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Numerical Investigation, Error Analysis and Application of Joint Quadrature Scheme in Physical Sciences

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Abstract:

In this work, a joint quadrature for numerical solution of the double integral is presented. This method is based on combining two rules of the same precision level to form a higher level of precision. Numerical results of the present method with a lower level of precision are presented and compared with those performed by the existing high-precision Gauss-Legendre five-point rule in two variables, which has the same functional evaluation. The efficiency of the proposed method is justified with numerical examples. From an application point of view, the determination of the center of gravity is a special consideration for the present scheme. Convergence analysis is demonstrated to validate the current method.

Keywords: Center of gravity, Convergence analysis, Density function, Degree of Precision, Joint quadrature, Maclaurin's series.
MSC 2010:65D30, 65D32

Introduction:

Numerical techniques are the best fit for solving different integral problems. Recently, different types of quadrature rules are used in the field of numerical integration for the benefits of science and technology. The method of mixing quadrature rule is based on constructing a mixed quadrature rule of higher precision by taking the combination of quadrature rules of lower precision. Many authors have developed joint quadrature for numerical evaluation of real definite integrals. The idea for joint quadrature rule is initiated [1]. Later on, the extension part for real definite integrals are executed [2 - 5]. The electromagnetic field problems, electric circuit problem, hybrid quadrature rule to find the approximate solution of nonlinear Fredholm integral equation with separable kernel and approximate evaluation of real definite integrals are demonstrated [6 - 7]. A mixed quadrature rule based on Gaussian quadrature [8 - 9] for approximate evaluation of real definite integrals.

Let us consider real definite integrals of the form

$$\int_a^b f(x)dx = I \qquad 1$$

Now splitting the range into n equal parts with step size h

$a = x_0, x_1 = x_0 + h, \dots x_n = b$

$$\int_a^b f(x)dx = I = h\int_0^n f(ph + x_0)dp, \qquad 2$$

Where, $x - x_0 = ph$

Using forward interpolation method

$$I = \int_a^b f(x)dx = \left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{n^2}{2}\left(\frac{n}{3} - \frac{1}{2}\right)\Delta^2 y_0 + \cdots\right]h \qquad 3$$

Different Newton's close type rules such as Trapezoidal rule, Simpson's $\frac{1}{3}$rd rule, Simpson's $\frac{3}{8}$th rule and Weddle's rule are obtained by taking $n = 1,2,3$ and 6 respectively [10]. The Gaussian rule is also represented by

$$\int_a^b f(x)dx = I = 0.5(b - a)\int_{-1}^{1} f\left[\frac{(b+a)+(b-a)t}{2}\right]dt \qquad 4$$

Where, $[a, b] = [-1,1]$.

In the double integration the function f bounded over the square region

$$R = \{(x,y): a \le x \le b, c \le y \le d\}$$

Where,

$$\int_a^b \int_c^d f(x,y)dydx = I = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} f(\zeta,\eta)d\zeta d\eta \qquad 5$$

$\eta = 0.5[(d-c)y + (d+c)]$, $\zeta = 0.5[(b-a)x + (b+a)]$

Double integrals have various physical and geometric applications, such as area, volume, and moment of inertia about various axes. Certain gravitational effects on an ejected object can be described by the gravitational force on a point particle located at the object's center of mass. Centroid of the region can be seen as the center of mass vector of that region in the case that the mass density is constant.

When the mass density is constant it cancel s out from the numerator and denominator of the ce nter of mass.

Let $f(x,y)$ is the density function of a distribution o f mass M in a certain domain R [11]

$$M = \iint_R dx\,dy \qquad 6$$

The co-ordinate $(\bar{x},\bar{y})$ represents the center of gravity.

$$\bar{x} = \frac{1}{M}\iint_R y f(x,y)\,dx\,dy \qquad 7$$

$$\bar{y} = \frac{1}{M}\iint_R x f(x,y)\,dx\,dy \qquad 8$$

Encouraged by the excellent performance of these methods, here is a plan for joint quadrature rule for higher precision as well as application in physical sciences for real definite integrals in two variables. Here the novelty of our paper is that a quadrature rule of lower precision seven has constructed taking the joint effort of Lobatto four-point rule and Gauss Legendre- three-point rule each of precision level five which has compared with Gauss Legendre five-point rule with higher precision nine and the former rule produces better approximation than the later even if the lower precision along with same functional evaluations. Several numerical methods like block method [12-15.] finite element and transformation technique [16 - 19], B-spline collocation [20- 23] and higher degree B-spline and block method [24-26] are also highly essential to the present topic.

The content of this paper is organized in eight parts. Construction of basic quadrature rules, Gauss Legendre five-point rule and joint quadrature rule are appeared in second, third and fourth part respectively. Fifth part contains error analysis. Numerical results are displayed in sixth part. The application part is focused on in the seventh part and the eighth part contains the conclus ions.

Basic Quadrature Rule

Let us discuss some basic Gaussian and Lobatto quadrature rules [27-28]

Gauss Legendre Three-point Quadrature Rule

$$R_{GL3}(f) = \int_{-1}^{1} f(x)dx = \left[\frac{5}{9}\left\{f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right\} + \frac{8}{9}f(0)\right] \qquad 9$$

Gauss Legendre Five-point Rule

$$R_{GL5}(f) \cong \int_{-1}^{1} f(x)dx = \frac{1}{900}\left[\left(322+13\sqrt{70}\right)\{f(-\delta)+f(\delta)\} + 512f(0) + \left(322-13\sqrt{70}\right)\{f(-\sigma)+f(\sigma)\}\right] \quad 10$$

where

$$\delta = \sqrt{\frac{5-2\sqrt{\frac{10}{7}}}{9}},\quad \sigma = \sqrt{\frac{5+2\sqrt{\frac{10}{7}}}{9}}$$

Lobatto-four-point Quadrature Rule

$$\int_{-1}^{1} f(x)dx \cong R_{L4}(f) = \frac{1}{6}\left[f(-1) + f(1) + 5\left\{f\left(\frac{1}{\sqrt{5}}\right) + f\left(-\frac{1}{\sqrt{5}}\right)\right\}\right] \qquad 11$$

In the double integration the function f bounded over the square region

Joint Quadrature Rule

Let us consider double integration for real definite integral in two variables

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy = I(f) \qquad 12$$

Lobatto-four-point Quadrature Rule

The two variables representation of Lobatto-four-point quadrature rule is obtained by [29-30]

$$I(f) \cong R_{L4}(f) = \frac{1}{36}\left[\left\{f(-1,-1) + 5f\left(-1,-\frac{1}{\sqrt{5}}\right) + 5f\left(-1,\frac{1}{\sqrt{5}}\right) + f(-1,1)\right\} + 5\left\{f\left(-\frac{1}{\sqrt{5}},-1\right) + 5f\left(-\frac{1}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right) + 5f\left(-\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}}\right) + f\left(-\frac{1}{\sqrt{5}},1\right)\right\} + 5\left\{\begin{matrix} f\left(\frac{1}{\sqrt{5}},-1\right) + 5f\left(\frac{1}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right) + 5f\left(\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}}\right) + \\ +f\left(\frac{1}{\sqrt{5}},1\right)\end{matrix}\right\} + \left\{f(1,-1) + 5f\left(1,\frac{-1}{\sqrt{5}}\right) + 5f\left(1,\frac{1}{\sqrt{5}}\right) + f(1,1)\right\}\right] \quad 13$$

Gauss Legendre-three-point Quadrature Rule

The two variables representation of Gauss Legendre-three-point quadrature rule are given by [31-34]

$$I(f) \cong R_{GL3}(f) = \frac{1}{81}\left[5\left\{5f\left(-\sqrt{\frac{3}{5}},-\sqrt{\frac{3}{5}}\right)+8f\left(-\sqrt{\frac{3}{5}},0\right)+5f\left(-\sqrt{\frac{3}{5}},\sqrt{\frac{3}{5}}\right)\right\}+8\left\{5f\left(0,-\sqrt{\frac{3}{5}}\right)+8f(0,0)+5f\left(0,\sqrt{\frac{3}{5}}\right)\right\}+5\left\{5f\left(\sqrt{\frac{3}{5}},-\sqrt{\frac{3}{5}}\right)+8f\left(\sqrt{\frac{3}{5}},0\right)+5f\left(\sqrt{\frac{3}{5}},\sqrt{\frac{3}{5}}\right)\right\}\right] \quad 14$$

Where, Eq. 13 and Eq. 14 are quadrature rule of precision five. Hence

$$R_{GL3}(f) + E_{GL3}(f) = I(f) \quad 15$$

$$R_{L4}(f) + E_{L4}(f) = I(f) \quad 16$$

Where, $E_{L4}(f)$ and $E_{GL3}(f)$ are the errors due to Lobatto-four-point quadrature rule $R_{L4}(f)$ and Gauss Legendre-three-point quadrature rule $R_{GL3}(f)$ for approximating $I(f)$ by Eq. 13 and Eq. 14 respectively.

Rewriting Eq. 12 by Maclaurin's series

$$I(f) = \int_{-1}^{1}\int_{-1}^{1}\Big[f(0,0) + \{xf_{1,0}(0,0) + yf_{0,1}(0,0)\} + \frac{1}{2!}\{x^2f_{2,0}(0,0) + 2xyf_{1,1}(0,0) + y^2f_{0,2}(0,0)\} + \frac{1}{3!}\{x^3f_{3,0}(0,0) + 3x^2yf_{2,1}(0,0) + 3xy^2f_{1,2}(0,0) + y^3f_{0,3}(0,0)\} + \frac{1}{4!}\{x^4f_{4,0}(0,0) + 4x^3yf_{3,1}(0,0) + 6x^2y^2f_{2,2}(0,0) + 4xy^3f_{1,3}(0,0) + y^4f_{0,4}(0,0)\} + \frac{1}{5!}\{x^5f_{5,0}(0,0) + 5x^4yf_{4,1}(0,0) + 10x^3y^2f_{3,2}(0,0) + 10x^2y^3f_{2,3}(0,0) + 5xy^4f_{1,4}(0,0) + y^5f_{0,5}(0,0)\} + \frac{1}{6!}\{x^6f_{6,0}(0,0) + 6x^5yf_{5,1}(0,0) + 15x^4y^2f_{4,2}(0,0) + 20x^3y^3f_{3,3}(0,0) + 15x^2y^4f_{2,4}(0,0) + 6xy^5f_{1,5}(0,0) + y^6f_{0,6}(0,0)\} + \frac{1}{7!}\{x^7f_{7,0}(0,0) + 7x^6yf_{6,1}(0,0) + 21x^5y^2f_{5,2}(0,0) + 35x^4y^3f_{4,3}(0,0) + 35x^3y^4f_{3,4}(0,0) + 21x^2y^5f_{2,5}(0,0) + 7xy^6f_{1,6}(0,0) + y^7f_{0,7}(0,0)\} + \frac{1}{8!}\{x^8f_{8,0}(0,0) + 8x^7yf_{7,1}(0,0) + 28x^6y^2f_{6,2}(0,0) + 56x^5y^3f_{5,3}(0,0) + 70x^4y^4f_{4,4}(0,0) + 56x^3y^5f_{3,5}(0,0) + 28x^2y^6f_{2,6}(0,0) + 8xy^7f_{1,7}(0,0) + y^8f_{0,8}(0,0)\}\Big]dxdy \quad 17$$

Integrating Eq. 17

$$I(f) = 4f_{0,0}(0,0) + \frac{2}{3}\left[f_{2,0}(0,0) + f_{0,2}(0,0)\right] + \frac{1}{30}\left[f_{4,0}(0,0) + f_{0,4}(0,0)\right] + \frac{1}{9}f_{2,2}(0,0) + \frac{1}{180}\left[f_{4,2}(0,0) + f_{2,4}(0,0)\right] + \frac{4}{7!}\left[f_{6,0}(0,0) + f_{0,6}(0,0)\right] + \frac{1}{3600}f_{4,4}(0,0) + \frac{1}{7560}\left[f_{6,2}(0,0) + f_{2,6}(0,0)\right] + \frac{4}{9!}\left[f_{8,0}(0,0) + f_{0,8}(0,0)\right] + \frac{4}{11!}\left[f_{10,0}(0,0) + f_{0,10}(0,0)\right] + \frac{20}{3\times10!}\left[f_{8,2}(0,0) + f_{2,8}(0,0)\right] + \frac{24}{10!}\left[f_{6,4}(0,0) + f_{4,6}(0,0)\right] \quad 18$$

Eq. 13 can also be represented as

$$R_{L4}(f) = 4f_{0,0}(0,0) + \frac{2}{3}\left[f_{2,0}(0,0) + f_{0,2}(0,0)\right] + \frac{1}{30}\left[f_{4,0}(0,0) + f_{0,4}(0,0)\right] + \frac{1}{9}f_{2,2}(0,0) + \frac{1}{180}\left[f_{4,2}(0,0) + f_{2,4}(0,0)\right] + \frac{52}{75\times6!}\left[f_{6,0}(0,0) + f_{0,6}(0,0)\right] + \frac{1}{3600}f_{4,4}(0,0) + \frac{1456}{225\times8!}\left[f_{6,2}(0,0) + f_{2,6}(0,0)\right] + \frac{84}{125\times8!}\left[f_{8,0}(0,0) + f_{0,8}(0,0)\right] \quad 19$$

Associating Eq. 18 and Eq. 19 in Eq. 15,

$$E_{L4}(f) = I(f) - R_{L4}(f) = -\frac{64}{75\times7!}\left[f_{6,0}(0,0) + f_{0,6}(0,0)\right] - \frac{256}{25\times9!}\left[\left[f_{6,2}(0,0) + f_{2,6}(0,0)\right] + \frac{1}{5}\left[f_{8,0}(0,0) + f_{0,8}(0,0)\right]\right] \quad 20$$

Eq. 14 can be expressed as

$$R_{GL3}(f) = 4f_{0,0}(0,0) + \frac{2}{3}\left[f_{2,0}(0,0) + f_{0,2}(0,0)\right] + \frac{1}{30}\left[f_{4,0}(0,0) + f_{0,4}(0,0)\right] + \frac{1}{9}f_{2,2}(0,0) + \frac{1}{180}\left[f_{4,2}(0,0) + f_{2,4}(0,0)\right] + \frac{12}{25\times6!}\left[f_{6,0}(0,0) + f_{0,6}(0,0)\right] + \frac{1}{3600}f_{4,4}(0,0) + \frac{112}{25\times8!}\left[f_{6,2}(0,0) + f_{2,6}(0,0)\right] + \frac{2916}{78125\times8!}\left[f_{8,0}(0,0) + f_{0,8}(0,0)\right] \quad 21$$

Eq. 22 is obtained by associating Eq. 18 and Eq. 21 with Eq. 16

$$E_{GL3}(f) = I(f) - R_{GL3}(f) = \frac{16}{25\times7!}\left[f_{6,0}(0,0) + f_{0,6}(0,0)\right] + \frac{64}{75\times8!}\left[f_{6,2}(0,0) + f_{2,6}(0,0)\right] + \frac{286256}{703125\times8!}\left[f_{8,0}(0,0) + f_{0,8}(0,0)\right] \quad 22$$

Eq. 22 is quadrature rule of precision five.

Gauss -Legendre Five-point Rule

[33-34] developed the compact form of Gauss-Legendre five-point rule.

$$I(f) \cong R_{GL5}(f) = \frac{1}{810000}\Big[\Big\{\alpha\{\alpha\{f(P,-P) + f(P,P)\} + 512f(P,0) + \beta\{f(P,Q) + f(P,-Q)\}\}\Big\} + \Big\{\alpha\{\alpha\{f(-P,P) + f(-P,-P)\} + 512f(-P,0) + \beta\{f(-P,Q) + f(-P,-Q)\}\}\Big\} + \Big\{512\{\alpha\{f(0,P) + f(0,-P)\} + 512f(0,0) + \beta\{f(0,Q) + f(0,-Q)\}\}\Big\} + \Big\{\beta\{\alpha\{f(Q,P) + f(Q,-P)\} + 512f(Q,0) + \beta\{f(Q,Q) + f(Q,-Q)\}\}\Big\} + \Big\{\beta\{\alpha\{f(-Q,P) + f(-Q,-P)\} +$$

$512f(-Q,0)+\beta\{f(-Q,Q)+f(-Q,-Q)\}\}\}\Big]$
23

where $P=\sqrt{\frac{5+2\sqrt{\frac{10}{7}}}{9}}$, $Q=\sqrt{\frac{5-2\sqrt{\frac{10}{7}}}{9}}$, $\alpha=\left(322-13\sqrt{70}\right)$ and $\beta=\left(322+13\sqrt{70}\right)$

The two variable mathematical representations of Eq. 23 are denoted by Eq. 24

$$R_{GL5}(f)=4f_{0,0}(0,0)+\frac{2}{3}\left[f_{0,2}(0,0)+f_{2,0}(0,0)\right]+\frac{1}{30}\left[f_{0,4}(0,0)+f_{4,0}(0,0)\right]+\frac{1}{9}f_{2,2}(0,0)+\frac{1}{180}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{4}{7!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\frac{1}{3600}f_{4,4}(0,0)+\frac{1}{7560}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right]+\frac{4}{9!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]+\frac{1}{544320}\left[f_{8,2}(0,0)+f_{2,8}(0,0)\right]+\frac{1}{151200}\left[f_{6,4}(0,0)+f_{4,6}(0,0)\right]+\frac{771}{2155\times 10!}\left[f_{10,0}(0,0)+f_{0,10}(0,0)\right] \quad 24$$

The error $E_{GL5}(f)$ due to Gauss Legendre-five-point quadrature rule is expressed in Eq. 24

$$E_{GL5}(f)=I(f)-R_{GL5}(f)=\frac{139}{2155\times 11!}\left[f_{10,0}(0,0)+f_{0,10}(0,0)\right] \quad 25$$

Construction of Joint (Mixed) Quadrature Rule

Multiplying Eq. 22 by $\left(\frac{4}{3}\right)$ and adding it to Eq. 20, the required joint (mixed) quadrature scheme is

$$I(f)=\frac{1}{7}\left[3R_{L4}(f)+4R_{GL3}(f)\right]+\frac{1}{7}\left[3E_{L4}(f)+4E_{GL3}(f)\right] \quad 26$$

Where,

$$I(f)=R_{L4GL3}(f)+E_{L4GL3}(f) \quad 27$$

$$R_{L4GL3}(f)=\frac{1}{7}\left[3R_{L4}(f)+4R_{GL3}(f)\right] \quad 28$$

$$E_{L4GL3}(f)=\frac{1}{7}\left[3E_{L4}(f)+4E_{GL3}(f)\right] \quad 29$$

Eqs. 28 and 29 are known as joint rule and error due to joint rule respectively.

Determination of Error

The error analysis is governed by Theorem 1 [35]

Theorem 1:

The absolute error connected with $R_{L4GL3}(f)$ is expressed as

$$|E_{L4GL3}(f)|=\frac{2048}{525\times 8!}\left|f_{6,2}(0,0)+f_{2,6}(0,0)\right|+\frac{665024}{4921875\times 8!}\left|f_{8,0}(0,0)+f_{0,8}(0,0)\right|$$

Proof:

Associating Eqs. 20 and 22 in Eq. 29

$$|E_{L4GL3}(f)|=|I(f)-R_{L4GL3}(f)|=\frac{2048}{525\times 8!}\left|f_{6,2}(0,0)+f_{2,6}(0,0)\right|+\frac{665024}{4921875\times 8!}\left|f_{8,0}(0,0)+f_{0,8}(0,0)\right|$$

Lemma 1:

The truncated error bound [36]

$$|E_{L4GL3}(f)|\le\frac{64M}{175\times 7!}|\eta_2-\eta_1|$$

Where, $\eta_1,\eta_2\in[-1,1]$ and $M=\max_{-1\le x,y\le 1}\left|f_{7,0}(x,*)+f_{0,7}(*,y)\right|$

Proof:

$$E_{L4}(f)=-\frac{64}{75\times 7!}\left[f_{6,0}(\eta_1)+f_{0,6}(\eta_1)\right]$$

$$E_{GL3}(f)=\frac{16}{25\times 7!}\left[f_{6,0}(\eta_2)+f_{0,6}(\eta_2)\right]$$

$$E_{L4GL3}(f)=\frac{1}{7}\left[3E_{L4}(f)+4E_{GL3}(f)\right]=\frac{64}{175\times 7!}\left[f_{6,0}(\eta_2,0)+f_{0,6}(0,\eta_2)-f_{6,0}(\eta_1,0)-f_{0,6}(0,\eta_1)\right]=\frac{64}{175\times 7!}\left[\int_{\eta_1}^{\eta_2}f_{7,0}(x,0)dx+\int_{\eta_1}^{\eta_2}f_{0,7}(0,y)dy\right]=\frac{64}{175\times 7!}\int_{\eta_1}^{\eta_2}\int_{\eta_1}^{\eta_2}\left[f_{0,7}(x,*)+f_{7,0}(*,y)\right]dxdy$$

$$|E_{L4GL3}(f)|\le\frac{64M}{175\times 7!}|\eta_2-\eta_1|$$

Eq. 30 is known as the error bound at points η_1 and η_2 in the domain$[-1,1]$.

Corollary 1:

$$\frac{128M}{175\times 7!}\ge|E_{L4GL3}(f)|$$

Proof:

From Eq.30 and $|\eta_1-\eta_2|\le 2$ [37-38]

$$\frac{128M}{175\times 7!}\ge|E_{L4GL3}(f)|$$

Numerical Verification of the Current Rule

It is found that joint quadrature rule executes a close approximation to the analytical solution than Gauss Legendre-five-point rules for different integrals with the same twenty-five-point functional evaluations.

Example -1

The numerical results along with errors of Examle-1 are reported in Table. 1. Analytical result of joint rule and Gauss-Legendre five-point quadrature rule are depicted in Fig 1. The exact results of different integrals are given below referring [10].

$$I_1 = \int_0^1 \int_0^1 \frac{\sin(\pi x)}{1+\cos^2(y)} dxdy = 0.375228493605129$$

$$I_2 = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \cos^2(x+y)\, dxdy = 0.308425137534042$$

$$I_3 = \int_0^{\pi} \int_0^1 \cos(x)\sin(y^2)\, dxdy = 0$$

$$I_4 = \int_0^{\frac{\pi}{2}} \int_0^1 x^3 \cos(y)\sin(y)\, dxdy = 0.125000000000000$$

Table 1. (Comparison of numerical results of joint quadrature rule $R_{L4GL3}(f)$ with $R_{GL5}(f)$ and corresponding error of Example-1)

Integrals (I)	Lobatto-4-pointrule $R_{L4}(f)$	GauusLegendre-3-pointrule $R_{GL3}(f)$	joint quadrature rule $R_{L4GL3}(f)$	Error due to joint quadrature rule $E_{L4GL3}(f)$	GauusLegendre-5-pointrule $R_{GL5}(f)$	Error due to GauusLegendre-5-point rule $E_{GL5}(f)$
I_1	0.233981688988131	0.234684670677885	0.234383392810848	0.140845100794281	0.234283667883072	0.140944825722052
I_2	0.308425137534042	0.308425137534042	0.308425137534042	0.000000000000000	0.308425137534042	0.000000000000000
I_3	0.000000000000000	0.000000000000000	0.000000000000000	0.000000000000000	0.000000000000000	0.000000000000000
I_4	0.300660234425586	0.300660234425586	0.300660234425586	0.175660234425586	0.300660234425586	0.175660234425586

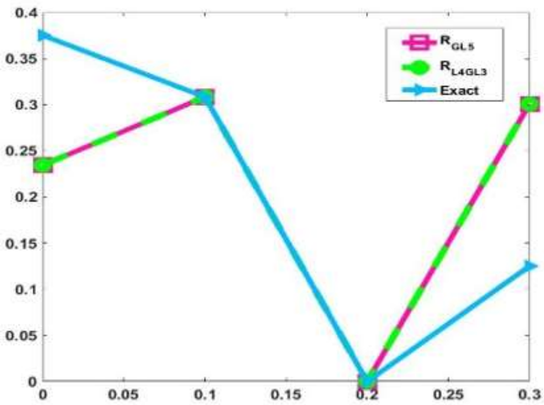

Figure 1. Analytical result of joint rule and Gauss-Legendre five-point quadrature rule

Example-2

In this section, application of joint quadrature rule is implemented to determine the center of gravity $(\bar{x}, \bar{y})$ with different density function $f(x,y)$.Table. 2, reports numerical results for exact and approxima te
 results of center of gravity $E(\bar{x}, \bar{y})$ and $A(\bar{x}, \bar{y})$ of Example-2.

$$M_1 = \int_0^1 \int_0^1 \frac{1}{y+1} dydx, \qquad M_2 = \int_0^{\ln 4} \int_0^{\ln 3} e^{x+y}\, dxdy$$

$$M_3 = \int_0^1 \int_0^2 (y+e^x) dydx, \quad M_4 = \int_0^2 \int_0^3 xy\, dydx$$

Table 2. (Numerical results for exact and approximate value of Center of gravity $E(\bar{x},\bar{y})$ and $A(\bar{x},\bar{y})$) of Example-2)

Center of gravity M due to different density function $f(x,y)$	Exact result M_{exact}	Joint quadrature rule M_{L4GL3}	Exact value of center of gravity $E(\bar{x},\bar{y})$	Approximate value of Center of gravity $A(\bar{x},\bar{y})$
M_1	0.6931471 80559945	0.6931426 15089163	(0.5000000000000000, 0.442695040888964)	(0.499907443859020, 0.442704543380822)
M_2	5.999999 999999998	6.0544074 13522095	(0.848392481493188, 0.647918433002165)	(0.854782304520975, 0.650593621205011)
M_3	5.436563 656918091	5.4365636 46046245	(0.551819161757163, 1.122626480390481)	(0.551819145832902, 1.131156444843434)
M_4	9.000000 000000000	9.000000 000000000	(1.333333333333333, 2.000000000000000)	(1.33333333333333, 2.000000000000000)

Discussion:

The joint quadrature method is based on combining two rules of the same precision level to form a higher level of precision. This scheme is based on the mixing of two constituent rule of same precision level seven to develop a higher degree of precision nine. The present scheme may be extended to obtain approximate solution of volume integrals in Mechanics and also surface integral in Physics. The approximate solution of analytic function in complex plane is also obtained by current method. Aigher precision rule always produces less error which is clearly reporTable in ted s. 1 and 2. Fig. 1 depicts the comparison of analytical results of joint quadrature rule with Gauss-Legendre five-point quadrature rule. From Fig. 1 and reported data of Table. 1, it is observed that thejoint quadrature rule executes less error than all constituent rules and good agreement with Gaussian quadrature. The order of accuracy of various quadrature is obtained as follows:

$$|E_{L4}(f)| \geq |E_{GL3}(f)| \geq |E_{GL5}(f)| \geq |E_{L4GL3}(f)|$$

The efficiency of proposed approach is justified by determining the center of gravity. Convergence analysis is demonstrated to validate the current approach. The present scheme is also extended to obtain approximate solution of analytic function in complex plane.

Conclusions:

This paper is based on the mixing of two constituent rules of same precision level seven to develop a higher degree of precision nine. Numerical results by present approach with lower precision rule have presented and compared with the results performed by existing Gauss Legendre five-point rule with same functional evaluation with higher degree of precision in two variables and noticed that joint rule contributes better approximation to constituent rules and excellent coincident with analytical solution. The efficiency of proposed approach is justified with numerical tests. Particularly, the determination of center of gravity is taken into account for application point of view. Convergence analysis is executed to validate the current approach. The present scheme may be extended to obtain approximate solution of moment of inertia, surface integrals and in volume integrals in Mechanics and also in Physics. The approximate solution of analytic function in complex plane is also obtained by current method

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in department of Environmental Engineering, KIIT Deemed to be University.

Authors' contributions statement:

SR J developed the idea and worked out the numerical part by MATLAB. D N. initiates the Theorems and Lemmas which is best fit to this manuscript. MM A wrote the manuscript and interpreted the data. SK M wrote another part of the article and revised it. All Authors read the manuscript minutely and approved the final version of their manuscript.

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التحقيق العددي وتحليل الأخطاء وتطبيق مخطط التربيع المشترك في العلوم الفيزيائية

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الخلاصة:
تم تقديم تربيع مشترك للحل العددي للتكامل المزدوج في هذا البحث. تعتمد هذه الطريقة على الجمع بين قاعدتين من نفس مستوى الدقة لتشكيل مستوى أعلى من الدقة. تم عرض النتائج العددية للطريقة الحالية بمستوى أقل من الدقة ومقارنتها مع تلك التي يتم إجراؤها بواسطة قاعدة النقاط الخمس عالية الدقة Gauss-Legendre الحالية ، والتي لها نفس التقييم الوظيفي. تم تبرير كفاءة الطريقة المقترحة بأمثلة عددية. من وجهة نظر التطبيق ، يعد تحديد مركز الثقل اعتبارًا خاصًا للمخطط الحالي. تم إثبات تحليل التقارب للتحقق من صحة الطريقة الحالية.

الكلمات المفتاحية: مركز الثقل ، تحليل التقارب، دالة الكثافة، درجة الدقة، التربيع المشترك، سلسلة ماكلاورين