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Effective Computational Methods for Solving the Jeffery-Hamel Flow Problem

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Abstract

In this paper, the effective computational method (ECM) based on the standard monomial polynomial has been implemented to solve the nonlinear Jeffery-Hamel flow problem. Moreover, novel effective computational methods have been developed and suggested in this study by suitable base functions, namely Chebyshev, Bernstein, Legendre, and Hermite polynomials. The utilization of the base functions converts the nonlinear problem to a nonlinear algebraic system of equations, which is then resolved using the Mathematica®12 program. The development of effective computational methods (D-ECM) has been applied to solve the nonlinear Jeffery-Hamel flow problem, then a comparison between the methods has been shown. Furthermore, the maximum error remainder (MER_n) has been calculated to exhibit the reliability of the suggested methods. The results persuasively prove that ECM and D-ECM are accurate, effective, and reliable in getting approximate solutions to the problem.

Keywords: Approximate solution, Bernstein polynomials, Chebyshev polynomials, Hermite polynomials, Legendre polynomials.

Introduction:

In several fields of engineering and applied sciences, nonlinear ordinary differential equations (NODE) play a significant role in simulating many real-life issues. Many phenomena, including engineering, fluid mechanics, physics, chemical matters, biology, and electrostatics, have been mathematically formulated using these types of equations. The exact solution for nonlinear problems is difficult or sometimes cannot be obtainable. Therefore authors want to develop efficient either numerical or approximate methods to solve these types of problems [1-4].

Several analytical and approximate methods have been proposed by researchers to solve nonlinear differential equations, such as the Adomian decomposition method (ADM) and Direct Homotopy Analysis Method (DHAM) [5], the Bernoulli collocation method [6], the Hemite polynomial method [7], the Taylor collocation method [8], and the Gegenbauer wavelet method [9]. In particular, Singh [10] has used the Jacobi collocation method to solve the fractional advection-dispersion equation. Ganji et al. [11] have used the fifth-kind Chebyshev polynomials to solve differential equations with multiple variable orders and non-local and non-singular kernels. Also, Singha et al. [12] used Boubaker polynomials to solve a class of fractional optimal control problems. Yuttanan et al. [13] solved the non-linear distributed fractional differential equations using the Legendre wavelets method and some other approximation methods, see [14-16].

One of the most important applications in fluid mechanics and biomechanical engineering is the flow between two nonparallel plates [17]. Jeffery[18] and Hamel[19] introduced incompressible viscous fluid movement in convergent and divergent channels, and this is known as Jeffery-Hamel flow.

Many researchers have attempted to develop analytical approximations methods to solve the Jeffery-Hamel flow: such as optimal iterative perturbation technique [20], Bernstein collocation method (BCM) [21], modified Adomian decomposition method (MADM) [22,23], Homotopy analysis method (HAM) [24], Homotopy perturbation method (HPM) [25], Bernoulli collocation method [26], Hermite wavelet method [27], differential transform method (DTM) [28]. More recently, AL-Jawary et al. [29], has implemented three semi-analytical iterative

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methods namely the Daftardar-Jafari method (DJM), Temimi-Ansari method (TAM), and Banach contraction method (BCM) to obtain the solution for this problem. In addition, AL-Jawary et al. [30], has employed two operational matrices techniques (OMM) based on Bernstein and Chebyshev polynomials to solve a similar problem.

More recently, the Turkyilmazoglu has proposed an analytic approximate method namely the effective computational method (ECM), and implemented it to solve various types of problems. For example, Lane-Emden-Fowler singular nonlinear equations [31], high-order Fredholm integro-differential equations [32], high-order Volterra-Fredholm-Hammerstein integro-differential equations [33], heat transfer of fin problems [34], and initial and boundary value problems for linear differential equations of any order with difficult exact solutions[35]. The approach was based on well-chosen general-type basis functions, such as classical polynomials, and that exact solution is obtained under particular conditions. A nonlinear equation's solution is also converted into a nonlinear algebraic equations system that can be solved numerically.

Recently, orthogonal functions and polynomials have received a lot of attention from researchers since they are very useful tools and techniques in dealing with many different problems in approximation theory as well as numerical analysis [30]. On the other hand, these techniques are mainly characterized by simplifying the required solution effectively by transforming the problem into a system of algebraic equations, where it can be solved simply by using any computational program[36-39]. Accordingly, the problems are simplified substantially and the unknown function is approximated using a series of powers of polynomials. Thus, all integrals and differentials are eliminated by using the operational matrices procedure. Furthermore, the literature is full of the applications that have been discussed by OMM of orthogonal polynomials, for instance, see [40-43].

The motivation for this research work is our great interest in finding the approximate solutions of the nonlinear ordinary differential equations, in particular the Jeffery-Hamel flow problem, which is one of the most important applications in fluid mechanics and biomechanics. Moreover, this study aims to implement the ECM based on the standard polynomial to solve the Jeffrey-Hamel problem, and another aim is to develop and suggest a novel ECM based on various orthogonal polynomials such as Chebyshev, Bernstein, Legendre, and Hermite polynomials, and then D-ECM has been applied to solve the Jeffery-Hamel flow problem.

This paper is organized as follows: The mathematical description of the Jeffery-Hamel flow problem is presented in section two. Section three explains the basic concepts of the proposed methods. Solving the Jeffery-Hamel flow problem by the proposed methods will be given in section four. In section five, the numerical results will be displayed and explained. Finally, in section six, a conclusion will be presented.

The Mathematical Formulation of Jeffrey Hamel's Flow Problem

The Jeffrey-Hamel flow problem represented by the NODE is the steady flow of a viscous, conductive, incompressible fluid in two dimensions at the intersection of two plane rigid and non-parallel walls that get together at an angle 2α [21]. It is assumed that the flow is perfectly radial and symmetric. Therefore, the velocity field is only along the radial direction and depends on r and θ, so it can be given by $V(u(r,\theta),0)$, as illustrated in (Fig. 1) [30].

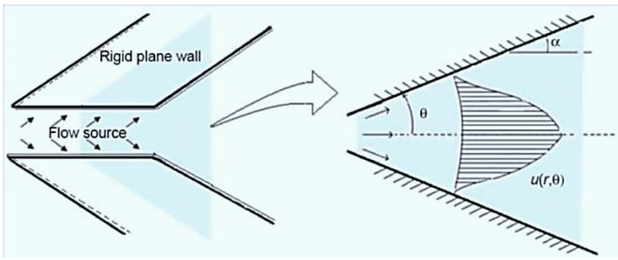

Figure 1. Jeffry-Hamel flow's geometry [30].

The continuity equations and the Navier-Stokes equations can be expressed in polar coordinates as follows:

$$\frac{\rho}{r}\frac{\partial}{\partial r}\left(ru(r,\theta)\right)=0, \quad (1)$$

$$u(r,\theta)\frac{\partial u(r,\theta)}{\partial r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+v\left[\frac{\partial^2 u(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial\theta^2}-\frac{u(r,\theta)}{r^2}\right]-\frac{\sigma B_0^2}{\rho r^2}u(r,\theta), \quad (2)$$

$$-\frac{1}{\rho r}\frac{\partial P}{\partial\theta}+\frac{2v}{r^2}\frac{\partial u(r,\theta)}{\partial\theta}=0, \quad (3)$$

where $u(r,\theta)$ is the radial velocity, B_0 is denoted by the electromagnetic induction and σ is a fluid's conductivity, P is the pressure of the fluid, ρ is the fluid density constant, and v is the kinematic viscosity parameter.

Eq.1 can be written as:

$$g(\theta)=ru(r,\theta), \quad (4)$$

By using dimensionless parameters [29], so

$$w(x)=\frac{g(\theta)}{g_{max}}, \qquad \text{where, } x=\frac{\theta}{\alpha}. \quad (5)$$

By eliminating P term from Eq.2 and Eq.3, and using the formulas given in Eq.4 and Eq.5, a nonlinear third-order ODE is obtained:

$$w'''(x) + 2\alpha\, Re\, w(x)\, w'(x) + (4-Ha)\,\alpha^2\, w'(x) = 0, \qquad 6$$

with the boundary conditions as follows:

$$w(0) = 1, \quad w'(0) = 0, \qquad w(1) = 0, \qquad 7$$

where, $Re = \frac{\alpha\, U_{max}}{v}$, and $Ha^2 = \frac{\sigma\, B_0^2}{\rho\, v}$, are the Reynolds number and the Hartmann number's square, respectively.

The Basic Concepts of the Proposed Methods

A description of the suggested methods will be presented in this section. Also, orthogonal polynomials and the operational matrices will be offered, which are used in the development of the ECM algorithm to get the approximate solution to the problem.

The Basic Concepts of ECM

Consider m^{th}-order non-linear ODE as follows [34],

$$f\left(x, y, y', y'', \dots, y^{(m)}\right) = h(x), \quad \alpha \le x \le \beta. \; 8$$

with either the I.C:

$$y^{(i)}(\alpha) = \omega_i, \quad 0 \le i \le m-1, \qquad 9$$

or the following B.C:

$$y^{(i)}(\alpha) = \mu_i, y^{(i)}(\beta) = \delta_i, 0 \le i \le \frac{m}{2} - 1, \; 10$$

where $h(x)$ is a function that is known and ω_i, μ_i, δ_i, are constants. The essential assumption is that Eq.8 has a unique solution with the initial or boundary conditions given in Eq.9 or Eq.10. Moreover, a function $y(x) \in L^2[0,1]$ can be expressed by a linear combination of m^{th}-order function series based on the classical standard monomial polynomials as:

$$y(x) = \sum_{i=0}^{m} c_i\, \varphi_i(x), \qquad 11$$

where c_i, are the coefficients whose values will be found by giving the following definitions

$$\boldsymbol{X} = [\varphi_0 \;\; \varphi_1 \;\; \varphi_2 \dots \varphi_m], \boldsymbol{C} = [c_0\; c_1\; c_2 \dots c_m]^T$$

where φ_m represents the base functions from the classical polynomials [31]. By using the dot product, the m^{th} order approximation of the series solution provided in Eq.11 is as follows:

$$y(x) = \sum_{i=0}^{m} c_i\, \varphi_i(x) = \boldsymbol{X}\,\boldsymbol{C}, \qquad 12$$

Assume that the derivative of vector $\boldsymbol{X}$ will be defined as below

$$D[\boldsymbol{X}] = \boldsymbol{X}\,\boldsymbol{B},$$

where $\boldsymbol{B}_{(m+1)\times(m+1)}$ is the operational auxiliary matrix with the given entries in classical monomials:

$$\boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & m \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(m+1)\times(m+1)}$$

Also, the higher derivatives can be written as,

$$D^m[\boldsymbol{X}] = \boldsymbol{X}\,\boldsymbol{B}^m \quad \text{where} \quad m = 1,2, \dots \qquad 13$$

Therefore, Eq.13 can be used to write the derivatives in the following format:

$$y^{(m)}(x) = \boldsymbol{X}\,\boldsymbol{B}^m\,\boldsymbol{C} \qquad m \ge 1. \qquad 14$$

Now, substituting the Eqs.12, and 14 in Eqs.8-10, the matrix equation with the restrictions [31], can be obtained:

$$f(x,\; \boldsymbol{X}\,\boldsymbol{C},\; \boldsymbol{X}\,\boldsymbol{B}\,\boldsymbol{C},\; \boldsymbol{X}\,\boldsymbol{B}^2\,\boldsymbol{C}, \dots, \boldsymbol{X}\,\boldsymbol{B}^m\,\boldsymbol{C}) = h(x), \quad m = 1,2, \dots.$$

$$\boldsymbol{X}(0)\,\boldsymbol{B}^i\,\boldsymbol{C} = \omega_i, \quad 0 \le i \le m-1, \qquad 15$$

and

$$\boldsymbol{X}(0)\,\boldsymbol{B}^i\,\boldsymbol{C} = \mu_i, \qquad \boldsymbol{X}(1)\,\boldsymbol{B}^i\,\boldsymbol{C} = \delta_i. \quad 0 \le i \le \frac{m}{2} - 1, \qquad 16$$

Consider the Hilbert space $H = L^2[0,1]$, which has the inner product as follows:

$$\langle f_1, f_2 \rangle = \int_0^1 f_1(x)\, f_2(x) dx\,, \qquad 17$$

Assume a set of functions that are linearly independent in H

$$\boldsymbol{\psi} = \{\psi_0, \psi_1, \dots, \psi_m\}, \qquad 18$$

where ψ_m be the base function of a standard monomial polynomials $x^i, \forall\, i = 0,1,2, \dots, m$ or any other type of polynomial [31,32]. Then, by applying the inner product given in Eq.17 with the elements of $\boldsymbol{\psi}$ defined in Eq.18, the following matrix equation [33] will be shown:

$$\boldsymbol{G} = \boldsymbol{E}, \qquad 19$$

The i^{th} row of $\boldsymbol{G}$ and $\boldsymbol{E}$, respectively, is made up of:

$$\langle \psi_i, f(x,\; \boldsymbol{X}\,\boldsymbol{C},\; \boldsymbol{X}\,\boldsymbol{B}\,\boldsymbol{C},\; \boldsymbol{X}\,\boldsymbol{B}^2\,\boldsymbol{C}, \dots, \boldsymbol{X}\,\boldsymbol{B}^m\,\boldsymbol{C}) \;\rangle, \quad \langle \psi_i, h(x) \rangle, \;\; 0 \le i \le m. \qquad 20$$

In addition, by applying the initial or boundary conditions in Eqs.15, and 16, some entries of Eq.19 are modified from the left-hand side $\boldsymbol{G}$ and the corresponding right-hand side $\boldsymbol{E}$ [35]. Thus, a system of $(m+1)$ nonlinear algebraic equations for unknown $\boldsymbol{C}$ will be obtained. By solving the resulting system numerically or sometimes analytically, unique values can be obtained for unknown elements $c_0, c_1, c_2, \dots c_m$, this will be substituted in Eq.12 to obtain an approximate solution to Eq.8.

First Kind Chebyshev Polynomials

The first kind of Chebyshev polynomials $\boldsymbol{T}_i(x)$ of degree i is defined by:

$$\boldsymbol{T}_i\,(x) = \sum_{j=0}^{i}(-1)^{i-j}\,2^j\,\frac{(i+j-1)!}{(i-j)!\,(2j)!}(x+1)^j. \qquad 21$$

The unknown function $y(x)$ can be represented as:

$$y(x) = \sum_{i=0}^{\infty} c_i\,\boldsymbol{T}_i(x),$$

where,

$c_i = \langle y, \boldsymbol{T}_i\rangle = (2\,i+1)\int_0^1 y(x)\,P_i(x)dx;\ i \geq 0.$

In general, only the first $(m+1)$ terms of the Chebyshev polynomials have been expressed [39], so

$$y(x) = \sum_{i=0}^{m} c_i\,\boldsymbol{T}_i(x) = \boldsymbol{C}^T\,\emptyset(x), \qquad 22$$

where, $\boldsymbol{C}^T = [c_0\; c_1\; c_2 \ldots c_m]$ and $\emptyset(x) = [\boldsymbol{T}_0(x), \boldsymbol{T}_1(x), \ldots, \boldsymbol{T}_m(x)]^T$. Moreover, the derivatives of $\emptyset(x)$ can be considered as:

$$D[\emptyset(x)] = \boldsymbol{D_T}\emptyset(x),\ D^2[\emptyset(x)] = \boldsymbol{D_T}^2\emptyset(x), \ldots, D^m[\emptyset(x)] = \boldsymbol{D_T}^m\emptyset(x), \qquad 23$$

where $\boldsymbol{D_T}_{\,(m+1)\times(m+1)}$, is the operational matrix of the provided derivative, which is defined as follows:

$$\boldsymbol{D_T} = (di, j) = \begin{cases} \dfrac{2i}{\rho_j}, & for\ j = i-k, \\ 0 & \text{otherwise}, \end{cases},$$

where, $k = 1, 3, 5, \ldots, m-1$ if m is even, or $k = 1, 3, 5, \ldots, m$ if m is odd, $\rho_0 = 2$, and $\rho_k = 1$ for all $k \geq 1$.

For example, if m is even then the $\boldsymbol{D_T}$ is expressed as follows:

$$\boldsymbol{D_T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 3 & 0 & 6 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & \ldots & 0 & 0 & 0 \\ 5 & 0 & 10 & 0 & 10 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 \\ m-1 & 0 & 2(m-1) & 0 & 2(m-1) & \ldots & 2(m-1) & 0 & 0 \\ 0 & 2m & 0 & 2m & 0 & \ldots & 0 & 2m & 0 \end{pmatrix} \qquad 24$$

In addition, if m is odd then the matrix $\boldsymbol{D_T}$ is defined as follows:

$$\boldsymbol{D_T} = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 3 & 0 & 6 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 8 & 0 & 8 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 2(m-1) & 0 & 2(m-1) & \ldots & 2(m-1) & 0 & 0 \\ m & 0 & 2m & 0 & \ldots & 0 & 2m & 0 \end{pmatrix} \qquad 25$$

Hence, the derivatives can be written by using Eq.23 in the following form:

$$\frac{dy}{dx} = \boldsymbol{C}^T\boldsymbol{D_T}\,\emptyset(x),\ \frac{d^2y}{dx^2} = \boldsymbol{C}^T\boldsymbol{D_T}^2\,\emptyset(x), \ldots, \frac{d^my}{dx^m} = \boldsymbol{C}^T\boldsymbol{D_T}^m\emptyset(x). \qquad 26$$

Bernstein Polynomials

The degree n Bernstein polynomials in $[0, 1]$ are defined by [44]:

$$\boldsymbol{B}_{j,n}(x) = \binom{n}{j}\,x^j\,(1-x)^{n-j},\ 0 \leq j \leq n \qquad 27$$

There is $(n+1)$ degree of the Bernstein Polynomials. Also, these polynomials have two most significant properties [30]:

i) Property of unity partition, $\sum_{j=0}^{n}\boldsymbol{B}_{j,n}(x) = 1,\quad 0 \leq x \leq 1$

ii) Positivity property, $\boldsymbol{B}_{j,n}(x) \geq 0,\ for\ \ 0 \leq j \leq n$ and $\boldsymbol{B}_{j,n}(x) = 0\ if\ \ j < 0\ or\ n < j.$

In general, the $y(x)$ can be approximated by the linear combination of Bernstein polynomial shown in the following formula below:

$$y(x) = \sum_{j=0}^{n} c_j\,\boldsymbol{B}_{j,n}(x) = \boldsymbol{C}^T\emptyset(x), \qquad 28$$

where $\boldsymbol{C}^T = [c_0\; c_1\; c_2 \ldots c_n]$, and $\emptyset(x) = \left[\boldsymbol{B}_{0,n}, \boldsymbol{B}_{1,n}, \boldsymbol{B}_{2,n}, \ldots, \boldsymbol{B}_{n,n}\right]^T$.

Moreover, the vector $\emptyset(x)$ can be decomposed as a square matrix multiplication $\boldsymbol{A}_{(n+1)x(n+1)}$ and a vector $\boldsymbol{X}_{(n+1)x1}$ as:

$$\emptyset(x) = \boldsymbol{A}\,\boldsymbol{X},\quad \boldsymbol{X} = [1, x, x^2, \ldots, x^n]^T,$$

Define the vector $\boldsymbol{A}_{j+1}$ as[30]:

$$\boldsymbol{A}_{j+1}=\left[\overbrace{0,0,\ldots,0}^{j-times},(-1)^0\binom{n}{j},(-1)^1\binom{n}{j}\binom{n-j}{1},\ldots,(-1)^{n-j}\binom{n}{j}\binom{n-j}{n-j}\right], \text{ for, } 0\le j\le n.$$

Also, if $\boldsymbol{A}_{(n+1)\times(n+1)}$ such that $\boldsymbol{A}=[\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_{n+1}]^T$ the following matrix will be exposed [44]:

$$\boldsymbol{A}=\begin{bmatrix}(-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1} & \ldots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0}\\ 0 & (-1)^0\binom{n}{j} & \ldots & (-1)^{m-j}\binom{n}{j}\binom{n-j}{n-j}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & (-1)^0\binom{n}{n}\end{bmatrix}_{(n+1)\times(n+1)}$$

Therefore, the derivatives of $\emptyset(x)$ can be defined by:

$$D[\emptyset(x)]=\boldsymbol{D_B}\,\emptyset(x),\qquad x\epsilon[0,1],\qquad \text{and}\qquad \boldsymbol{D_B}=\boldsymbol{A}\,\boldsymbol{U}\,\boldsymbol{B}^* \qquad 29$$

$$\text{where, } \boldsymbol{U}=\begin{bmatrix}0&0&0&\ldots&0\\1&0&0&\ldots&0\\0&2&0&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&n\end{bmatrix}_{(n+1)\times n}, \text{ and } \boldsymbol{B}^*=\begin{bmatrix}A_1^{-1}\\A_2^{-1}\\A_3^{-1}\\ \vdots\\A_n^{-1}\end{bmatrix}_{n\times(n+1)}$$

and the higher derivatives can be defined as follows:

$$D^n[\emptyset(x)]=\boldsymbol{D_B}^n\emptyset(x),\quad n=1,2,\ldots.$$

Therefore, the derivatives can be expressed as follows:

$$\frac{d^ny}{dx^n}=\boldsymbol{C}^T\,\boldsymbol{D_B}^n\emptyset(x)=\boldsymbol{C}^T(\boldsymbol{A}\,\boldsymbol{U}\,\boldsymbol{B}^*)^n\,\emptyset(x)\quad \text{where } n=1,2,\ldots. \qquad 30$$

Legendre Polynomials

The Legendre polynomials, $\boldsymbol{P}_m(x)$, on $[-1,1]$ of m^{th}-order are defined as [41,42]:

$$\boldsymbol{P}_0(x)=1,\qquad \boldsymbol{P}_1(x)=x,$$

$$\boldsymbol{P}_{m+1}(x)=\frac{2\,m+1}{m+1}x\,\boldsymbol{P}_m(x)-\frac{m}{m+1}\,\boldsymbol{P}_{m-1}(x),\quad m\ge 1$$

Also, the Legendre polynomials $\boldsymbol{P}_m(x)$ can be obtained in the analytical formula by the following:

$$\boldsymbol{P}_m(x)=\sum_{j=0}^{m}(-1)^{m+j}\frac{(m+j)!}{2^j(m-j)!\,(j!)^2}(x+1)^j. \qquad 31$$

Furthermore, the$(m+1)-$terms of polynomials $\boldsymbol{P}_m(x)$ can be used to approximate the function $y(x)$ as:

$$y(x)=\sum_{j=0}^{m}c_j\,\boldsymbol{P}_j(x)=\boldsymbol{C}^T\emptyset(x), \qquad 32$$

where, $\boldsymbol{C}^T=[c_0\ c_1\ c_2\ldots c_m]$ and $\emptyset(x)=[\boldsymbol{P}_0(x),\boldsymbol{P}_1(x),\ldots,\boldsymbol{P}_m(x)]^T$.

The derivatives of $\emptyset(x)$ can be defined by:

$$D[\emptyset(x)]=\boldsymbol{D_P}\,\emptyset(x),\ D^2[\emptyset(x)]=\boldsymbol{D_P}^2\emptyset(x),\ldots,D^m[\emptyset(x)]=\boldsymbol{D_P}^m\emptyset(x),$$

where $\boldsymbol{D_P}_{\,(m+1)\times(m+1)}$, is the operational matrix of the given derivative and is defined as follows:

$$\boldsymbol{D_P}=\begin{cases}(2j-1), & j=i-k,\ \text{where, } \begin{cases}k=1,3,\ldots,m, & if\ m\ odd,\\ k=1,3,\ldots,m-1, & if\ m\ even\end{cases}\\ 0 & Otherwise.\end{cases} \qquad 33$$

Therefore, the derivatives can be expressed as follows:

$$\frac{d^my}{dx^m}=\boldsymbol{C}^T\,\boldsymbol{D_p}^m\,\emptyset(x),\quad \text{where, } m\ge 1 \qquad 34$$

Hermite Polynomials

The Hermite polynomials, $\boldsymbol{H}_m(x)$, on $(-\infty,\infty)$ of m^{th}-order are defined as [36]:

$$\boldsymbol{H}_m(x)=m!\sum_{j=0}^{K}\frac{(-1)^j}{j!\,(m-2\,j)!}(2\,x)^{m-2j}. \qquad 35$$

where $K=\frac{m-1}{2}$ if m is odd and $K=\frac{m}{2}$ if m is even. Also, the Hermite polynomials $\boldsymbol{H}_m(x)$ can be written as follows:

$$\boldsymbol{H}_m(x)=\sum_{j=0}^{K}\frac{(-1)^j}{j!}m(m-1)\ldots(m-2\,j+1)(2\,x)^{m-2j}$$

The function $y(x)$ is defined by a truncated Hermite polynomials $\boldsymbol{H}_m(x)$, as:

$$y(x)=\sum_{j=0}^{K} c_j\, \boldsymbol{H}_j(x)=\emptyset(x)\,\boldsymbol{C}, \qquad 36$$

where, $\emptyset(x)=[\boldsymbol{H}_0(x),\boldsymbol{H}_1(x),\dots,\boldsymbol{H}_K(x)]$ and, $\boldsymbol{C}=[c_0\; c_1\; c_2 \dots c_K]^T$. On the other hand, Hermite polynomials $\boldsymbol{H}_m(x)$ and the powers x^m are related to the following relation [45],

$$x^{2m}=\frac{(2m\,)!}{2^{2m}}\sum_{m=0}^{s}\frac{\boldsymbol{H}_{2m}(x)}{(s-m)!\,(2m\,)!}, \quad 0\le x \le 1, \qquad 37$$

and,

$$x^{2m+1}=\frac{(2m+1\,)!}{2^{2m+1}}\sum_{m=0}^{s}\frac{\boldsymbol{H}_{2m+1}(x)}{(s-m)!\,(2m+1\,)!}, \quad 0\le x \le 1 \qquad 38$$

Therefore, when using the expressions in the Eqs.37, 38, and by taking $m=0,1,\dots,K$, the corresponding matrix relationship can be achieved as follows:

$$\left(\boldsymbol{X}(x)\right)^T=\boldsymbol{D_M}\left(\emptyset(x)\right)^T \;\; and \;\; \boldsymbol{X}(x)=\emptyset(x)\,(\boldsymbol{D_M})^T,$$

where $\boldsymbol{X}(x)=[1,x,\dots,x^K]$, and for odd K, then the matrix $\boldsymbol{D_M}$ defined as follows [45]:

$$\boldsymbol{D_M}=\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 & \dots & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{8} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{K!}{2^K\left(\frac{K-1}{2}\right)!1!} & 0 & \frac{K!}{2^K\left(\frac{K-1}{2}-1\right)!3!} & \dots & 0 \end{pmatrix} \qquad 39$$

and for even K, then the matrix $\boldsymbol{D_M}$ is defined as follows [45]:

$$\boldsymbol{D_M}=\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \dots & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 & \dots & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{8} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{K!}{2^K\left(\frac{K}{2}\right)!0!} & 0 & \frac{K!}{2^K\left(\frac{K}{2}-1\right)!2!} & 0 & \dots & \frac{K!}{2^K(0)!K!} \end{pmatrix} \qquad 40$$

From above, the expression of $\emptyset(x)$ will be written as follows:

$$\emptyset(x)=\boldsymbol{X}(x)((\boldsymbol{D_M})^{-1})^T$$

and,

$$\left(\emptyset(x)\right)^{(n)}=\boldsymbol{X}^{(n)}(x)((\boldsymbol{D_M})^{-1})^T \qquad n=1,2,\dots.$$

Furthermore, the below relation can be applied to obtain the $\boldsymbol{X}^{(n)}(x)$ by using terms of the $\boldsymbol{X}(x)$ [36]:

$$\boldsymbol{X}^{(1)}(x)=\boldsymbol{X}(x)\,\boldsymbol{G}, \qquad \boldsymbol{X}^{(2)}(x)=\boldsymbol{X}(x)\,\boldsymbol{G}^2, \qquad \boldsymbol{X}^{(n)}(x)=\boldsymbol{X}(x)\,\boldsymbol{G}^n$$

where $\boldsymbol{G}=\begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & K \\ 0 & 0 & 0 & & 0 \end{bmatrix}_{(K+1)\times(K+1)}$

Similarly, the derivatives $y^{(n)}(x)$ can be expressed as:

$$\frac{d^n y}{dx^n}=\left(\emptyset(x)\right)^{(n)}\boldsymbol{C}=\boldsymbol{X}(x)\,\boldsymbol{G}^n\,((\boldsymbol{D_M})^{-1})^T\boldsymbol{C}, \quad \text{where } n=1,2,\dots. \qquad 41$$

Solving the Jeffery-Hamel Flow Problem by the ECM and D-ECM

The proposed methods from section three will be implemented in this section to provide accurate approximation solutions to the Jeffery-Hamel flow problem.

The D-ECM depends on the base functions of different polynomials such as Chebyshev, Bernstein, Legendre, and Hermite polynomials that are given in the Eqs.21, 27, 31, 35, respectively, and applying the operational matrices corresponding to these polynomials represented on Eqs.24, 25, 29, 33, 39, 40, respectively. To increase the accuracy and efficiency of ECM, these polynomials are used in two steps of the suggested approach procedure. Firstly, to describe the unknown function $y(x)$ and its derivatives; secondly, to process of calculating the inner product to solve the left and right sides of the matrix equation, which are given in Eq.19.

By substituting the initial or boundary conditions in Eqs.15, and 16, some entries of Eq.19 are modified. Thereafter, $(m+1)$ nonlinear algebraic equations for unknown $\boldsymbol{C}$ can be obtained by solving this system numerically by Mathematica®12, where unique values are given for unknown elements $c_0, c_1, c_2, \dots c_m$, to achieve the approximate solution to the problem.

The ECM and D-ECM procedures can be used to solve Eq.6 with boundary conditions Eq.7, by using Eqs.12, 14, replacing unknown function $w(x)$ with its derivatives as matrices, for ECM:

$$\boldsymbol{X}\,\boldsymbol{B}^3\,\boldsymbol{C} + 2\alpha\, Re\,(\boldsymbol{X}\,\boldsymbol{C})(\boldsymbol{X}\,\boldsymbol{B}\,\boldsymbol{C}) + (4-Ha)\,\alpha^2\,(\boldsymbol{X}\,\boldsymbol{B}\,\boldsymbol{C}) = 0,\quad (\boldsymbol{X}\,\boldsymbol{C})(0) = 1,\ (\boldsymbol{X}\,\boldsymbol{B}\,\boldsymbol{C})(0) = 0,\ (\boldsymbol{X}\,\boldsymbol{C})(1) = 0 \qquad 42$$

Then, the process has been used as presented in Eqs.19, 20, so:

$$\langle x^i,\ \boldsymbol{X}\,\boldsymbol{B}^3\,\boldsymbol{C} + 2\alpha\, Re\,(\boldsymbol{X}\,\boldsymbol{C})(\boldsymbol{X}\,\boldsymbol{B}\,\boldsymbol{C}) + (4-Ha)\,\alpha^2\,(\boldsymbol{X}\,\boldsymbol{B}\,\boldsymbol{C})\rangle = \langle x^i, 0\rangle,\quad \forall\, i = 0,1,2,\dots,m. \qquad 43$$

Applying Eqs.22, 26 for D-ECM based on the first kind of Chebyshev polynomials, it follows:

$$\boldsymbol{C}^T\,\boldsymbol{D_T}^3\,\emptyset(x) + 2\alpha\, Re\,\left(\boldsymbol{C}^T\,\emptyset(x)\right)\left(\boldsymbol{C}^T\,\boldsymbol{D_T}\,\emptyset(x)\right) + (4-Ha)\,\alpha^2\,\left(\boldsymbol{C}^T\,\boldsymbol{D_T}\,\emptyset(x)\right) = 0,\quad \boldsymbol{C}^T\,\emptyset(0) = 1,\ \boldsymbol{C}^T\,\boldsymbol{D_T}\,\emptyset(0) = 0,\ \boldsymbol{C}^T\,\emptyset(1) = 0 \qquad 44$$

Using the procedures as given in the Eqs.19, and 20, hence:

$$\langle \boldsymbol{T}_i(x), \boldsymbol{C}^T\,\boldsymbol{D_T}^3\,\emptyset(x) + 2\alpha\, Re\left(\boldsymbol{C}^T\,\emptyset(x)\right)\left(\boldsymbol{C}^T\,\boldsymbol{D_T}\,\emptyset(x)\right) + (4-Ha)\,\alpha^2\left(\boldsymbol{C}^T\,\boldsymbol{D_T}\,\emptyset(x)\right)\rangle = \langle \boldsymbol{T}_i(x), 0\rangle,\quad \forall\, 0 \le i \le m \qquad 45$$

By setting the Eqs.28, and 30 for D-ECM based on the Bernstein polynomials, the following is obtained:

$$\boldsymbol{C}^T\boldsymbol{D_B}^3\emptyset(x) + 2\alpha\, Re\,\left(\boldsymbol{C}^T\,\emptyset(x)\right)\left(\boldsymbol{C}^T\boldsymbol{D_B}\,\emptyset(x)\right) + (4-Ha)\,\alpha^2\,\left(\boldsymbol{C}^T\boldsymbol{D_B}\,\emptyset(x)\right) = 0,\quad \boldsymbol{C}^T\,\emptyset(0) = 1,\ \boldsymbol{C}^T\,\boldsymbol{D_B}\,\emptyset(0) = 0,\ \boldsymbol{C}^T\,\emptyset(1) = 0 \qquad 46$$

By implementing the processes as presented in Eqs.19, 20, Eq.47 will be shown

$$\langle \boldsymbol{B}_{j,n}(x),\ \boldsymbol{C}^T\boldsymbol{D_B}^3\emptyset(x) + 2\alpha\, Re\,\left(\boldsymbol{C}^T\,\emptyset(x)\right)\left(\boldsymbol{C}^T\boldsymbol{D_B}\,\emptyset(x)\right) + (4-Ha)\,\alpha^2\,\left(\boldsymbol{C}^T\boldsymbol{D_B}\,\emptyset(x)\right)\rangle = \langle \boldsymbol{B}_{j,n}(x), 0\rangle,\quad \forall\, j = 0,1,2,\dots,n. \qquad 47$$

Substituting the Eqs.32, and 34 for D-ECM based on the Legendre polynomials, it follows that:

$$\boldsymbol{C}^T\,\boldsymbol{D_P}^3\,\emptyset(x) + 2\alpha\, Re\,\left(\boldsymbol{C}^T\,\emptyset(x)\right)\left(\boldsymbol{C}^T\,\boldsymbol{D_P}\,\emptyset(x)\right) + (4-Ha)\,\alpha^2\,\left(\boldsymbol{C}^T\,\boldsymbol{D_P}\,\emptyset(x)\right) = 0,\quad \boldsymbol{C}^T\,\emptyset(0) = 1,\ \boldsymbol{C}^T\,\boldsymbol{D_P}\,\emptyset(0) = 0,\ \boldsymbol{C}^T\,\emptyset(1) = 0 \qquad 48$$

Moreover, using the techniques given in the Eqs.19, and 20, the following equation will be obtained:

$$\langle \boldsymbol{P}_i(x),\ \boldsymbol{C}^T\,\boldsymbol{D_P}^3\,\emptyset(x) + 2\alpha\, Re\,\left(\boldsymbol{C}^T\,\emptyset(x)\right)\left(\boldsymbol{C}^T\,\boldsymbol{D_P}\,\emptyset(x)\right) + (4-Ha)\,\alpha^2\,\left(\boldsymbol{C}^T\,\boldsymbol{D_P}\,\emptyset(x)\right)\rangle = \langle \boldsymbol{P}_i(x), 0\rangle,\quad \forall\, 0 \le i \le m \qquad 49$$

Furthermore, applying Eqs.36, 41 for D-ECM based on the Hermite polynomials, it follows:

$$\boldsymbol{X}(x)\boldsymbol{G}^3\,((\boldsymbol{D_M})^{-1})^T\boldsymbol{C} + 2\alpha\, Re\,(\emptyset(x)\,\boldsymbol{C})\,(\boldsymbol{X}(x)\boldsymbol{G}\,((\boldsymbol{D_M})^{-1})^T\boldsymbol{C}) + (4-Ha)\,\alpha^2\,(\boldsymbol{X}(x)\,\boldsymbol{G}\,((\boldsymbol{D_M})^{-1})^T\boldsymbol{C}) = 0,\quad \emptyset(0)\,\boldsymbol{C} = 1,\ \boldsymbol{X}(0)\,\boldsymbol{G}\,((\boldsymbol{D_M})^{-1})^T\boldsymbol{C} = 0,\ \emptyset(1)\,\boldsymbol{C} = 0 \qquad 50$$

Then, using the procedures as given in Eqs.19, 20, so:

$$\langle \boldsymbol{H}_i(x),\ \boldsymbol{X}(x)\boldsymbol{G}^3\,((\boldsymbol{D_M})^{-1})^T\boldsymbol{C} + 2\alpha\, Re\,(\emptyset(x)\,\boldsymbol{C})\,(\boldsymbol{X}(x)\,\boldsymbol{G}\,((\boldsymbol{D_M})^{-1})^T\boldsymbol{C}) + (4-Ha)\,\alpha^2\,(\boldsymbol{X}(x)\,\boldsymbol{G}\,((\boldsymbol{D_M})^{-1})^T\boldsymbol{C})\rangle = \langle \boldsymbol{H}_i(x), 0\rangle, \forall\, i = 0,1,\dots,K. \qquad 51$$

Then, the values of $\boldsymbol{C} = [c_0\ c_1\ c_2 \dots c_m]^T$ are calculated by solving the algebraic system obtained by the inner product for the left and right sides, from Eqs.43, 45, 47, 49, and 51, respectively. Subsequently, applying the boundary conditions on the Eqs.42, 44, 46, 48, and 50 leads to obtaining the approximate solution.

The approximate polynomials for the Jeffery-Hamel flow problem when the parameter values are as follows: $\alpha = 5°, Re = 10, Ha = 0$ as in [30], with n=12, will be:

- By using ECM based on the standard monomial polynomial,

$$w(x) \approx 1. - 1.12597\,x^2 + 8.4681 * 10^{-7}x^3 + 0.166615\,x^4 + 0.0000643873\,x^5 - 0.0470176\,x^6 + 0.000792892\,x^7 + 0.00575024\,x^8 + 0.00218839\,x^9 - 0.0035073\,x^{10} + 0.00126349\,x^{11} - 0.000175929\,x^{12}.$$

- By using D-ECM based on the first kind of the Chebyshev polynomials,

$$
\begin{aligned}
w(x) \approx 1. -1.12597\, x^2 + 8.79819 * 10^{-8} x^3 &\\
+ 0.166622\, x^4 &\\
+ 0.000024296\, x^5 &\\
- 0.046882\, x^6 &\\
+ 0.000494395\, x^7 &\\
+ 0.00618549\, x^8 &\\
+ 0.00177057\, x^9 &\\
- 0.00325326\, x^{10} &\\
+ 0.00117476\, x^{11} &\\
- 0.000162363\, x^{12}. &
\end{aligned}
$$

- By using D-ECM based on the Bernstein polynomials,

$$
\begin{aligned}
w(x) \approx 1. -1.12597\, x^2 + 5.81122 * 10^{-7} x^3 &\\
+ 0.166618\, x^4 &\\
+ 0.0000498819\, x^5 &\\
- 0.0469669\, x^6 &\\
+ 0.000676843\, x^7 &\\
+ 0.00592638\, x^8 &\\
+ 0.00201225\, x^9 &\\
- 0.00339574\, x^{10} &\\
+ 0.00122292\, x^{11} &\\
- 0.000169476\, x^{12}. &
\end{aligned}
$$

- By using D-ECM based on the Legendre polynomials,

$$
\begin{aligned}
w(x) \approx 1. -1.12597\, x^2 + 8.93943 * 10^{-8} x^3 &\\
+ 0.166622\, x^4 &\\
+ 0.0000245082\, x^5 &\\
- 0.0468829\, x^6 &\\
+ 0.000496724\, x^7 &\\
+ 0.00618166\, x^8 &\\
+ 0.0017746\, x^9 &\\
- 0.0032559\, x^{10} &\\
+ 0.00117574\, x^{11} &\\
- 0.000162521\, x^{12}. &
\end{aligned}
$$

- By using D-ECM based on the Hermite polynomials,

$$
\begin{aligned}
w(x) \approx 1. -1.12597\, x^2 + 6.8438 * 10^{-8} x^3 &\\
+ 0.166623\, x^4 &\\
+ 0.0000209462\, x^5 &\\
- 0.0468671\, x^6 &\\
+ 0.000454542\, x^7 &\\
+ 0.00625298\, x^8 &\\
+ 0.00169763\, x^9 &\\
- 0.00320442\, x^{10} &\\
+ 0.00115627\, x^{11} &\\
- 0.000159336\, x^{12}. &
\end{aligned}
$$

The Numerical Results and Discussion:

In this section, an example is presented when the value of $n = 3, \alpha = 5°, Re = 10,$ and $Ha = 0$, to illustrate the approach of the proposed methods to solve the Jeffery-Hamel flow problem.

To explain the technique of ECM, by using the Eqs.12, and 14, it follows:

$$
\begin{aligned}
w(x) &= \boldsymbol{X\, C} \\
&= \varphi_0 c_0 + \varphi_1 c_1 + \varphi_2 c_2 \\
&+ \varphi_3 c_3, \qquad 52
\end{aligned}
$$

where, $\varphi_0 = 1$, $\varphi_1 = x$, $\varphi_2 = x^2$, $\varphi_3 = x^3$, and the derivatives of $w(x)$ as matrices, expressed as:

$$
w'(x) = \boldsymbol{X\, B\, C} = [\varphi_0 \ \ \varphi_1 \ \ \varphi_2 \ \ \varphi_3] \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} [c_0\ c_1\ c_2\ c_3]^T,
$$

and,

$$
w'''(x) = \boldsymbol{X\, B}^3\, \boldsymbol{C} = [\varphi_0 \ \ \varphi_1 \ \ \varphi_2 \ \ \varphi_3] \begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} [c_0\ c_1\ c_2\ c_3]^T.
$$

Now, substituting the $w'(x)$, $w'''(x)$ in Eqs.6, 7, and applying the inner product to solve the left and right sides of the matrix equation given in Eq.43, with boundary conditions Eq.42, four nonlinear algebraic equations for unknown $c_0, c_1, c_2, c_3,$ can be obtained as:

$$
\begin{aligned}
\frac{100}{3}{}^{\circ 2}\, c_1 + \frac{100}{3}{}^{\circ}\, c_0\, c_1 + 25^{\circ}\, {c_1}^2 + 50^{\circ 2}\, c_2 &\\
+ 50^{\circ}\, c_0 c_2 + 60^{\circ}\, c_1\, c_2 + \frac{5}{27}\pi\, {c_2}^2 &\\
+ 2\, c_3 + 60^{\circ 2}\, c_3 + 60^{\circ}\, c_0 c_3 &\\
+ \frac{10}{27}\pi\, c_1\, c_3 + \frac{25}{63}\pi\, c_2\, c_3 &\\
+ \frac{5}{24}\pi\, {c_3}^2 = 0, &\\
c_0 = 1, &\\
c_1 = 0, &
\end{aligned}
$$

$$c_0 + c_1 + c_2 + c_3 = 0.$$

By solving this system numerically by Mathematica®12, unique values of c_0, c_1, c_2, c_3 are given as follows:

$c_0 = 1, c_1 = 0,\ c_2 = -1.14626,\ c_3 = 0.146262.$

Hence, the values of c_0, c_1, c_2, c_3, will be substituted in Eq.52 to obtain an approximate solution to the Eq.6, as:

$$w(x) \approx 1. -1.14626\, x^2 + 0.146262\, x^3.$$

Using the Eqs.22, and 26, the following is a description of D-ECM based on the first kind of the Chebyshev polynomials technique:

$$w(x) = \boldsymbol{C}^T\, \emptyset(x)$$
$$= c_0\boldsymbol{T}_0(x) + c_1\boldsymbol{T}_1(x) + c_2\boldsymbol{T}_2(x) + c_3\boldsymbol{T}_3(x), \qquad 53$$

where, $\boldsymbol{T}_0(x) = 1$, $\boldsymbol{T}_1(x) = x$, $\boldsymbol{T}_2(x) = -1 + 2x^2$, $\boldsymbol{T}_3(x) = -3x + 4x^3$, and the derivatives $w'(x)$, $w'''(x)$ as matrices, can be given as:

$$w'(x) = \boldsymbol{C}^T \boldsymbol{D}_{\boldsymbol{T}}\, \emptyset(x) = [c_0\ c_1\ c_2\ c_3]\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 3 & 0 & 6 & 0 \end{pmatrix}[\boldsymbol{T}_0(x), \boldsymbol{T}_1(x), \boldsymbol{T}_2(x), \boldsymbol{T}_3(x)]^T,$$

Also,

$$w'''(x) = \boldsymbol{C}^T {\boldsymbol{D}_{\boldsymbol{T}}}^3\, \emptyset(x) = [c_0\ c_1\ c_2\ c_3]\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 \end{pmatrix}[\boldsymbol{T}_0(x), \boldsymbol{T}_1(x), \boldsymbol{T}_2(x), \boldsymbol{T}_3(x)]^T.$$

Therefore, substituting the $w'(x)$, $w'''(x)$ in Eqs.6, 7, and employing the inner product of the matrix equation given in Eq.45, with boundary conditions Eq.44, four nonlinear algebraic equations for unknown c_0, c_1, c_2, c_3, are achieved as:

$$-\frac{100}{3}{}^{\circ 2}\, c_1 - \frac{100}{3}{}^{\circ}\, c_0\, c_1 + \frac{220}{3}{}^{\circ}\, c_1\, c_2 - 200^{\circ}\, {c_2}^2 + \frac{40}{27}\pi\, {c_2}^2 - 8\, c_3 + 180^{\circ 2}\, c_3 + 180^{\circ}\, c_0 c_3 - 400^{\circ}\, c_1 c_3 + \frac{80}{27}\pi a\, c_1 c_3 - 1100^{\circ}\, c_2\, c_3 + \frac{400}{63}\pi\, c_2\, c_3 + 1200^{\circ}\, {c_3}^2 - \frac{20}{3}\pi\, {c_3}^2 = 0,$$
$$c_0 - c_2 = 1,$$
$$c_1 - 3c_3 = 0,$$
$$c_0 + c_1 + c_2 + c_3 = 0.$$

Solving this system numerically by Mathematica®12, the following unique values of c_0, c_1, c_2, c_3 will be obtained:

$$c_0 = 0.419839,\ c_1 = 0.120242,\ c_2 = -0.580161,\ \ c_3 = 0.0400806.$$

Hence, the values of c_0, c_1, c_2, c_3, will be substituted in Eq.53 to obtain an approximate solution to the Eq.6, as:

$$w(x) \approx 1. - 1.16032\, x^2 + 0.160322\, x^3.$$

By implementing the Eqs.28, 30, for D-ECM based on the Bernstein polynomials, the following is obtained:

$$w(x) = \boldsymbol{C}^T\, \emptyset(x)$$
$$= c_0\boldsymbol{B}_{0,3} + c_1\boldsymbol{B}_{1,3} + c_2\boldsymbol{B}_{2,3} + c_3\boldsymbol{B}_{3,3}, \qquad 54$$

where, $\boldsymbol{B}_{0,3} = 1 - 3x + 3x^2 - x^3$, $\boldsymbol{B}_{1,3} = 3x - 6x^2 + 3x^3$, $\boldsymbol{B}_{2,3} = 3x^2 - 3x^3$, $\boldsymbol{B}_{3,3} = x^3$, as matrices, the derivatives $w'(x)$, $w'''(x)$ may be written as:

$$w'(x) = \boldsymbol{C}^T \boldsymbol{D}_{\boldsymbol{B}}\, \emptyset(x) = [c_0\ c_1\ c_2\ c_3]\begin{pmatrix} -3 & -1 & 0 & 0 \\ 3 & -1 & -2 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 3 \end{pmatrix}\left[\boldsymbol{B}_{0,3}, \boldsymbol{B}_{1,3}, \boldsymbol{B}_{2,3}, \boldsymbol{B}_{3,3}\right]^T,$$

and,

$$w'''(x) = \boldsymbol{C}^T {\boldsymbol{D}_{\boldsymbol{B}}}^3\, \emptyset(x) = [c_0\ c_1\ c_2\ c_3]\begin{pmatrix} -6 & -6 & -6 & -6 \\ 18 & 18 & 18 & 18 \\ -18 & -18 & -18 & -18 \\ 6 & 6 & 6 & 6 \end{pmatrix}\left[\boldsymbol{B}_{0,3}, \boldsymbol{B}_{1,3}, \boldsymbol{B}_{2,3}, \boldsymbol{B}_{3,3}\right]^T.$$

Thus, if the $w'(x)$, $w'''(x)$ substituting into Eqs.6, 7, and using the inner product of the matrix equation from Eq.47 with the boundary conditions from Eq.46, four nonlinear algebraic equations for unknown c_0, c_1, c_2, c_3, are attained as follows:

$$-\frac{3c_0}{2}-15^{\circ 2}\,c_0-\frac{25}{7}{}^{\circ}\,{c_0}^2+\frac{9c_1}{2}-15^{\circ 2}\,c_1-\frac{75}{14}{}^{\circ}\,c_0\,c_1-\frac{45}{14}{}^{\circ}\,{c_1}^2-\frac{9c_2}{2}-\frac{15}{7}{}^{\circ}\,c_0\,c_2-\frac{45}{14}{}^{\circ}\,c_1\,c_2+\frac{3c_3}{2}+30^{\circ 2}\,c_3-\frac{5}{14}{}^{\circ}\,c_0\,c_3+\frac{75}{14}{}^{\circ}\,c_2\,c_3+\frac{25}{2}{}^{\circ}\,{c_3}^2=0,$$

$$c_0=1,$$

$$-3c_0+3c_1=0,$$

$$c_3=0.$$

The following unique values of c_0, c_1, c_2, c_3 will be found by numerically solving this system with Mathematica®12:

$$c_0=1,\ \ c_1=1,\ \ c_2=0.60497,\ \ \ c_3=0.$$

To achieve an approximate solution to Eq.6, the values of c_0, c_1, c_2, c_3 will be substituted in Eq.54, as follows:

$$w(x)\approx 1.-1.18509\,x^2+0.185091\,x^3$$

By applying the Eqs.32, and 34, for D-ECM based on the Legendre polynomials, the following is achieved:

$$w(x)=\boldsymbol{C}^T\,\emptyset(x)=c_0\boldsymbol{P}_0(x)+c_1\boldsymbol{P}_1(x)+c_2\boldsymbol{P}_2(x)+c_3\boldsymbol{P}_3(x), \qquad 55$$

where, $\boldsymbol{P}_0(x)=1,\ \boldsymbol{P}_1(x)=x,\ \boldsymbol{P}_2(x)=-\frac{1}{2}+\frac{3x^2}{2}$, $\boldsymbol{P}_3(x)=-\frac{3x}{2}+\frac{5x^3}{2}$, and the derivatives of $w(x)$, can be written as matrices:

$$w'(x)=\boldsymbol{C}^T\boldsymbol{D_P}\,\emptyset(x)=[c_0\ c_1\ c_2\ c_3]\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&3&0&0\\1&0&5&0\end{pmatrix}[\boldsymbol{P}_0(x),\boldsymbol{P}_1(x),\boldsymbol{P}_2(x),\boldsymbol{P}_3(x)]^T,$$

and,

$$w'''(x)=\boldsymbol{C}^T{\boldsymbol{D_p}}^3\,\emptyset(x)=[c_0\ c_1\ c_2\ c_3]\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\15&0&0&0\end{pmatrix}[\boldsymbol{P}_0(x),\boldsymbol{P}_1(x),\boldsymbol{P}_2(x),\boldsymbol{P}_3(x)]^T.$$

In addition, four nonlinear algebraic equations with unknowns c_0, c_1, c_2, c_3, are obtained by substituting $w'(x)$ and $w'''(x)$ in Eqs. 6, 7, and applying the inner product of the matrix equation from Eq.49 with the boundary conditions from Eq.48:

$$\frac{25}{2}{}^{\circ}\,{c_1}^2+\frac{75}{2}{}^{\circ 2}\,c_2+\frac{75}{2}{}^{\circ}\,c_0\,c_2+60^{\circ}\,c_1\,c_2+\frac{75}{2}{}^{\circ}\,{c_2}^2+100^{\circ 2}\,c_3+100^{\circ}\,c_0c_3+\frac{175}{2}{}^{\circ}\,c_1c_3+\frac{520}{7}{}^{\circ}\,c_2\,c_3+\frac{575}{16}{}^{\circ}\,{c_3}^2=0,$$

$$c_0-\frac{c_2}{2}=1,$$

$$c_1-\frac{3\,c_3}{2}=0,$$

$$c_0+\ c_1+\ c_2+\ c_3=0.$$

Then, using Mathematica®12, to solve this system numerically, the following unique values of c_0, c_1, c_2, c_3, will be obtained:

$$c_0=0.648645,\ \ c_1=0.0324384,\ \ c_2=-0.702709,\ \ \ c_3=0.0216256.$$

As a result, the values c_0, c_1, c_2, c_3, will be substituted in Eq.55 to give an approximate solution to Eq.6, as follows:

$$w(x)\approx 1.-1.05406\,x^2+0.0540639\,x^3$$

Moreover, by using the Eqs. 36, 41, a description of the D-ECM based on the Hermite polynomials procedure follows:

$$w(x)=\emptyset(x)\,\boldsymbol{C}=\boldsymbol{H}_0(x)c_0+\boldsymbol{H}_1(x)c_1+\boldsymbol{H}_2(x)c_2+\boldsymbol{H}_3(x)c_3, \qquad 56$$

where, $\boldsymbol{H}_0(x)=1,\ \boldsymbol{H}_1(x)=2\,x,\ \boldsymbol{H}_2(x)=-2+4x^2,\ \boldsymbol{H}_3(x)=-12x+8x^3$, and the derivatives $w'(x), w'''(x)$ as matrices can be obtained as:

$$w'(x)=[\boldsymbol{H}_0(x)\ \boldsymbol{H}_1(x)\ \boldsymbol{H}_2(x)\ \boldsymbol{H}_3(x)]\begin{pmatrix}0&2&0&0\\0&0&4&0\\0&0&0&6\\0&0&0&0\end{pmatrix}[c_0\ c_1\ c_2\ c_3]^T,$$

Also,

$$w'''(x) = [\boldsymbol{H}_0(x)\ \boldsymbol{H}_1(x)\ \boldsymbol{H}_2(x)\ \boldsymbol{H}_3(x)] \begin{pmatrix} 0 & 0 & 0 & 48 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} [c_0\ c_1\ c_2\ c_3]^T$$

Substituting $w'(x), w'''(x)$ into Eqs. 6, 7, and using the inner product of the matrix equation from Eq. 51 with the boundary conditions from Eq. 50, yields four nonlinear algebraic equations with unknowns c_0, c_1, c_2, c_3:

$$-\frac{400}{3}{}^{\circ 2}\ c_1 - \frac{400}{3}{}^{\circ}\ c_0\ c_1 + \frac{1760}{3}{}^{\circ}\ c_1\ c_2 + \frac{1600}{3}{}^{\circ}\ c_2{}^2 - 32c_3 + 1120^{\circ 2}\ c_3 + 1120^{\circ}\ c_0\ c_3 + \frac{3200}{3}{}^{\circ}\alpha\ c_1\ c_3 - \frac{9920}{7}{}^{\circ}\ c_2\ c_3 - 3200^{\circ}\ c_3{}^2 = 0,$$
$$c_0 - 2\ c_2 = 1,$$
$$2c_1 - 12c_3 = 0,$$
$$c_0 + 2c_1 + 2c_2 - 4c_3 = 0.$$

Then, using Mathematica®12, solve this system numerically to acquire the following unique values of c_0, c_1, c_2, c_3:

$$c_0 = 0.419839,\ \ c_1 = 0.120242,\ \ c_2 = -0.290081,\ \ c_3 = 0.0200403.$$

As a consequence, the values c_0, c_1, c_2, c_3, will be swapped in Eq.56 to get the following approximate solution to Eq.6:

$$w(x) \approx 1. -1.16032\ x^2 + 0.160322\ x^3.$$

Furthermore, the maximal error remainder MER_n has been introduced in this section because there is no exact solution available to the problem, as well as to verify the accuracy and reliability of the approximate solution obtained by ECM and D-ECM. The MER_n is calculated by:

$$MER_n = \max_{0\le x\le 1} |w'''(x) + 2\alpha\ Re\ w(x)\ w'(x) + (4-Ha)\ \alpha^2\ w'(x)|$$

Fig. 2 presents the logarithmic plots for the MER_n values, obtained by the ECM based on the standard monomial polynomial, as well as, by the D-ECM based on the Chebyshev, Bernstein, Legendre, and Hermite polynomials, for parameters $Re = 10$, $Ha = 0$ and $\alpha = 5^\circ$ according to previous studies [30], which showed the efficiency of these methods by observing the error values for $n =$ 4 to 12, the error was observed to be lower when the value of n increased.

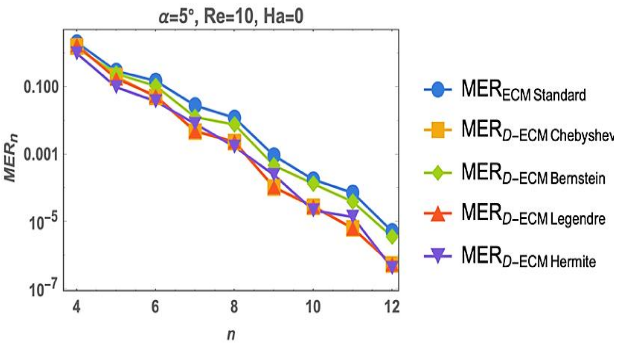

Figure 2. Logarithmic plots for MER_n by proposed methods.

A comparison of the approximate solutions obtained using the proposed techniques is also shown in Fig. 3 for $n = 12$, $Re = 10$, $Ha = 0$, and $\alpha = 5^\circ$, as is evident from the figure, good agreements have been obtained for all proposed methods.

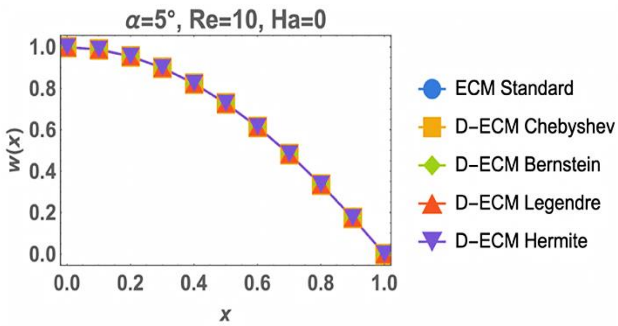

Figure 3. Solutions of the Jeffery–Hamel by proposed methods for $n = 12$.

Moreover, in Table 1 the values of MER_n for the approximate solution is given by using ECM and D-ECM with $n = 12$ and parameters $Re = 10, Ha = 0$ and versus the value of α, which appears the efficiency of these methods. In addition, it can be noted that D-ECM based on the Hermite polynomials method produces better accuracy with the lowest errors compared to the other methods.

Table 1. The MER_{12} when $Re = 10, Ha = 0$ and versus the value of α, for Jeffery– Hamel flow

α	ECM Standard	D-ECM Chebyshev	D-ECM Bernstein	D-ECM Legendre	D-ECM Hermite
3°	$1.78573 * 10^{-6}$	$1.55044 * 10^{-7}$	$1.07042 * 10^{-6}$	$1.57835 * 10^{-7}$	$1.16736 * 10^{-7}$
-3°	$3.09536 * 10^{-6}$	$2.15838 * 10^{-7}$	$1.55861 * 10^{-6}$	$2.20333 * 10^{-7}$	$1.60927 * 10^{-7}$
-5°	0.0000152937	$1.01151 * 10^{-6}$	$7.35397 * 10^{-6}$	$1.03335 * 10^{-6}$	$8.15967 * 10^{-7}$

Furthermore, in Table 2 the comparisons of MER_{12} values are presented when $Re = 10, Ha = 0, \alpha = 5°$, for the solutions by proposed methods and by the Chebyshev and the Bernstein operational matrices methods according to previous studies [30]. Better accuracy can be realized by using the suggested methods.

Table 2. The comparison between the MER_{12} when $Re = 10, Ha = 0, \alpha = 5°$ by proposed methods and by Chebyshev and Bernstein [30].

ECM Standard	D-ECM Chebyshev	D-ECM Bernstein	D-ECM Legendre	D-ECM Hermite	Chebyshev [30]	Bernstein [30]
$\mathbf{5.08086 * 10^{-6}}$	$5.27892 * 10^{-7}$	$3.48673 * 10^{-6}$	$5.36366 * 10^{-7}$	$4.10628 * 10^{-7}$	$3.3003 * 10^{-5}$	$9.68873 * 10^{-6}$

Also, Figs.4-7 illustrate the velocity profiles for the Jeffery–Hamel problem in the cases $\alpha = 5°, \alpha = -5°$ with fixed $Re = 50$ and increasing values of Ha, as chosen in [17]. The velocity is noted to be increased by increasing Ha values in all the figures. The curvature of the curves also increases with increasing Ha values.

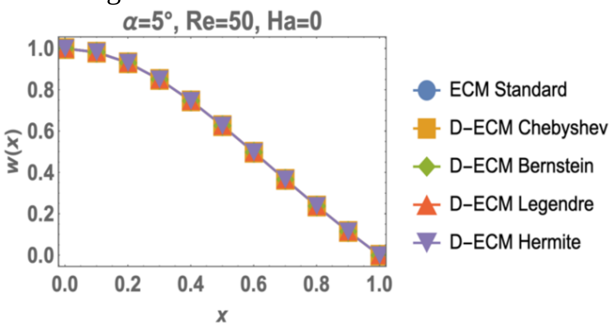

Figure 4. The velocity plot for Jeffery–Hamel by proposed methods for $Ha = 0$.

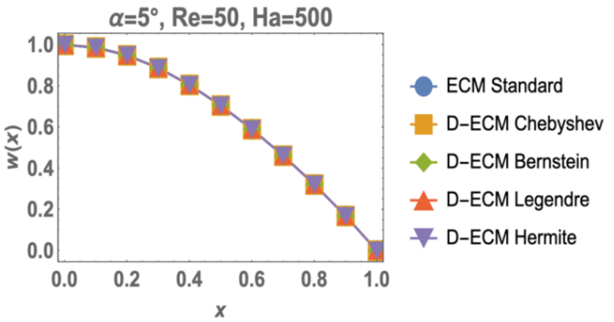

Figure 5. The velocity plot for Jeffery–Hamel by proposed methods for $Ha = 500$.

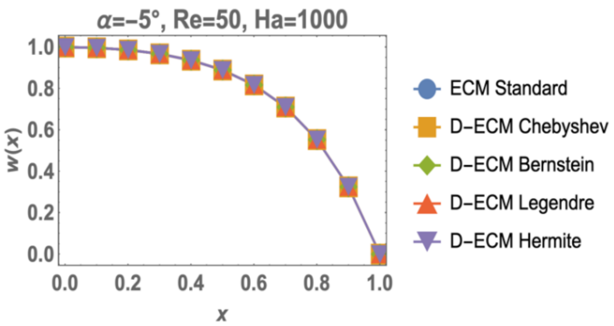

Figure 6. The velocity plot for Jeffery–Hamel by proposed methods for $Ha = 1000$.

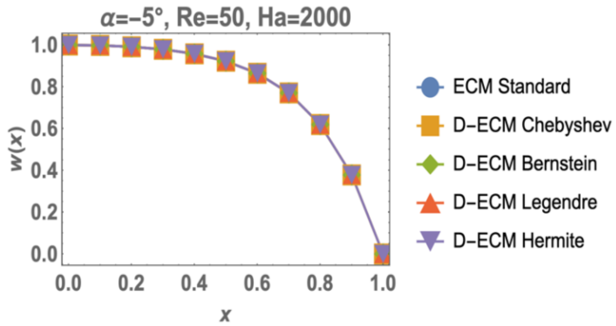

Figure 7. The velocity plot for Jeffery–Hamel by proposed methods for $Ha = 2000$.

Conclusion:

The effective computational method and novel computational methods with suitable base functions, namely Chebyshev, Bernstein, Legendre, and Hermite polynomials, have been presented in this paper for solving the Jeffery-Hamel problem. The nonlinear problems are reduced to the solution of a nonlinear algebraic system of equations, which is processed using Mathematica®12. The approximate solution is accurate and efficient even within a few orders of polynomials. In addition, the MER_n has been calculated for the proposed methods and compared with the Chebyshev and the Bernstein operational matrices methods that are available in the literature, the results obtained showed that the proposed methods have produced better accuracy with less errors. Moreover, it can be concluded that the results of the MER_n by the proposed methods D-ECM decreased significantly compared to ECM, which gives higher accuracy and efficiency. Furthermore, it was found that the results of D-ECM based on the Hermite polynomials are better than the other methods.

The present methods can also be extended to partial differential equations and fractional differential equations, which certainly require extensive further analysis.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

Authors' Contributions:

OMS contributed to the design and implementation of the research, the analysis of the results, and the writing of the manuscript. MA.AJ interpretation, drafting, revision, proofreading, and verifying the analytic approximate methods of the manuscript. The authors discussed the results and contributed to the final manuscript.

References:

1. Hermann M, Saravi M. Nonlinear Ordinary Differential Equations: Analytical Approximation and Numerical Methods. India: Springer; 2016. 320 p.
2. AL-Jawary MA, Salih OM. Reliable iterative methods for 1D Swift–Hohenberg equation. Arab J Basic Appl Sci. 2020; 27(1): 56-66.
3. Abed AM, Younis MF, Hamoud AA. Numerical solutions of nonlinear Volterra-Fredholm integro-differential equations by using MADM and VIM. Nonlinear Funct Anal Appl. 2022; 27(1): 189-201.
4. Hasan PM, Sulaiman NA. Convergence Analysis for the Homotopy Perturbation Method for a Linear System of Mixed Volterra-Fredholm Integral Equations. Baghdad Sci J. 2020; 17(3(Suppl.): 1010-1018.
5. Hamoud AA, Hussain KH, Mohammed NM, Ghadle KP. Solving Fredholm integro-differential equations by using numerical techniques. Nonlinear Funct. Anal Appl. 2019; 24(3): 533-542.
6. Adel W, Sabir Z. Solving a new design of nonlinear second-order Lane–Emden pantograph delay differential model via Bernoulli collocation method. Eur Phys J Plus. 2020; 135(5): 1-12.
7. Yari A, Mirnia MK. Solving optimal control problems by using Hermite polynomials. Comput methods differ equ. 2020; 8(2): 314-329.
8. Laib H, Bellour A, Boulmerka A. Taylor collocation method for a system of nonlinear Volterra delay integro-differential equations with application to COVID-19 epidemic. Int J Comput Math. 2022; 99(4): 852-876.
9. Srivastava HM, Shah FA, Abass R. An application of the Gegenbauer wavelet method for the numerical solution of the fractional Bagley-Torvik equation. Russ. J Math Phys. 2019; 26(1): 77-93.
10. Singh H. Jacobi collocation method for the fractional advection-dispersion equation arising in porous media. Numer. Methods Partial Differ Equ. 2022; 38(3): 636-653.
11. Ganji RM, Jafari H, Baleanu D. A new approach for solving multi variable orders differential equations with Mittag–Leffler kernel. Chaos Solit Fractals. 2020; 130: 109405.
12. Singha N, Nahak C. A numerical method for solving a class of fractional optimal control problems using Boubaker polynomial expansion scheme. Filomat. 2018; 32(13): 4485-4502.
13. Yuttanan B, Razzaghi M. Legendre wavelets approach for numerical solutions of distributed order fractional differential equations. Appl Math Model. 2019; 70: 350-364.
14. Salih OM, AL-Jawary MA. Reliable iterative methods for solving 1D, 2D and 3D Fisher's equation. IIUM Eng J. 2021; 22(1): 138-166.
15. Hamoud AA, Mohammed NM, Ghadle KP. Some powerful techniques for solving nonlinear Volterra-Fredholm integral equations. J Appl Nonlinear Dyn. 2021; 10(3): 461-469.
16. Hussin CHC, Azmi A, Ismail AIM, Kilicman A, Hashim I. Approximate Analytical Solutions of Bright Optical Soliton for Nonlinear Schrödinger Equation of Power Law Nonlinearity. Baghdad Sci J. 2021 March 30; 18(1(Suppl.)): 836-845.
17. Bougoffa L, Mziou S, Rach RC. Exact and approximate analytic solutions of the Jeffery-Hamel flow problem by the Duan-Rach modified Adomian decomposition method. Math Model Anal. 2016; 21(2): 174-187.
18. Jeffery GBL. The two-dimensional steady motion of a viscous fluid. Lond. Edinb. Dublin philos. Mag J sci. 1915; 29(172): 455-465.
19. Li Z, Khan I, Shafee A, Tlili I, Asifa T. Energy transfer of Jeffery–Hamel nanofluid flow between non-parallel walls using Maxwell–Garnetts (MG) and Brinkman models. Energy Rep. 2018; 4: 393-399.
20. Bildik N, Deniz S. Optimal iterative perturbation technique for solving Jeffery–Hamel flow with high magnetic field and nanoparticle. J Appl Anal Comput. 2020; 10(6): 2476-2490.
21. Bataineh AS, Isik OR, Hashim I. Bernstein Collocation Method for Solving MHD Jeffery–Hamel Blood Flow Problem with Error Estimations. Int J Differ Equ. 2022; 2022: 9 pages.
22. Nourazar SS, Nazari-Golshan A, Soleymanpour F. On the expedient solution of the magneto-hydrodynamic Jeffery-Hamel flow of Casson fluid. Sci Rep. 2018; 8(1): 1-16.
23. Patel HS, Meher R. Analytical Investigation of Jeffery–Hamel Flow by Modified Adomian Decomposition. Ain Shams Eng J. 2018; 9(4): 599-606.
24. Biswal U, Chakraverty S. Investigation of Jeffery-Hamel flow for nanofluid in the presence of magnetic field by a new approach in the optimal homotopy analysis method. J Appl Comput Mech. 2022; 8(1): 48-59.
25. Ahmad I, Ilyas H. Homotopy Perturbation Method for the nonlinear MHD Jeffery–Hamel blood flows problem. Appl Numer Math. 2019; 141: 124-132.
26. Adel W, Biçer KE, Sezer M. A Novel Numerical Approach for Simulating the Nonlinear MHD

Jeffery–Hamel Flow Problem. Int J Appl Comput Math. 2021; 7(3): 1-15.
27. Kumbinarasaiah S, Raghunatha KR. Numerical solution of the Jeffery–Hamel flow through the wavelet technique. Heat Transf. 2022; 51(2): 1568-1584.
28. Meher R, Patel ND. Analytical Investigation of MHD Jeffery–Hamel flow problem with heat transfer by differential transform method. SN. Appl Sci. 2019; 1(7): 1-12.
29. AL-Jawary MA, Abdul Nabi AJ. Three Iterative Methods for Solving Jeffery-Hamel Flow Problem. Kuwait J Sci. 2020; 47(1): 1-13.
30. AL-Jawary MA, Ibraheem GH. Two meshless methods for solving nonlinear ordinary differential equations in engineering and applied sciences. Nonlinear Eng. 2020; 9(1): 244-255.
31. Turkyilmazoglu M. Effective computation of exact and analytic approximate solutions to singular nonlinear equations of Lane–Emden–Fowler type. Appl Math Model. 2013; 37(14-15): 7539-7548.
32. Turkyilmazoglu M. An effective approach for numerical solutions of high-order Fredholm integro-differential equations. Appl Math Comput. 2014; 227(15): 384-398.
33. Turkyilmazoglu M. High-order nonlinear Volterra–Fredholm-Hammerstein integro-differential equations and their effective computation. Appl Math Comput. 2014; 247: 410-416.
34. Turkyilmazoglu M. Effective Computation of Solutions for Nonlinear Heat Transfer Problems in Fins. J Heat Transfer. 2014; 136(9): 091901.
35. Turkyilmazoglu M. Solution of initial and boundary value problems by an effective accurate method. Int J Comput Methods. 2017; 14(06): 1750069.
36. Secer A, Ozdemir N, Bayram M. A Hermite Polynomial Approach for Solving the SIR Model of Epidemics. Mathematics. 2018; 6(12): 305.
37. Al-Humedi HO, Al-Saadawi FA. The Numerical Technique Based on Shifted Jacobi-Gauss-Lobatto Polynomials for Solving Two Dimensional Multi-Space Fractional Bioheat Equations. Baghdad Sci J. 2020; 17(4): 1271-1282.
38. Talaei Y, Asgari M. An operational matrix based on Chelyshkov polynomials for solving multi-order fractional differential equations. Neural Comput Appl. 2018; 30(5): 1369-1376.
39. Sharma B, Kumar S, Paswan MK, Mahato D. Chebyshev Operational Matrix Method for Lane-Emden Problem. Nonlinear Eng. 2019; 8(1): 1-9.
40. Khataybeh SN, Hashim I, Alshbool M. Solving directly third-order ODEs using operational matrices of Bernstein polynomials method with applications to fluid flow equations. J King Saud Univ Sci. 2019; 31(4): 822-826.
41. Pourbabaee M, Saadatmandi A. A novel Legendre operational matrix for distributed order fractional differential equations. Appl Math Comput. 2019; 361: 215-231.
42. Ibraheem GH, AL-Jawary MA. The operational matrix of Legendre polynomials for solving nonlinear thin film flow problems. Alexandria Eng J. 2020; 59(5): 4027-4033.
43. Kosunalp HY, Gülsu M. Operational Matrix by Hermite Polynomials for Solving Nonlinear Riccati Differential Equations. Int J Math Comput Sci. 2021; 16(2): 525-536.
44. Kadkhoda N. A numerical approach for solving variable order differential equations using Bernstein polynomials. Alexandria Eng J. 2020; 59(5): 3041-3047.
45. Gülsu M, Yalman H, Öztürk Y, Sezer M. A New Hermite Collocation Method for Solving Differential Difference Equations. Appl Appl Math. 2011; 6(1): 116-129.

طرق حسابية فعالة لحل مشكلة تدفق جيفري-هامل

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الخلاصة:

في هذا البحث، تم تنفيذ الطريقة الحسابية الفعالة (ECM) المستندة إلى متعددة الحدود القياسية الأحادية لحل مشكلة تدفق جيفري-هامل غير الخطية. علاوة على ذلك، تم تطوير واقتراح الطرق الحسابية الفعالة الجديدة في هذه الدراسة من خلال وظائف أساسية مناسبة وهي متعددات الحدود تشيبشيف، بيرنشتاين، ليجندر، هيرمت. يؤدي استخدام الدوال الأساسية إلى تحويل المسألة غير الخطية إلى نظام جبري غير خطي من المعادلات، والذي يتم حله بعد ذلك باستخدام برنامج ماثماتيكا®١٢. تم تطبيق تطوير طرق حسابية فعالة (D-ECM) لحل مشكلة تدفق جيفري-هامل غير الخطية، ثم تم عرض مقارنة بين الطرق. علاوة على ذلك، تم حساب الحد الأقصى للخطأ المتبقي (MER_n)، لإظهار موثوقية الطرق المقترحة. تثبت النتائج بشكل مقنع أن ECM و D-ECM دقيقة وفعالة وموثوقة للحصول على حلول تقريبية للمشكلة.

الكلمات المفتاحية: الحل التقريبي، متعددات حدود بيرنشتاين، متعددات حدود تشيبشيف، متعددات حدود هيرمت، متعددات حدود ليجندر.