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Modeling and Analyzing the Influence of Fear on the Harvested Modified Leslie-Gower Model

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Abstract:

A modified Leslie-Gower predator-prey model with a Beddington-DeAngelis functional response is proposed and studied. The purpose is to examine the effects of fear and quadratic fixed effort harvesting on the system's dynamic behavior. The model's qualitative properties, such as local equilibria stability, permanence, and global stability, are examined. The analysis of local bifurcation has been studied. It is discovered that the system experiences a saddle-node bifurcation at the survival equilibrium point whereas a transcritical bifurcation occurs at the boundary equilibrium point. Additionally established are the prerequisites for Hopf bifurcation existence. Finally, using MATLAB, a numerical investigation is conducted to verify the validity of the theoretical analysis and visualize the model dynamics.

Keywords: Bifurcation, Fear, Leslie-Gower model, Permanence, Quadratic fixed effort harvesting, Stability.

Introduction:

It is well understood, that any full assessment of a complex event involves a comprehensive examination of essential systems, not just sections of them. Therefore, many scientists have researched prey-predator interactions in theoretical ecology and evolutionary biology over the last few decades, and mathematical models have contributed significantly to better comprehending these complicated settings [1]. In the early nineteenth century, Malthus appears to have constructed mathematical models to depict the patterns of prey-predator relationships. The well-known Lotka-Volterra model was then adjusted to include a logistic growth factor for the prey and a number of population-dependent response functions, enabling realistic prey-predator interaction simulations [2, 3].

The application of fear impacts also enhanced the birth rate of prey animals. In fundamental ecology and environmental biology, the impact of fear has been extensively investigated [4-6]. Fear of predators may be equally as significant as direct consumption; in addition to direct ingestion, fear of predators has been demonstrated in wild animals living in social groups to increase awareness and reduce time spent hunting when population size drops[7-10]. When averaged across numerous trials, the effect of population-level anxiety on prey survival could be equivalent to that of direct predator eating. Prey species' physiological states may be influenced by predator anxiety, and prey species may face problems and losses as a result. In the Greater Yellowstone ecosystem, for example, wolves (Canis lupus) alter the reproductive physiology of deer (Cervus elaphus).

Recently, the impact of anxiety on the system dynamics of prey-predator models has piqued researchers' interest [11, 12]. Sarkar et al [13] constructed and tested a prey-predator system with Holling type-II functional response that included the cost of fear in prey reproduction using a new suggested fear function, which is used in our investigation. They demonstrated that prey-predator interactions can be stabilized by significant anti-predator reflexes. With fear, Maghool et al [14] built and investigated a tritrophic Leslie-Gower food-web system. Because of the prey group's protective capability, the Sokol-Howell type of function response is employed to describe the predation process. The influence of predation anxiety on the dynamics of the three species food chain system at the first and second levels was proposed and examined by Maghool et al [15]. However, Rahi et al [16] looked at the dynamics of a prey-predator model that included two facts: the fear effect and predator stage structure.

On the other hand, harvesting, as well as other biological factors such as the Allee effect and refuge, are important and common events[17-20]. Recently, researchers have looked into how refuge, Allee, and intra-specific competition among predators may impact the stability, bifurcation, and equilibria that affect the dynamic of prey-predator systems [21]. They concluded that an increase in intraspecific competition coefficient values stabilizes the dynamic system. On the other hand, it is noted that the dynamical system becomes unstable as the Allee effect intensity values increase. Because natural systems are mostly renewable, harvesting is widely used in fisheries. In an exploited fishing system with two interacting species, researchers are looking into harvesting either prey or predator species, or both prey and predator species. There have been a variety of harvesting tactics used. Some of them used fixed harvesting, fixed effort harvesting (or proportional harvesting), and continuous threshold harvesting, while others examined nonlinear harvesting [22-25].

Leslie-Gower is a prey-predator model where the predator's carrying capacity is dependent on the amount of prey, highlighting that both prey and predator growth rates have upper limits. In the Lotka-Volterra model, this concept was not considered. These upper limits can be reached in perfect conditions: for predators, whenever the quantity of prey per predator is large, and for prey, whenever the quantity of predators (and presumably prey) is low. The study of the behavior of Leslie-Gower prey-predator systems [22, 26-28] has been taken up by a number of scholars.

It is common knowledge that changing a fundamental parameter through a critical value causes a dynamic system's stability to change. A bifurcation occurs when the parameter crosses a critical value. When a bifurcation occurs close to an attractor, it is referred to be local. If only one parameter causes the bifurcation, it is referred to as codimension one. It is referred to as codimension two when the bifurcation is caused by the simultaneous change of two parameters. The local bifurcation caused by one parameter value is explored in this work since the biological elements in the environment are always changing. As a consequence, a prey-predator system based on the Beddington-DeAngelis functional response will be built in this work. The Leslie-Gower notion, which assumes the predator has access to extra food sources, will increase the predator population's birth rate. Our goal is to explore how fear and fixed effort harvesting affect the recommended model's dynamical behavior through the investigation of their stability and bifurcation.

Model Construction

The construction of the mathematical model that simulates the harvested Leslie-Gower prey-predator real-world system is described as follows. Let $N(T)$ and $P(T)$ represent the density of prey and predator at time T respectively. It started with the modified Lotka-Volterra type prey-predator system:

$$\begin{aligned}\frac{dN}{dT} &= (r - bN)N - f(N,P)P,\\ \frac{dP}{dT} &= f(N,P)P - dp,\end{aligned} \qquad 1$$

where the function $f(N,P)$ represents the functional response and the parameters $r, b,$ and d stand for intrinsic growth rate, intraspecific competition rate, and predator natural death rate respectively. Model 1 is studied by many researchers using different types of functional responses. Later on, many authors studied the following famous Leslie prey-predator system[26-28]:

$$\begin{aligned}\frac{dN}{dT} &= (r - bN)N - f(N,P)P,\\ \frac{dP}{dT} &= P\left[u - \frac{vP}{N}\right].\end{aligned} \qquad 2$$

The parameter u is the intrinsic growth rate of the predator. The parameter v is a measure of the food quantity that the prey provides converted to predator birth. The term P/N is the Leslie-Gower term that measures the loss in the predator population due to the rarity (per capita P/N) of its favorite food. The predator consumes the prey according to the functional response $f(N,P)$ and carries capacity N/v.

The predator P can transfer over to other populations in the event of acute scarcity, according to Aziz-Alaoui et al [26], but its growth will be limited because its favorite food N is not abundant. They offered the following prey-predator model with a modified Leslie-Gower strategy to handle such a problem by adding a positive constant K to the denominator:

$$\begin{aligned}\frac{dN}{dT} &= (r - bN)N - f(N,P)P,\\ \frac{dP}{dT} &= P\left[u - \frac{vP}{K+N}\right].\end{aligned} \qquad 3$$

Here, the parameter K measures the extent to which the environment protects the predator P. System 3 with Holling type-II functional response was investigated in [26], but system 3 with Bedington-DeAngelis type of functional response was investigated in [27]. In light of the above discussion, model 3 provided in [27] is modified to include the fear cost proposed by Sarkar et al [13] in the prey equation, as well as assessing the impact of quadratic fixed effort harvesting on the overall model. As a result, the dynamics of a prey-predator system like this can be described as follows:

$$\frac{dN}{dT} = N\left[r_1\left(m + \frac{n(1-m)}{n+P}\right) - d - bN - \frac{aP}{c+N+eP} - q_1EN\right] = Nf(N,P),$$
$$\frac{dP}{dT} = P\left[r_2\left(1 - \frac{eP}{K+N}\right) - q_2EP\right] = Pg(N,P), \qquad 4$$

where $N(0) \geq 0$, and $P(0) \geq 0$. While the parameters are described in Table 1:

Table 1. Parameters description

Parameter	Description
r_1, r_2	The birth rate of prey population and predator population respectively.
m	The minimum cost of fear with $m \in [0,1]$.
n	The level of fear.
d	The natural death rate of the prey.
b	Decay rate due to intraspecific competition.
a	Attack rate.
c	Half saturation constant.
e	A measure of the food quantity that the prey provides converted to predator birth.
q_1, q_2	The catchability coefficients of the prey and predator respectively.
E	The effort that is applied to harvest the prey and predator.
K	The carrying capacity of the predator in the absence of their prey.

Theorem 1: System 4 is a positively invariant.

Proof: Solving the system 4, using the positive conditions $\left(N(0), P(0)\right)$ yields that:

$$N(T) = N(0)\, e^{\int_0^t \left[\begin{matrix} r_1\left(m + \frac{n(1-m)}{n+P(s)}\right) - d - bN(s) \\ -\frac{aP(s)}{c+N(s)+eP(s)} - q_1EN(s) \end{matrix}\right] ds}$$
$$P(T) = P(0)e^{\int_0^t \left[r_2\left(1 - \frac{eP(s)}{K+N(s)}\right) - q_2EP(s)\right] ds}$$

It can be deduced from the preceding two equations that any solution in the interior of $\mathbb{R}_+^2$ that starts with positive initial conditions $\left(N(0), P(0)\right)$ remains there for all future time. ■

Theorem 2: All the solutions of system 4 are uniformly bounded.

Proof: From the first equation of system 4, it is observed that

$$\frac{dN}{dt} \leq (r_1 - d)N - bN^2.$$

Hence, solving the above differential inequality gives that:

$$N(t) \leq \frac{r_1 - d}{b\left[1 - e^{-(r_1-d)t}\right] + (r_1 - d)N(0)e^{-(r_1-d)t}}.$$

Therefore, as $t \to \infty$, it is obtained $N(t) \leq \frac{r_1-d}{b} = \beta > 0$, because the birth rate is biologically bigger than the death rate for the survival species. Now, using the bound of the prey in the second equation of system 4, it is obtained that:

$$\frac{dP}{dt} \leq r_2P - \left[\frac{r_2e}{K+\beta} + q_2E\right]P^2 = r_2P - \delta P^2.$$

Similarly, solving the last differential inequality gives:

$$P(t) \leq \frac{r_2}{\delta\,[1 - e^{-r_2t}] + r_2N(0)e^{-r_2t}}.$$

Therefore, as $t \to \infty$, it is obtained $P(t) \leq \frac{r_2}{\delta}$. Consequently, the proof is done. ■

Keeping the above in mind, system 4 has continuous with continuous partial derivative interaction functions in their domain $\mathbb{R}_+^2$, Therefore, system 4 with a given initial condition has a unique solution.

Equilibria and Their Stability

There are at most four nonnegative equilibrium points in the system 4, and the present conditions and their forms are as follows:

The trivial equilibrium point (TEP) is always present when $Z_1 = (0,0)$.

The predator-free equilibrium point (PDFEP) given by $Z_2 = (\bar{N}, 0)$, where $\bar{N} = \frac{r_1-d}{b+q_1E}$, exists solely on the horizontal axis if and only if

$$d < r_1. \qquad 5$$

While the prey-free equilibrium point (PYFEP) is expressed by $Z_3 = \left(0, \tilde{P}\right)$, where $\tilde{P} = \frac{r_2K}{r_2e+Kq_2E}$, always exists.

Finally, the survival equilibrium point (SEP), which is written as $Z_4 = (N^*, P^*)$, where

$$N^* = \frac{er_2P^* - K(r_2 - Eq_2P^*)}{r_2 - Eq_2P^*}. \qquad 6$$

While P^* represents the positive root of the following fourth-order polynomial equation.

$$A_4P^4 + A_3P^3 + A_2P^2 + A_1P + A_0 = 0, \qquad 7$$

where the coefficients $A_i;\ i = 0,1,2,3,4$ of Eq.7 are given as:

$$\begin{aligned} A_4 = {} & Eq_2[Eq_2(a - e(mr_1 - d)) \\ & -e(b + Eq_1)(Eq_2K + er_2)] \end{aligned},$$

$$\begin{aligned} A_3 = {} & E^2{q_2}^2[(c - K)[d - K(b + Eq_1) - mr_1] + n[a - e(r_1 - d)] \\ & -eKn(b + Eq_1)] - Eq_2[2ar_2 + bcer_2 + be^2nr_2 - 4eKr_2(b + Eq_1) \\ & +eEq_1r_2(c + en) - 3er_2(mr_1 - d)] + 2e^2{r_2}^2(b + Eq_1), \end{aligned}$$

$$\begin{aligned} A_2 &= E^2{q_2}^2 n(c-K)[d-K(b+Eq_1)-r_1] \\ &\quad -Eq_2[2r_2(c-K)(d-K(b+Eq_1)-mr_1)+2anr_2+bcenr_2 \\ &\quad +3enr_2(d-r_1)-4eKnr_2(b+Eq_1)+ceEnq_1r_2]+{r_2}^2[a+bce+2de \\ &\qquad -3beK+2be^2n+ceEq_1-3eEKq_1+2e^2Enq_1-2emr_1], \\ A_1 &= -2Enq_2r_2(c-K)[d-K(b+Eq_1)-r_1] \\ &\quad +{r_2}^2[(c-K)[d-K(b+Eq_1)-mr_1] \\ &\quad +an+bcen-3beKn+2en(d-r_1)+eEnq_1(c-3K)], \\ A_0 &= n{r_2}^2(c-K)[d-K(b+Eq_1)-r_1]. \end{aligned}$$

The discard rule of the sign will be used in Eq.7 in order to determine the sufficient criteria for having one positive real root. Consequently, the SEP exists in the interior of $\mathbb{R}_+^2$ uniquely if and only if the following requirements are met.

$$0 < K(r_2 - Eq_2P^*) < er_2P^*, \tag{8}$$

with one set of the following sets of sufficient conditions.

$$\left.\begin{matrix} A_4 < 0, A_3 < 0, A_1 > 0, A_0 > 0 \\ A_4 > 0, A_3 > 0, A_1 < 0, A_0 < 0 \\ A_4 < 0, A_3 < 0, A_2 < 0, A_1 < 0, A_0 > 0 \\ A_4 < 0, A_3 > 0, A_2 > 0, A_1 > 0, A_0 > 0 \\ A_4 > 0, A_3 > 0, A_2 > 0, A_1 > 0, A_0 < 0 \\ A_4 > 0, A_3 < 0, A_2 < 0, A_1 < 0, A_0 < 0 \\ A_4 < 0, A_3 < 0,\ A_1 > 0, A_0 = 0 \\ A_4 < 0,\ A_2 > 0, A_1 > 0, A_0 = 0 \\ A_4 = 0, A_3 < 0, A_2 < 0,\ A_0 > 0 \\ A_4 = 0, A_3 < 0,\ A_1 > 0, A_0 > 0 \end{matrix}\right\}. \tag{9}$$

Other sets of conditions may exist too, depending on the sign of the coefficients of Eq. 7.

The local stability analysis is investigated in the following by computing the Jacobian matrix of system 4 at each equilibrium point. The Jacobian matrix at the point (N, P) is calculated as follows:

$$J(N,P) = \begin{bmatrix} N\frac{\partial f}{\partial N} + f(N,P) & N\frac{\partial f}{\partial P} \\ P\frac{\partial g}{\partial N} & P\frac{\partial g}{\partial P} + g(N,P) \end{bmatrix}, \tag{10}$$

where $\frac{\partial f}{\partial N} = -b + \frac{aP}{\Lambda^2} - q_1E$, $\frac{\partial f}{\partial P} = -\frac{r_1n(1-m)}{(n+P)^2} - \frac{a(c+N)}{\Lambda^2}$, $\frac{\partial g}{\partial N} = \frac{r_2eP}{(K+N)^2}$, and $\frac{\partial g}{\partial P} = -q_2E - \frac{r_2e}{K+N}$ with $\Lambda = c + N + eP$.

Accordingly, for TEP the Jacobian matrix becomes

$$J(Z_1) = \begin{bmatrix} r_1 - d & 0 \\ 0 & r_2 \end{bmatrix}.$$

Hence the eigenvalues of $J(Z_1)$ are $\lambda_{11} = r_1 - d$, and $\lambda_{12} = r_2 > 0$. Thus, the TEP is an unstable node when $r_1 > d$, and it is a saddle point when $r_1 < d$.

For the PDFEP, the Jacobian can be written as:

$$J(Z_2) = \begin{bmatrix} -(r_1 - d) & \bar{N}\left(\frac{-r_1(1-m)}{n} - \frac{a}{c+\bar{N}}\right) \\ 0 & r_2 \end{bmatrix}.$$

Therefore, $J(Z_2)$ has the following eigenvalues $\lambda_{21} = -(r_1 - d) < 0$ due to the existing condition 5, and $\lambda_{22} = r_2 > 0$. Thus, the PDFEP is a saddle point.

The Jacobian matrix at the PYFEP can be written as:

$$J(Z_3) = \begin{bmatrix} r_1\left(m + \frac{n(1-m)}{n+\tilde{P}}\right) - d - \frac{a\tilde{P}}{c+e\tilde{P}} & 0 \\ \frac{r_2e\tilde{P}^2}{K^2} & -r_2 \end{bmatrix}. \tag{11}$$

Therefore, $J(Z_3)$ has the following eigenvalues $\lambda_{31} = r_1\left(m + \frac{n(1-m)}{n+\tilde{P}}\right) - d - \frac{a\tilde{P}}{c+e\tilde{P}}$, and $\lambda_{31} = -r_2 < 0$. Accordingly, the PYFEP is locally asymptotically stable if and only if the following requirement is met.

$$r_1\left(m + \frac{n(1-m)}{n+\tilde{P}}\right) < d + \frac{a\tilde{P}}{c+e\tilde{P}}. \tag{12}$$

Finally, system 4 has the following Jacobian matrix at the SEP:

$$J(Z_4) = \begin{bmatrix} N^*\left(-b + \frac{aP^*}{\Lambda^{*2}} - q_1E\right) & -N^*\left(\frac{r_1n(1-m)}{(n+P^*)^2} + \frac{a(c+N^*)}{\Lambda^{*2}}\right) \\ \frac{r_2eP^{*2}}{(K+N^*)^2} & -P^*\left(q_2E + \frac{r_2e}{K+N^*}\right) \end{bmatrix}, \tag{13}$$

where $\Lambda^* = c + N^* + eP^*$. Direct computation shows that $J(Z_4)$ has two negative real parts eigenvalues, and hence the SEP is locally asymptotically stable if the following sufficient condition holds.

$$\frac{aP^*}{\Lambda^{*2}} < b + q_1E. \tag{14}$$

Permanence

A fundamental subject in mathematical biology is the long-term survival of each component of a system of interacting components, which is often a population in an ecological context. Long-term survival has been defined using a variety of criteria. The concept of permanency interests us, which is defined as the study of each species' long-term survival in a prey-predator system. In this section, the Gard technique, which is based on building Lyapunov-like persistence functions [29], is applied. The following theorem establishes the conditions that guarantee the persistence of the system 4.

Theorem 3: System 4 is uniformly persistent if and only if the following condition is met.

$$r_1\left(m + \frac{n(1-m)}{n+\tilde{P}}\right) > d + \frac{a\tilde{P}}{c+e\tilde{P}}. \tag{15}$$

Proof: Define the function $L(N,P) = N^\alpha P^\tau$, where α and τ are positive constants.

Clearly, $L(N,P) > 0$ for all $(N,P) \in \mathbb{R}_+^2$, and $L(N,P) = 0$ for all $(N,P) \in \partial\mathbb{R}_+^2$, where $\partial\mathbb{R}_+^2$ denotes the boundary of $\mathbb{R}_+^2$. Therefore, $L(N,P)$ is known as a claimed persistence function or termed average Lyapunov function in the sense of the Gard approach. Then, according to Gard, the proof is done if and only if $\varphi(N,P)=\frac{L'(N,P)}{L(N,P)}$ is positive for all points (N,P) that belong to the $\omega-$limit sets of the system 4 in the $\partial\mathbb{R}_+^2$. Direct computation gives that:

$$L'(N,P)=P^{\tau}\alpha N^{\alpha-1}\frac{dN}{dt}+N^{\alpha}\tau P^{\tau-1}\frac{dP}{dt}.$$

Then

$$\varphi(N,P)=\frac{L'(N,P)}{L(N,P)}=\frac{\alpha}{N}\frac{dN}{dt}+\frac{\tau}{P}\frac{dP}{dt}.$$

Now, since the TEP, PDFEP, and PYFEP are the only points that belong to the $\omega-$limit sets of system 4 in the boundary of the positive quadrant. Moreover, direct calculation shows that:

$$\varphi(Z_1)=\alpha(r_1-d)+r_2\tau.$$
$$\varphi(Z_2)=r_2\tau.$$
$$\varphi(Z_3)=\alpha\left[r_1\left(m+\frac{n(1-m)}{n+\tilde{P}}\right)-d-\frac{a\tilde{P}}{c+e\tilde{P}}\right].$$

Clearly, $\varphi(Z_1)>0$ for any suitable choice of positive constants α and τ. While $\varphi(Z_2)>0$ always. However, $\varphi(Z_3)>0$ provided that condition 15 is satisfied. Hence the proof is done.

Global Stability Analysis

The global dynamics of system 4 are investigated in the following theorems. The attractive basin of trajectories of a dynamical system is either the state space or a specific region in the state space that is the defining region of the system's state variables, according to global stability. In other words, global stability implies that, regardless of initial conditions, all trajectories eventually tend to the system's attractor. The global stability of most biological systems, such as gene regulatory systems, is required.

Theorem 4: Assume that the PYFEP is locally asymptotically stable then it is globally asymptotically stable if and only if the following requirement is satisfied

$$r_1<d. \qquad 16$$

Proof: Let $V_1(N,P)=N+\left[P-\tilde{P}-\tilde{P}ln\left(\frac{P}{\tilde{P}}\right)\right]$ be a positive definite real-valued function in the region $D_1=\{(N,P)\in\mathbb{R}_+^2: N\geq 0, P>0\}$. Then, it is deduced that

$$\frac{dV_1}{dT}=\frac{dN}{dT}+\left(\frac{P-\tilde{P}}{P}\right)\frac{dP}{dT}=r_1mN+r_1(1-m)N\frac{n}{n+P}-dN-bN^2-\frac{aNP}{c+N+eP}$$
$$-q_1EN^2-(P-\tilde{P})^2\left[\frac{r_2eK}{(K+P)(K+\tilde{P})}\right]$$

Further simplification yields:

$$\frac{dV_1}{dT}\leq(r_1-d)N-(P-\tilde{P})^2\left[\frac{r_2eK}{(K+P)(K+\tilde{P})}\right].$$

Clearly, $\frac{dV_1}{dT}$ is negative definite provided that condition 16 holds. Hence, the PYFEP is globally asymptotically stable. ■

Theorem 5: Assume that the SEP is locally asymptotically stable then it is globally asymptotically stable if and only if the following conditions are met.

$$\frac{aP^*}{c\Lambda^*}<b+q_1E, \qquad 17$$
$${\delta_{12}}^2<4\delta_{11}\delta_{22}, \qquad 18$$

where all the symbols are described in the proof.

Proof: Let $V_2(N,P)=\left[N-N^*-N^*ln\left(\frac{N}{N^*}\right)\right]+\left[P-P^*-P^*ln\left(\frac{P}{P^*}\right)\right]$ be a positive definite real-valued function on the region $D_2=\{(N,P)\in\mathbb{R}_+^2: N>0, P>0\}$. Then, after some calculation, it's reached that:

$$\frac{dV_2}{dT}=-\delta_{11}(N-N^*)^2-\delta_{22}(P-P^*)^2 \\ -\delta_{12}(N-N^*)(P-P^*),$$

where $\delta_{11}=\left[b+q_1E-\frac{aP^*}{\Lambda\Lambda^*}\right]$,

$$\delta_{22}=\left[q_2E+\frac{e}{K+N}\right],$$

$$\delta_{12}=\left[\frac{r_1n(1-m)}{(n+P)(n+P^*)}+\frac{a(c+N^*)}{\Lambda\Lambda^*}-\frac{eP^*}{(K+N)(K+N^*)}\right].$$

Therefore, using the above-given conditions 17-18, the value of $\frac{dV_2}{dT}$ becomes negative definite and then the proof is complete. ■

Bifurcation Analysis

Bifurcation theory is the study of changes in the qualitative structure of a set of curves, such as the integral curves of a set of vector fields or the solutions of a set of differential equations. A bifurcation occurs when a little smooth shift in a system's parameter values causes a significant qualitative change in its behavior. It is most commonly used in the mathematical study of dynamical systems. Bifurcation can be divided into two types. Local bifurcations, which can be studied in detail as parameters cross critical thresholds by observing changes in the local stability properties of equilibria, periodic orbits, or other invariant sets; and global bifurcations, which commonly occur when the system's larger invariant sets interfere with each other or with the system's equilibria. They can't be found only by looking at the stability of the equilibria.

In this section, an investigation of the occurrence of local bifurcation is carried out. Rewrite system 4 as: $\frac{dZ}{dT} = F(N,P)$, with $Z = \begin{pmatrix} N \\ P \end{pmatrix}$, and $F = \begin{pmatrix} Nf(N,P) \\ Pg(N,P) \end{pmatrix}$. Therefore, direct computation gives that, the second directional derivative of F can be written:

$$D^2F(Z,\mu).(V,V) = \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}, \qquad 19$$

where

$$c_{11} = -\frac{2E(b+q_1)\Lambda^3+2aP(2c+2eP+3N)}{\Lambda^3}{v_1}^2 - \frac{2r_1n(1-m)\Lambda^3+2a(c^2+2ceP-N^2+eNP+e^2P^2)(n+P)^2}{(n+P)^2\Lambda^3}v_1v_2 + \frac{2r_1n(1-m)N\Lambda^3+2aeN(c+N)(n+P)^3}{(n+P)^3\Lambda^3}{v_2}^2$$

$$c_{21} = -\frac{2r_2eP^2}{(K+N)^3}{v_1}^2 + \frac{4r_2eP}{(K+N)^2}v_1v_2 - \frac{2(q_2E(K+N)+r_2e)}{K+N}{v_2}^2 .$$

For any vector $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and any parameter μ. Consequently, the following theorems studied the local bifurcation in system 4.

Theorem 6: The system 4 at the PYFEP undergoes a transcritical bifurcation at $d = d^*$, where:

$$d^* = r_1\left(m + \frac{n(1-m)}{n+\tilde{P}}\right) - \frac{a\tilde{P}}{c+e\tilde{P}}.$$

Proof: It is observed that for $d = d^*$ the following is obtained

$$J_1 = J(Z_3,d^*) = \begin{pmatrix} 0 & 0 \\ \frac{r_2e\tilde{P}^2}{K^2} & -r_2 \end{pmatrix} = (d_{ij}).$$

Therefore, $J(Z_3,d^*)$ has eigenvalues given by ${\lambda_{31}}^* = 0$, and ${\lambda_{32}}^* = -r_2$. Thus, PYFEP becomes a non-hyperbolic point. Let $V_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$, and $U_1 = \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix}$ be the eigenvectors corresponding to the ${\lambda_{31}}^* = 0$ of the J_1 and their transpose respectively. Then, the direct calculation gives that:

$$V_1 = \begin{pmatrix} 1 \\ e\left(\frac{r_2}{r_2e+Kq_2E}\right)^2 \end{pmatrix} = \begin{pmatrix} 1 \\ \xi \end{pmatrix}, \text{ and } U_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Moreover, simple computation gives that:

$$F_d(Z,d) = \begin{pmatrix} -N \\ 0 \end{pmatrix} \Rightarrow F_d(Z_3,d^*) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, it is obtained that ${U_1}^{\mathrm{T}}F_d(Z_3,d^*) = 0$.

$${U_1}^{\mathrm{T}}[DF_d(Z_3,d^*)V_1] = -1 \neq 0.$$

$$D^2F(Z_3,d^*)(V_1,V_1) = \begin{pmatrix} -\frac{2\left(E\left(b+q_1\right)(c+e\tilde{P})^2+2a\tilde{P}\right)}{(c+e\tilde{P})^2} - \frac{2\left(r_1n(1-m)(c+e\tilde{P})+a(n+\tilde{P})^2\right)}{(n+\tilde{P})^2(c+e\tilde{P})}\xi \\ -\frac{2r_2e\tilde{P}^2}{K^3} + \frac{4r_2e\tilde{P}}{K^2}\xi - \frac{2(q_2EK+r_2e)}{K}\xi^2 \end{pmatrix}.$$

Hence, it is obtained that:

$${U_1}^{\mathrm{T}}[D^2F(Z_3,d^*)(V_1,V_1)] = -\frac{2\left(E(b+q_1)(c+e\tilde{P})^2+2a\tilde{P}\right)}{(c+e\tilde{P})^2} . - \frac{2\left(r_1n(1-m)(c+e\tilde{P})+a(n+\tilde{P})^2\right)}{(n+\tilde{P})^2(c+e\tilde{P})}\xi \neq 0$$

Thus according to the Sotomayor theorem of local bifurcation, see [30] system 4 undergoes a transcritical bifurcation at the PYFEP, which completes the proof. ■

Theorem 7: The system 4 at the SEP undergoes a saddle-node bifurcation at $b = b^*$ if and only if the following condition holds:

$$0 < N^*\left(-b^* + \frac{aP^*}{\Lambda^{*2}} - q_1E\right) < P^*\left(q_2E + \frac{r_2e}{K+N^*}\right), \qquad 20$$

$${c_{11}}^* + \eta_2{c_{21}}^* \neq 0, \qquad 21$$

where,

$$b^* = \frac{1}{q_2E+\frac{r_2e}{K+N^*}}\left[-q_1q_2E^2 - \frac{r_1r_2en(1-m)P^*}{(K+N^*)^2(n+P^*)^2} - \frac{r_2eq_1E(K+N^*)\Lambda^{*2}-aq_2E(K+N^*)^2P^*+r_2ea(c-K)P^*}{(K+N^*)^2\Lambda^{*2}}\right].$$

While all other symbols are determined in the proof.

Proof: Recall the Jacobian matrix of system 4 at the SEP that is given by Eq. 13 with $b = b^*$, then it can be written as:

$$J_2 = J(Z_4,b^*) = (a_{ij})_{2\times 2},$$

where a_{ij} for all $i,j = 1,2$ are the elements of $J(Z_4)$ at $b = b^*$. The determinant of J_2 can be written as:

$$Det = N^*P^*\left[b^*\left(q_2E + \frac{r_2e}{K+N^*}\right) + q_1q_2E^2 + \frac{r_1r_2en(1-m)P^*}{(n+P^*)^2(K+N^*)^2} + \frac{r_2eq_1E(K+N^*)\Lambda^{*2}}{(K+N^*)^2\Lambda^{*2}} - \frac{aq_2E(K+N^*)^2P^*}{(K+N^*)^2\Lambda^{*2}} + \frac{r_2ea(c-K)P^*}{(K+N^*)^2\Lambda^{*2}}\right]$$

Hence, substituting the value of b^* leads to $Det = 0$, then J_2 has a zero eigenvalue (${\lambda_{41}}^* = 0$) with the second eigenvalue ${\lambda_{42}}^* = N^*\left(-b^* + \frac{aP^*}{\Lambda^{*2}} - q_1E\right) - P^*\left(q_2E + \frac{r_2e}{K+N^*}\right) < 0$ under condition 17. Thus, the SEP is a non-hyperbolic point when $b = b^*$.

Let $V_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$, and $U_2 = \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix}$ be the eigenvectors corresponding to the ${\lambda_{41}}^* = 0$ of the J_2 and their transpose respectively. Then, the direct calculation gives that:

$$V_2 = \begin{pmatrix} 1 \\ -\frac{a_{11}}{a_{12}} \end{pmatrix} = \begin{pmatrix} 1 \\ \eta_1 \end{pmatrix}, \text{ and } U_2 = \begin{pmatrix} 1 \\ -\frac{a_{11}}{a_{21}} \end{pmatrix} = \begin{pmatrix} 1 \\ \eta_2 \end{pmatrix}.$$

Note that, according to the Jacobian elements it is obtained that $\eta_1 > 0$, and $\eta_2 < 0$ due to condition 20. Moreover, simple computation gives that:

$$F_b(Z,b) = \begin{pmatrix} -N^2 \\ 0 \end{pmatrix} \Rightarrow F_b(Z_4,b^*) = \begin{pmatrix} -N^{*2} \\ 0 \end{pmatrix}.$$

Hence, it is observed that ${U_2}^{\mathrm{T}}F_b(Z_4,b^*) = -N^{*2} \neq 0$. Moreover, Eq. 19 gives that:

$$D^2F(Z_4,b^*)(V_2,V_2) = \begin{pmatrix} {c_{11}}^* \\ {c_{21}}^* \end{pmatrix},$$

where:

$$
\begin{aligned}
{c_{11}}^{*} = &-\frac{2E(b^{*}+q_1){\Lambda^{*}}^{3}+2aP^{*}(2c+2eP^{*}+3N^{*})}{{\Lambda^{*}}^{3}} \\
&-\frac{2r_1 n(1-m){\Lambda^{*}}^{3}+2a\left(c^2+2ceP^{*}-{N^{*}}^{2}+eN^{*}P^{*}+e^2{P^{*}}^{2}\right)(n+P^{*})^2}{(n+P^{*})^2{\Lambda^{*}}^{3}}\eta_1 \\
&+\frac{2r_1 n(1-m)N^{*}{\Lambda^{*}}^{3}+2aeN^{*}(c+N^{*})(n+P^{*})^2}{(n+P^{*})^3{\Lambda^{*}}^{3}}{\eta_1}^2 \\
{c_{21}}^{*} = &-\frac{2r_2 e{P^{*}}^{2}}{(K+N^{*})^3}+\frac{4r_2 eP^{*}}{(K+N^{*})^2}\eta_1 \\
&-\frac{2(q_2E(K+N^{*})+r_2e)}{K+N^{*}}{\eta_1}^2
\end{aligned}.
$$

Hence, by using condition 21, it is obtained that:

$$ {U_2}^{\mathrm{T}}[D^2F(Z_4,b^{*})(V_2,V_2)] = {c_{11}}^{*}+\eta_2 {c_{21}}^{*} \neq 0. $$

Thus according to the Sotomayor theorem of local bifurcation, system 4 undergoes a transcritical bifurcation at the PYFEP, which completes the proof. ■

In the following theorem, the occurrence of Hopf-bifurcation, as described in[31], is investigated.

Theorem 8: The system 4 at the SEP undergoes a Hopf-bifurcation at $b = b^{**}$ if and only if the following condition holds:

$$ b^{**}+q_1E < \frac{aP^{*}}{{\Lambda^{*}}^{2}}, \quad 22 $$

$$
\begin{aligned}
&\left(-b^{**}+\frac{aP^{*}}{{\Lambda^{*}}^{2}}-q_1E\right)\left(q_2E+\frac{r_2e}{K+N^{*}}\right) \\
&< \frac{r_2eP^{*}}{(K+N^{*})^2}\left(\frac{r_1n(1-m)}{(n+P^{*})^2}+\frac{a(c+N^{*})}{{\Lambda^{*}}^{2}}\right)
\end{aligned}, \quad 23
$$

where,

$$ b^{**} = \frac{1}{N^{*}}\left[\frac{aN^{*}P^{*}}{{\Lambda^{*}}^{2}}-E(q_1N^{*}+q_2P^{*})-\frac{r_2eP^{*}}{K+N^{*}}\right]. $$

Proof: The characteristic equation of the Jacobian matrix of system 4 at the SEP that is given in Eq. 13 can be written as:

$$ \lambda^2-(Tr)\lambda+(Det) = 0, $$

where,

$$ Tr = N^{*}\left(-b+\frac{aP^{*}}{{\Lambda^{*}}^{2}}-q_1E\right)-P^{*}\left(q_2E+\frac{r_2e}{K+N^{*}}\right). $$

$$
\begin{aligned}
Det = N^{*}P^{*}\Bigg[&-\left(-b+\frac{aP^{*}}{{\Lambda^{*}}^{2}}-q_1E\right)\left(q_2E+\frac{r_2e}{K+N^{*}}\right) \\
&+\frac{r_2eP^{*}}{(K+N^{*})^2}\left(\frac{r_1n(1-m)}{(n+P^{*})^2}+\frac{a(c+N^{*})}{{\Lambda^{*}}^{2}}\right)\Bigg]
\end{aligned}.
$$

Consequently, the roots of the above second-order polynomial equation can be determined by $\lambda_{1,2} = \frac{Tr}{2} \pm \frac{\sqrt{(Tr)^2-4(Det)}}{2}$. Now, for $b = b^{**}$ it is observed that $Tr = 0$, and $Det > 0$. Hence the eigenvalues become $\lambda_{1,2}(b^{**}) = \pm i\sqrt{(Det)}$, which is purely imaginary. Furthermore, in the neighborhood of $b = b^{**}$, the eigenvalues are complex conjugates with real parts given by $Re(\lambda_{1,2}) = \frac{Tr}{2}$. Note that, since

$$ \frac{d}{db}\left(Re(\lambda_{1,2})\right)_{b=b^{**}} = \frac{d}{db}\left(\frac{Tr}{2}\right)\Big|_{b=b^{**}} = -\frac{N^{*}}{2} \neq 0. $$

Hence, according to the Hopf-bifurcation theorem, the system 4 has a Hopf-bifurcation at the $b = b^{**}$, and that complete the proof. ■

Numerical Analysis

In this part, a numerical simulation was performed to simulate the system's 4 global dynamics and to better understand the impact of changing parameter values on the system's 4 dynamical behavior. To model the system's dynamic, different initial values were used. To solve and portray the acquired results in the form of phase portrait figures, a numerical approach using MATLAB is used. To numerically solve the system, the following set of hypothetical parameter values is used, though other sets can be used as well. The parameter values were chosen in order to make the system biologically feasible.

$$
\begin{aligned}
&r_1 = 2, r_2 = 1, m = 0.5, n = 1, d = 0.05, \\
&b = 0.1, a = 0.75, c = 2, e = 0.2, \\
&q_1 = 0.1, q_2 = 0.2, E = 0.75, K = 2
\end{aligned} \quad 24
$$

It is observed that for the above set of data the trajectories of system 4 approach asymptotically to a global stable SEP as represented in Fig. 1.

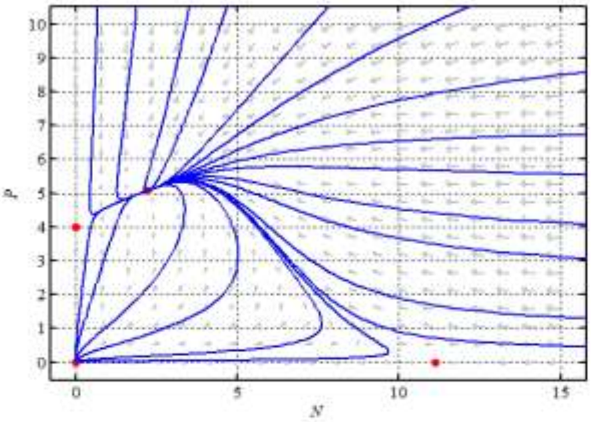

Figure 1. The trajectories of system 4 using set 24 approaches asymptotically to SEP that given by $Z_4 = (2.22, 5.06)$.

It is clear that all the equilibrium points exist $Z_1 = (0,0)$, $Z_2 = (11.14,0)$, $Z_3 = (0,4)$ and $Z_4 = (2.22,5.06)$, which are unstable point, saddle point, saddle point, and globally asymptotically stable respectively. Now, the influence of varying the parameter r_1 in the range $r_1 \leq 1.86$ is shown in Fig. 2 for a typical value of r_1.

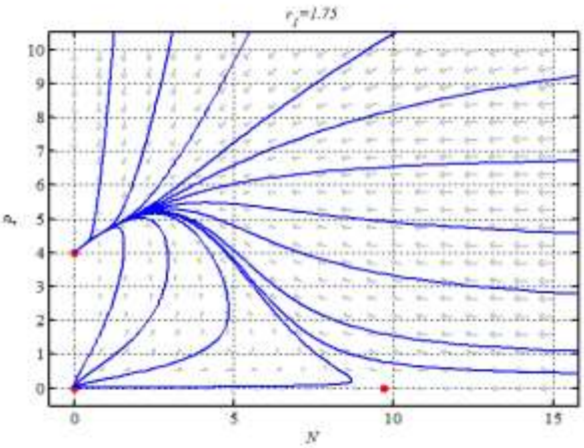

Figure 2. The trajectories of system 4 using set 24 approaches asymptotically to PYFEP that given by $Z_3 = (0, 4)$, when $r_1 = 1.75$.

Figure 2 shows that the range $r_1 \leq 1.86$, system 4 has only three equilibrium points, of which $Z_3 = (0,4)$ is globally asymptotically stable. In the range $r_1 > 1.86$, however, the system approaches a SEP.

The trajectories of system 4 approach SEP for the range $r_2 \leq 1.15$, otherwise the system confronts extinction in prey species, and the trajectories of system 4 approach PYFEP for the specific values of r_2, as illustrated in Figs. 3 and 4.

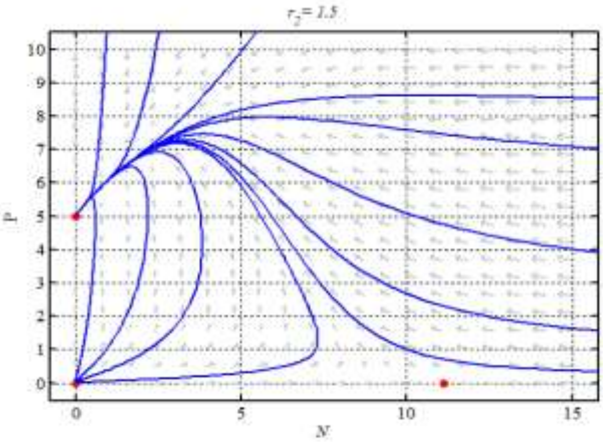

Figure 3. The trajectories of system 4 using set 24 approaches asymptotically to PYFEP that given by $Z_3 = (0, 5)$, when $r_2 = 1.5$.

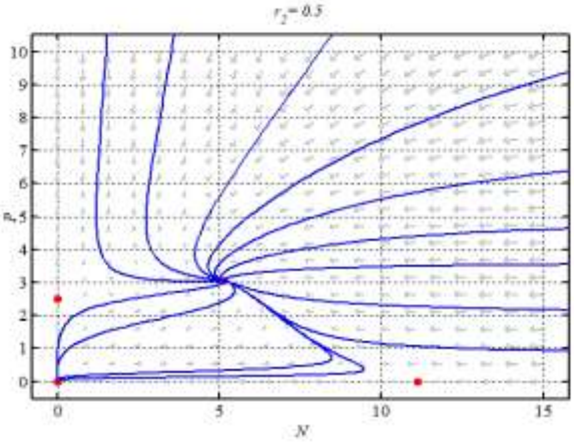

Figure 4. The trajectories of system 4 using set 24 approaches asymptotically to SEP that given by $Z_4 = (5.15, 3.04)$, when $r_2 = 0.5$.

The dynamic of system 4 is explored for various values of the parameter m, and the findings are shown in Fig. 5.

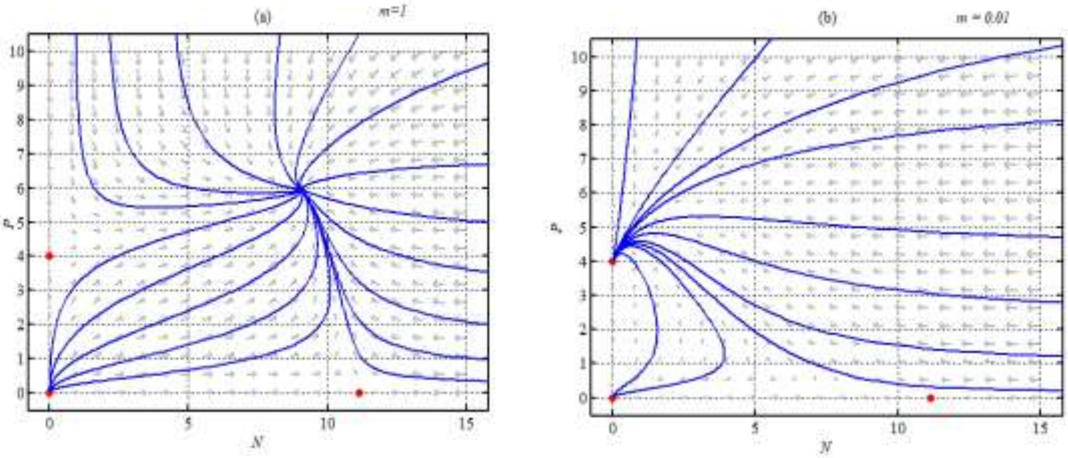

Figure 5. The trajectories of system 4 using set 24. (a) Approach asymptotically to SEP that given by $Z_4 = (9.06, 5.94)$, when $m = 1$. (b) Approach asymptotically to PYFEP that given by $Z_3 = (0, 4)$, when $m = 0.01$.

The trajectories of system 4 as a function of n are also numerically analyzed, and the resulting trajectories are shown in Fig. 6 for various n values. However, the trajectories of system 4 as a function of b are then studied numerically, and the obtained solutions are represented in Fig. 7 at a typical value of b.

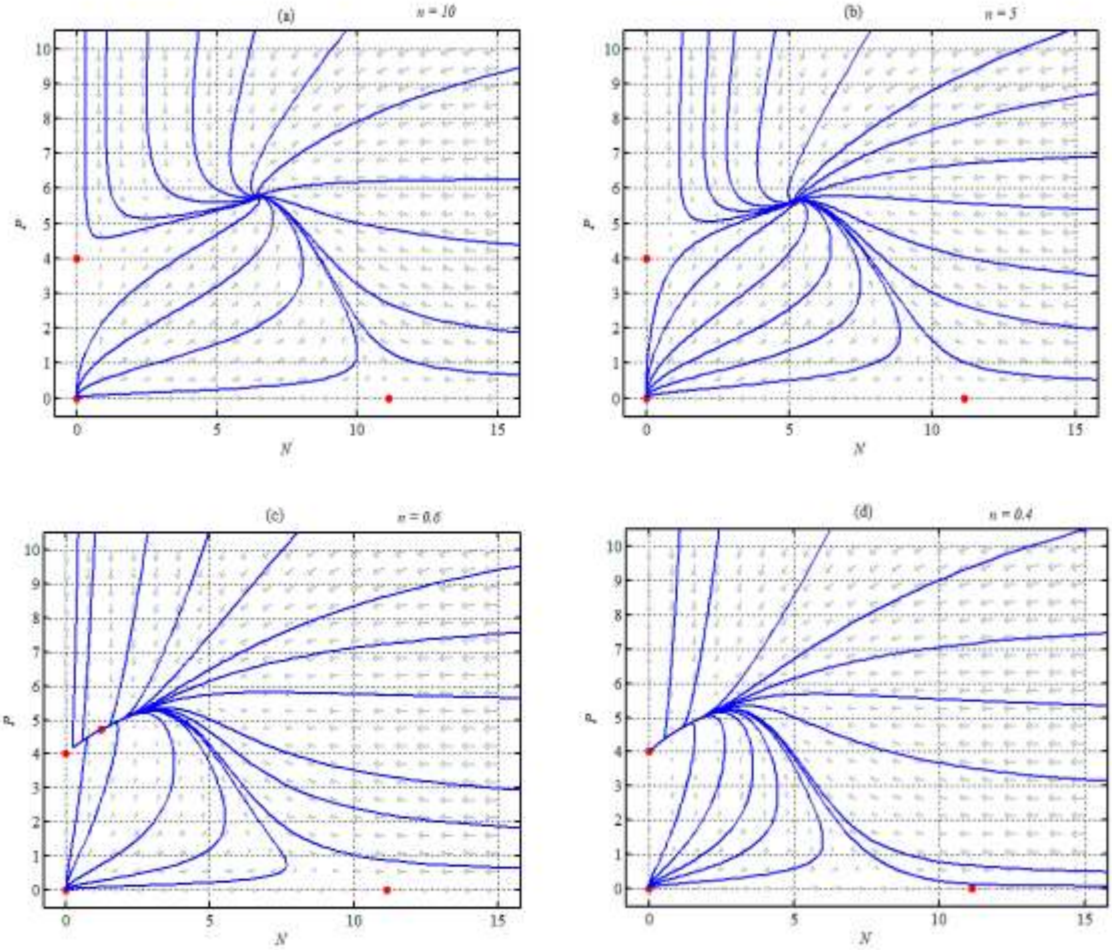

Figure 6. The trajectories of system 4 using set 24. (a) Approach asymptotically to SEP that given by $Z_4 = (6.49, 5.76)$, when $n = 10$. (b) Approach asymptotically to SEP that given by $Z_4 = (5.22, 5.62)$, when $n = 5$. (c) Approach asymptotically to SEP that given by $Z_4 = (1.22, 4.71)$, when $n = 0.6$. (d) Approach asymptotically to PYFEP that given by $Z_3 = (0, 4)$, when $n = 0.4$.

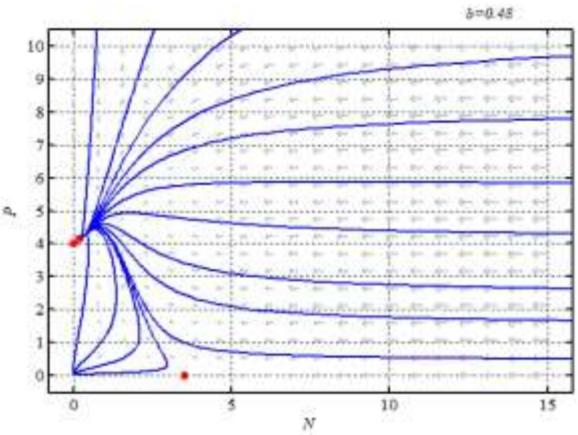

Figure 7. The trajectories of system 4 using set 24 approaches asymptotically to SEP that given by $Z_4 = (0.21, 4.16)$, when $b = 0.48$.

The following Table 2 summarizes the acquired results concerning the influence of modifying various parameters on the dynamics of the system 4.

Table 2. Description of the dynamical behavior of the system 4 as a function of parameters

Parameter	Range	Behavior
a	$a \geq 0.82$	The trajectories approach a PYFEP
	$a < 0.82$	The trajectories approach a SEP
c	$c \leq 1.7$	The trajectories approach a PYFEP
	$c > 1.7$	The trajectories approach a SEP
e	$e \leq 0.17$	The trajectories approach a PYFEP
	$e > 0.17$	The trajectories approach a SEP
q_1	$0 < q_1 < 1$	The trajectories approach a SEP
q_2	$q_2 \leq 0.17$	The trajectories approach a PYFEP
	$q_2 > 0.17$	The trajectories approach a SEP
E	$E \leq 0.75$	The trajectories approach a PYFEP
	$E > 0.75$	The trajectories approach a SEP
K	$K \leq 3$	The trajectories approach a SEP
	$K > 3$	The trajectories approach a PYFEP

Discussion:

In this paper, the effect of fear on the dynamics of the harvested Leslie-Gower prey-predator real-world system is investigated. The model's dynamical behavior is explored from both a theoretical and numerical standpoint. System 4 is discovered to contain a maximum of four nonnegative equilibrium points, three of which are on the boundary axis and the fourth in the positive quadrant's interior.

The model's solution's long-term behavior is investigated theoretically. If conditions 12 and 14 are met, the TEP and PDFEP are discovered to be unstable points, whereas the PYFEP and, respectively, SEP are asymptotically stable locally. The requirements for systems that are uniformly persistent have been established. The Lyapunov technique is used to determine the basin of attraction for each asymptotically stable equilibrium point. Local bifurcation around equilibrium points is also studied using Sotomayor's theorem.

The system's global dynamical behavior, including the influence of fear, is investigated via numerical simulation based on a set of hypothetical parameters. The following are the outcomes. Starting from various sets of initial points, the trajectories for set 24 of data approach SEP asymptotically. This confirms the theoretical conclusion about the equilibria's stability. Reducing the prey species' birth rate below a threshold value removes the SEP, and the system suffers extinction in prey species. Starting from varied initial values, the trajectories approach asymptotically to PYFEP.

However, increasing the predator species' birth rate above a certain threshold removes the SEP, the system suffers extinction in prey species, and the trajectories approach PYFEP asymptotically starting from various initial values. Now, since $\lim_{m\to 0}\left[m+\frac{n(1-m)}{n+P}\right]=\frac{n}{n+P}<1$, while $\lim_{m\to 1}\left[m+\frac{n(1-m)}{n+P}\right]=1$.

As a result, lowering the value of the minimum cost of fear lowers the birth rate of prey, which leads to prey extinction, and the solutions approach the PYFEP asymptotically. However, increasing the value of the minimum cost of fear maintains the system's persistence, and the solutions continue to approach a SEP asymptotically. On the other hand, given the fact that $\lim_{n\to 0}\left[m+\frac{n(1-m)}{n+P}\right]=m\leq 1$, while $\lim_{n\to \infty}\left[m+\frac{n(1-m)}{n+P}\right]=1$.

As the value of the level of the fear decreases, the SEP eventually approaches PYFEP, and they coincide, resulting in prey extinction and solutions approaching PYFEP asymptotically. Increasing the value of the amount of fear, on the other hand, keeps the system alive and the solutions close to SEP. Furthermore, even though increasing the decay rate due to intraspecific competition in prey species gradually lowers the value of the prey population, the system persists and the solutions approach SEP asymptotically.

It has been shown that increasing the predator's attack rate or carrying capacity above a threshold value has a similar effect to that explained by increasing the predator population's birth rate, and the system then loses its persistence. Reducing the half-saturation constant, conversion rate, predator catchability coefficients, or harvesting effort, on the other hand, has a similar effect as lowering the prey population's birth rate, and the system eventually loses its persistence. As a result, increasing prey and predator catchability coefficients, as well as harvesting effort, keep the system persists.

Conclusion:

The SEP is eliminated when the birth rate of the prey species falls below a certain level, and the system experiences prey species extinction. However, if the birth rate of the predator species rises over a specific point, the SEP is lost, the system experiences prey species extinction, and the trajectory approaches PYFEP asymptotically starting from a variety of initial values. The birth rate of prey decreases with decreasing minimal cost of fear, which causes prey to become extinct. As a result, solutions approach the PYFEP asymptotically. The system persists, nevertheless, as the solutions continue to asymptotically approach a SEP when the minimum cost of fear is increased. The SEP gradually approaches PYFEP and coincides with it, leading to prey extinction and solutions that approach PYFEP asymptotically as the value of the intensity of fear declines. On the other side, maintaining the system and keeping the solutions close to SEP requires increasing the value of the level of fear. The system continues to function by increasing the catchability coefficients of the predators and prey as well as the harvesting effort.

Data Availability

The data used to support the findings of this study are included in the article.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

Authors' contributions statement:

A.M. J. and R.K. N. conceived the presented idea. A.M. J. developed the theory and performed the computations. Both authors verified the analytical methods. R.K. N. encouraged A.M. J. to investigate [the impact of fear and harvest on the model's dynamics] and supervised the findings of this work. Both authors discussed the results and contributed to the final manuscript.

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نمذجة وتحليل تأثير الخوف على نموذج لزلي-كور المحور تحت تأثير الحصاد

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قسم الرياضيات، كلية العلوم، جامعة بغداد، العراق

الخلاصة:

تم اقتراح ودراسة نموذج لزلي كور المحور للفريسة والمفترس والمتضمن دالة الاستجابة الوظيفية لبدنكتن ديانجلز. الغرض من الدراسة هو اختبار تأثيرات كل من الخوف والحصاد الثنائي ذو الجهد الثابت على السلوك الديناميكي للنظام. الخواص النوعية للنموذج والمتضمنة الاستقرارية المحلية لنقاط التوازن ,الاصرار, الاستقرارية الشاملة اختبرت. تم دراسة تحليل التشعب المحلي. وقد لوحظ أن النظام يظهر تشعب العقدة السرج عند نقطة توازن البقاء بينما يحدث التشعب الحرج عند نقطة التوازن الحدودي. بالإضافة إلى ذلك ، تم تحديد المتطلبات الأساسية لوجود تشعب هوبف. واخيرا باستخدام برنامج الماثلاب تم تنفيذ المحاكات العددية للنظام للتحقق من صلاحية النتائج النظرية وتصور ديناميكيات النظام.

الكلمات المفتاحية: التشعب، الخوف، نوذج لزلي كور، الثبات، الحصاد التربيعي ثابت الجهد، الاستقرارية.