''(x)$ into Eqs. 6, 7, and using the inner product of the matrix equation from Eq. 51 with the boundary conditions from Eq. 50, yields four nonlinear algebraic equations with unknowns $c_0, c_1, c_2, c_3$: 

$$
\begin{align*}
-400 c_1 - 400 c_0 c_1 + \frac{1760}{3} c_1 c_2 \
+ \frac{1600}{3} c_2^2 - 32 c_3 + 1120 c_2 c_3 \
+ 1120 c_0 c_3 + \frac{3200}{3} \alpha c_1 c_3 \
- \frac{9920}{7} c_2 c_3 - 3200 c_3^2 = 0,
\end{align*}
\begin{align*}
0 = c_0 - 2 c_2, \\
2 c_1 - 12 c_3 = 0, \\
c_0 + 2 c_1 + 2 c_2 - 4 c_3 = 0.
\end{align*}
$$

Then, using Mathematica®12, solve this system numerically to acquire the following unique values of $c_0, c_1, c_2, c_3$:

$c_0 = 0.419839$,  
$c_1 = 0.120242$,  
$c_2 = -0.290081$,  
$c_3 = 0.0200403$.

As a consequence, the values $c_0, c_1, c_2, c_3$, will be swapped in Eq.56 to get the following approximate solution to Eq.6:

$$w(x) \approx 1. -1.16032 x^2 + 0.160322 x^3.$$

Furthermore, the maximal error remainder $MER_n$ has been introduced in this section because there is no exact solution available to the problem, as well as to verify the accuracy and reliability of the approximate solution obtained by ECM and D-ECM. The $MER_n$ is calculated by:

$$MER_n = \max_{0 \leq x \leq 1} |w''(x)| + 2 \alpha Re w(x) w'(x) + (4 - Ha) \alpha^2 w'(x)|$$

Fig. 2 presents the logarithmic plots for the $MER_n$ values, obtained by the ECM based on the standard monomial polynomial, as well as, by the D-ECM based on the Chebyshev, Bernstein, Legendre, and Hermite polynomials, for parameters $Re = 10$, $Ha = 0$ and $\alpha = 5^\circ$, as is evident from the figure, good agreements have been obtained for all proposed methods.

Moreover, in Table 1 the values of $MER_n$ for the approximate solution is given by using ECM and D-ECM with $n = 12$ and parameters $Re = 10, Ha = 0$ and versus the value of $\alpha$, which appears the efficiency of these methods. In addition, it can be noted that D-ECM based on the Hermite polynomials method produces better accuracy with the lowest errors compared to the other methods.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>ECM Standard</th>
<th>D-ECM Chebyshev</th>
<th>D-ECM Bernstein</th>
<th>D-ECM Legendre</th>
<th>D-ECM Hermite</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3^\circ$</td>
<td>1.78573 * 10^{-6}</td>
<td>1.55044 * 10^{-7}</td>
<td>1.07042 * 10^{-6}</td>
<td>1.57835 * 10^{-7}</td>
<td>1.16736 * 10^{-7}</td>
</tr>
<tr>
<td>$-3^\circ$</td>
<td>3.09536 * 10^{-6}</td>
<td>2.15838 * 10^{-7}</td>
<td>1.55861 * 10^{-6}</td>
<td>2.20333 * 10^{-7}</td>
<td>1.60927 * 10^{-7}</td>
</tr>
<tr>
<td>$-5^\circ$</td>
<td>0.0000152937</td>
<td>1.01151 * 10^{-6}</td>
<td>7.35397 * 10^{-6}</td>
<td>1.03335 * 10^{-6}</td>
<td>8.15967 * 10^{-7}</td>
</tr>
</tbody>
</table>
Furthermore, in Table 2 the comparisons of $MER_{12}$ values are presented when $Re = 10, Ha = 0, \alpha = 5^\circ$, for the solutions by proposed methods and by the Chebyshev and the Bernstein operational matrices methods according to previous studies \textsuperscript{30}. Better accuracy can be realized by using the suggested methods.

Table 2. The comparison between the $MER_{12}$ when $Re = 10, Ha = 0, \alpha = 5^\circ$ by proposed methods and by Chebyshev and Bernstein \textsuperscript{30}.

<table>
<thead>
<tr>
<th>ECM</th>
<th>D-ECM Chebyshev</th>
<th>D-ECM Bernstein</th>
<th>D-ECM Legendre</th>
<th>D-ECM Hermite</th>
<th>Chebyshev \textsuperscript{30}</th>
<th>Bernstein \textsuperscript{30}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard</td>
<td>5.08086</td>
<td>5.27892</td>
<td>3.48673</td>
<td>5.36366</td>
<td>4.10628</td>
<td>3.3003</td>
</tr>
<tr>
<td>$\times 10^{-6}$</td>
<td>$\times 10^{-7}$</td>
<td>$\times 10^{-6}$</td>
<td>$\times 10^{-7}$</td>
<td>$\times 10^{-7}$</td>
<td>$\times 10^{-5}$</td>
<td>$\times 10^{-6}$</td>
</tr>
</tbody>
</table>

Also, Figs.4-7 illustrate the velocity profiles for the Jeffery–Hamel problem in the cases $\alpha = 5^\circ, \alpha = -5^\circ$ with fixed $Re = 50$ and increasing values of $Ha$, as chosen in \textsuperscript{17}. The velocity is noted to be increased by increasing $Ha$ values in all the figures. The curvature of the curves also increases with increasing $Ha$ values.

Figure 7. The velocity plot for Jeffery–Hamel by proposed methods for $Ha = 2000$.

**Conclusion:**

The effective computational method and novel computational methods with suitable base functions, namely Chebyshev, Bernstein, Legendre, and Hermite polynomials, have been presented in this paper for solving the Jeffery-Hamel problem. The nonlinear problems are reduced to the solution of a nonlinear algebraic system of equations, which is processed using Mathematica\textsuperscript{®}12. The approximate solution is accurate and efficient even within a few orders of polynomials. In addition, the $MER_n$ has been calculated for the proposed methods and compared with the Chebyshev and the Bernstein operational matrices methods that are available in the literature, the results obtained showed that the proposed methods have produced better accuracy with less errors. Moreover, it can be concluded that the results of the $MER_n$ by the proposed methods D-ECM decreased significantly compared to ECM, which gives higher accuracy and efficiency. Furthermore, it was found that the results of D-ECM based on the Hermite polynomials are better than the other methods.

The present methods can also be extended to partial differential equations and fractional differential equations, which certainly require extensive further analysis.
Authors’ declaration:
- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for republication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

Authors’ Contributions:
OMS contributed to the design and implementation of the research, the analysis of the results, and the writing of the manuscript. MA.AJ interpreted, drafted, revised, proofread, and verifying the analytic approximate methods of the manuscript. The authors discussed the results and contributed to the final manuscript.

References:
طرق حسابية فعالة لحل مشكلة تدفق جيفري-هامل

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الخلاصة:

في هذا البحث، تم تشفير الطرق الحسابية الفعالة في حل مشكلة تدفق جيفري-هامل غير الخطية، بواسطة حل على ذلك، تم تعديل واجراء الطرق الحسابية المطلوبة في هذا الدراسة من خلال وظائف أساسية متعددة الحدود ودالة هيرمت، وتم استخدام الدوال المتعددة الحدود لتحويل المسألة غير الخطية إلى نظام جبري غير خطياً من المعادلات، والتي يتم حلها بعد ذلك باستخدام برنامج MATLAB.13 تم تعديل واجراء الطرق الحسابية الفعالة (D-ECM) موثوقية الطرق المتعددة، تثبت النتائج بشكل ملمع أن D-ECM و ECM دقيقة وفعالة وموثوقة الحصول على حل تقريبي للمشكلة.

الكلمات المفتاحية: الحل التقريبي، متعددة حدود بيرنشتاين، متعددة حدود تشيبشيف، متعددة حدود هيرمت، متعددة حدود ليجندر.