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The Necessary and Sufficient Optimality Conditions for a System of FOCPs with Caputo–Katugampola Derivatives

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Abstract:

The necessary optimality conditions with Lagrange multipliers $\lambda(t) \in R^n$ are studied and derived for a new class that includes the system of Caputo–Katugampola fractional derivatives to the optimal control problems with considering the end time free. The formula for the integral by parts has been proven for the left Caputo–Katugampola fractional derivative that contributes to the finding and deriving the necessary optimality conditions. Also, three special cases are obtained, including the study of the necessary optimality conditions when both the final time t_f and the final state $x(t_f)$ are fixed. According to convexity assumptions prove that necessary optimality conditions are sufficient optimality conditions.

Keywords: Calculus of variations, Caputo–Katugampola fractional derivative, Hamiltonian system, Necessary and sufficient optimality conditions, Optimal control.

Introduction:

In recent years, the topic of the fractional calculus with optimal control problems (OCPs) has become taking on a wide field and growing interest of many researchers and readers, the main reason for this is that solving problems for many natural systems, scientific problems, engineering, and biological applications is more accurate than classical OCPs ones.

To see some of these applications, the feedback control into the logistic model[1], nonanalytic dynamic systems [2], application to identification problems [3], economic growth model and so on[4, -7].

Fractional optimal control problems (FOCPs) are the generalization of the OCPs with fractional dynamical systems. The performance index of a FOCP is considered a function of both the state and the control variables, and the dynamic constraints are expressed by a set of fractional differential equations (FDEs).

Agrawal, O. P.[8], is using Riemann–Liouville FD in a general formulation and finds an approximate solution for a class of FOCPs.

The Modified Adomian decomposition method (MADM) has been used for finding solutions of a class from FOCPs with a Caputo FD type by Alizadeh, A et al.[9].

Approximate results and the necessary optimality conditions for a class composition FODEs corresponding to OCPs suggested and studied by Qasim Hasan S, et al.[10].

The variational approach has been applied to obtain the necessary optimality and transversality conditions for solving the FOCPs by Chiranjeevi T, et al [11]. Caputo–Fabrizio FD was used in a formulation of time FOCPs and deriving the optimality system in terms of Volterra integrals by Yildiz, T A, et al. [12] and for more about studying the FOCPs (see[13,14]). Also, it is possible to see articles that provide a study of solving fractional order Volterra–Fredholm integral equations [15,16].

New types of fractional operators were introduced by U. Katugampola, These are done by generalizing the Riemann–Liouville and Hadamard FIs[17] and generalizing the Riemann–Liouville and Hadamard FDs[18] and new generalizations of fractional derivatives can also be seen in[19,20].

This paper aims to study and derive the necessary and sufficient optimality conditions for a new system of FOCPs subjected to the dynamic control system from integer and Caputo–Katugampola FDs in the form:

$$\mathcal{A}\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} + \mathcal{B}\begin{bmatrix} {}^{CK}_{a}D^{\alpha,\rho}_{t} x_1(t) \\ {}^{CK}_{a}D^{\alpha,\rho}_{t} x_2(t) \\ \vdots \\ {}^{CK}_{a}D^{\alpha,\rho}_{t} x_n(t) \end{bmatrix} = g(t, x(t), u(t)),$$

Let ${}^{CK}_{a}D^{\alpha,\rho}_{t} x_i(t), i = 1,2,\dots,n,$ is the left Caputo–Katugampola FDs of order $\alpha \in (0,1), \rho > 0, a \in R$ and $(x(t), u(t))$ be state and control variables respectively considering $\mathcal{A}$ and $\mathcal{B}$ are a matrix of order $n \times n$. We proved a very important formula is the integration by parts formula for Caputo–Katugampola FD that contributes to the finding of the necessary optimality conditions and plays a major role in deriving these optimality conditions.

The other aspect of the paper is to study the sufficient optimality conditions under some convexity assumptions that have been obtained in fine detail for system FOCPs.

This paper contains six sections: In section 2, preliminaries of FDs. In section 3, prove integration by parts formula for Caputo–Katugampola FDs.

The necessary optimality conditions are studied for a class of system Caputo–Katugampola FOCPs in section 4. The sufficient optimality conditions for a class of system FOCPs are proven in section 5 and the conclusions are introduced in section 6.

Basic preliminaries

The basic definitions of FDs and integrals are presented with proof of some important theorems, which are used in later work:

Definition 1:[18, 21] Let $\alpha > 0$, $\rho > 0$, and an interval $[a, b]$ of R, where $0 < a < b$. The left and right Riemann–Katugampola FIs of a function $f \in L^1([a, b])$ are defined by

$${}^{RK}_{a}D^{-\alpha,\rho}_{t} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau^{1-\rho}}, \quad 1$$

$${}^{RK}_{t}D^{-\alpha,\rho}_{b} f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau^{1-\rho}}. \quad 2$$

where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1} dt, \alpha \in \mathbb{C}, (Re(\alpha) > 0)$.

Definition 2: [18, 21] Let $\alpha > 0, \rho > 0$ and an interval $[a, b]$ of R, where $0 < a < b$. The left and right Riemann–Katugampola FDs are defined by

$${}^{RK}_{a}D^{\alpha,\rho}_{t} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\alpha} f(\tau)d\tau, \quad 3$$

$${}^{RK}_{t}D^{\alpha,\rho}_{b} f(t) = \frac{-\rho^\alpha}{\Gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_t^b \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^\alpha} f(\tau)d\tau. \quad 4$$

Definition 3: [18, 21] Let $\alpha \in (0,1)$, $\rho > 0$ and an interval $[a, b]$ of R, where $0 < a < b$. The left and right Caputo–Katugampola Katugampola FDs are defined by

$${}^{CK}_{a}D^{\alpha,\rho}_{t} f(t) = {}^{RK}_{a}D^{\alpha,\rho}_{t}[f(t) - f(a)]$$
$$= \frac{\rho^\alpha}{\Gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\alpha}[f(\tau) - f(a)]d\tau, \quad 5$$

$${}^{CK}_{t}D^{\alpha,\rho}_{b} f(t) = {}^{RK}_{t}D^{\alpha,\rho}_{b}[f(t) - f(b)]$$
$$= \frac{-\rho^\alpha}{\Gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_t^b \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^\alpha}[f(\tau) - f(b)]d\tau. \quad 6$$

Theorem 1: [22] A function $f\,(\mathcal{X})$ is convex if for any two points $\mathcal{X}_1$ and $\mathcal{X}_2$, then

$$f\,(\mathcal{X}_2) \geq f\,(\mathcal{X}_1) + \nabla f^T(\mathcal{X}_1)(\mathcal{X}_2 - \mathcal{X}_1)$$

where $\mathcal{X}_1 = \begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_n^1 \end{bmatrix}$ and $\mathcal{X}_2 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_n^2 \end{bmatrix}$

Theorem 2: Let $\alpha \in (0,1)$ and $\rho > 0$, then the left Caputo–Katugampola FD of a function $f \in C^1\,[a, b]$ is given by

$${}^{CK}_{a}D^{\alpha,\rho}_{t} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_t^b (t^\rho - \tau^\rho)^{-\alpha} f'(\tau)d\tau. \quad 7$$

and

The right Caputo–Katugampola FD of a function $f \in C^1\,[a, b]$ is given by

$${}^{CK}_{t}D^{\alpha,\rho}_{b} f(t) = \frac{-\rho^\alpha}{\Gamma(1-\alpha)}\int_t^b (\tau^\rho - t^\rho)^{-\alpha} f'(\tau)d\tau. \quad 8$$

Proof:

First, by proofing the left Caputo–Katugampola FD using Definition 3, in Eq.5, let

$$u = f(\tau) - f(a)$$
$$du = f'(\tau)d\tau$$
$$dv = \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\alpha}d\tau$$
$$v = \int_a^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{-\alpha}d\tau$$
$$= \left(\frac{-1}{\rho(1-\alpha)}(t^\rho - \tau^\rho)^{1-\alpha}\right),$$

$${}^{CK}_{a}D^{\alpha,\rho}_{t} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_a^t u\, dv, \quad 9$$

Now, using integration by parts, to obtain:

$$\int_a^t u dv = [uv]_a^t - \int_a^t v du,$$
$$= \underbrace{[f(\tau) - f(a)]\left(\frac{-1}{\rho(1-\alpha)}(t^\rho - \tau^\rho)^{1-\alpha}\right)\Big|_{\tau=a}^{\tau=t}}_{=0} -$$
$$\int_a^t \frac{-1}{\rho(1-\alpha)}(t^\rho - \tau^\rho)^{1-\alpha} f'(\tau)d\tau, \quad 10$$

By substituting the result of Eq.9 into Eq.10, to get

$$= \frac{\rho^\alpha}{\Gamma(1-\alpha)}.\,t^{1-\rho}\left[(1-\alpha)(\rho t^{\rho-1})\int_a^t \frac{1}{\rho(1-\alpha)}(t^\rho - \tau^\rho)^{-\alpha} f'(\tau)d\tau\right],$$

$$f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_a^t (t^\rho - \tau^\rho)^{-\alpha} f'(\tau)d\tau.$$

where $\alpha \in (0,1)$, $\rho > 0$ two fixed real and $f \in L^1([a, b])$.■

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Integration by Parts Formula for Caputo–Katugampola FDs

Integral formula with a transformational relation between Caputo–Katugampola FD and Riemann–Katugampola FD has been proven in Theorem 3 and will rely on it to derive the necessary optimality conditions.

Theorem 3: Let $f(t) \in C[a,b]$ and $g(t) \in C^1[a,b]$ be two functions and $\alpha \in (0,1)$ and $\rho > 0$. Then

$$\int_a^b f(t) \cdot {}^{CK}_{a}D_t^{\alpha,\rho} g(t)dt = \int_a^b (g(t)t^{\rho-1})\, {}^{RK}_{t}D_b^{\alpha,\rho}\left(t^{1-\rho} f(t)\right)dt + \left[g(t)\, {}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho} f(t)\right)\right]_{t=a}^{t=b}.$$

Proof:
By using the definition of the left Caputo–Katugampola FDs by Theorem 2 of $f(t)$ of order (α,ρ), to obtain:

$$\int_a^b f(t) \cdot {}^{CK}_{a}D_t^{\alpha,\rho} g(t)dt = \int_a^b f(t)\left[\frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_a^t (t^\rho - \tau^\rho)^{-\alpha}\frac{d}{d\tau}g(\tau)d\tau\right]dt, \quad 11$$

By using Dirichlet's formula for Eq.11, to get

$$= \int_a^b \frac{d}{dt}g(t)\underbrace{\left[\frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_t^b (\tau^\rho - t^\rho)^{-\alpha} f(\tau)d\tau\right]}dt,$$

$$= \int_a^b \frac{d}{dt}g(t)\underbrace{\left[\frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_t^b (\tau^\rho - t^\rho)^{-\alpha}\frac{\tau^{1-\rho} f(\tau)}{\tau^{1-\rho}}d\tau\right]}dt, \quad 12$$

By using the definition of the right Riemann–Katugampola FI of $(t^{1-\rho} f(t))$ of order $(1-\alpha,\rho)$ in Eq.12, to get:

$$= \int_a^b \frac{d}{dt}g(t)\, {}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho} f(t)\right)dt,$$

Let $h(t) = {}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho} f(t)\right)$, to obtain:

$$= \int_a^b \frac{d}{dt}g(t)h(t)dt \quad 13$$

Now, using integration by parts of Eq.13, to obtain:

$$= \left[g(t)\, {}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho} f(t)\right)\right]_{t=a}^{t=b}$$

$$- \int_a^b g(t)\frac{t^{1-\rho}}{t^{1-\rho}}\frac{d}{dt}\left[\frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_t^b (\tau^\rho - t^\rho)^{-\alpha} f(\tau)\frac{\tau^{1-\rho} f(\tau)}{\tau^{1-\rho}}d\tau\right]dt,$$

$$= \left[g(t)\, {}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho} f(t)\right)\right]_{t=a}^{t=b} - \int_a^b g(t)\frac{1}{t^{1-\rho}}\left(t^{1-\rho}\frac{d}{dt}\right)\underbrace{\left[\frac{\rho^\alpha}{\Gamma(1-\alpha)}\int_t^b (\tau^\rho - t^\rho)^{-\alpha} f(\tau)\frac{\tau^{1-\rho} f(\tau)}{\tau^{1-\rho}}d\tau\right]}dt,$$

$$= \left[g(t)\, {}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho} f(t)\right)\right]_{t=a}^{t=b}$$

$$- \int_a^b (g(t)t^{\rho-1})(-1)\underbrace{\left[\frac{-\rho^\alpha}{\Gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_t^b (\tau^\rho - t^\rho)^{-\alpha} f(\tau)\frac{\tau^{1-\rho} f(\tau)}{\tau^{1-\rho}}d\tau\right]}dt, \quad 14$$

By using the definition of the right Riemann–Katugampola FD of $\left(t^{1-\rho} f(t)\right)$ of order $(1-\alpha,\rho)$ in Eq.14, to get:

$$= \int_a^b (g(t)t^{\rho-1})\, {}^{RK}_{t}D_b^{\alpha,\rho}\left(t^{1-\rho} f(t)\right)dt + \left[g(t)\, {}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho} f(t)\right)\right]_{t=a}^{t=b},$$

Thus,

$$\int_a^b f(t) \cdot {}^{CK}_{a}D_t^{\alpha,\rho} g(t)dt = \int_a^b (g(t)t^{\rho-1})\, {}^{RK}_{t}D_b^{\alpha,\rho}\left(t^{1-\rho} f(t)\right)dt + \left[g(t)\, {}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho} f(t)\right)\right]_{t=a}^{t=b}. \quad \blacksquare$$

Studying the Necessary Optimality Conditions for a Class of System Caputo–Katugampola FOCPs

Let f, g are two differentiable functions with domain $[a,+\infty) \times R^{m+n}$, and $\psi: [a,+\infty) \times R^n \to R$ is a differentiable function. Consider the system of Caputo–Katugampola FOCPs, in the form:

Minimize $J(x,u,t_f)$

$$= \int_a^{t_f} f(x(t),u(t),t)dt + \psi\left(t_f, x(t_f)\right), \quad 15$$

Subject to dynamic control system

$$\mathcal{A}\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} + \mathcal{B}\begin{bmatrix} {}^{CK}_{a}D_t^{\alpha,\rho} x_1(t) \\ {}^{CK}_{a}D_t^{\alpha,\rho} x_2(t) \\ \vdots \\ {}^{CK}_{a}D_t^{\alpha,\rho} x_n(t) \end{bmatrix} = g(x(t),u(t),t), \quad 16$$

and the initial boundary conditions

$$x(a) = x_a, \quad 17$$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix},$

$$g(t)=\begin{bmatrix} g_1\\ g_2\\ \vdots\\ g_n\end{bmatrix},$$

$$\mathcal{A}_{n\times n}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix},$$

$$\mathcal{B}_{n\times n}=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ b_{n1} & b_{n2} & \cdots & b_{nn}\end{bmatrix},$$

and $x_a=(x_{1a},x_{1a},\dots,x_{1a})$ the fixed real number with $\alpha\in(0,1),\rho>0,a\in R$. Considering the end time t_f free and t_f is a variable number with $a<t_f<\infty$.

Constructing the problem as minimizing by using Lagrange multipliers

$$\lambda(t)\in R^n,\ \lambda(t)=\begin{bmatrix}\lambda_1(t)\\ \lambda_2(t)\\ \vdots\\ \lambda_n(t)\end{bmatrix}$$, as follows:

Now, the Hamiltonian function is defined by:

$$\mathbf{H}=H(x(t),u(t),\lambda(t),t)=f(x(t),u(t),t)+\lambda^T(t)g(x(t),u(t),t), \quad 18$$

Thus,

$$J^*(x,u,t_f,\lambda)=\int_a^{t_f}\left[\mathbf{H}-\lambda^T(t)\left\{\mathcal{A}\begin{bmatrix}x_1'(t)\\ x_2'(t)\\ \vdots\\ x_n'(t)\end{bmatrix}+\mathcal{B}\begin{bmatrix}{}^{CK}_{a}D_t^{\alpha,\rho}x_1(t)\\ {}^{CK}_{a}D_t^{\alpha,\rho}x_2(t)\\ \vdots\\ {}^{CK}_{a}D_t^{\alpha,\rho}x_n(t)\end{bmatrix}\right\}\right]dt+\psi\left(t_f,x(t_f)\right), \quad 19$$

Consider there are variations as follows:

$$x_i+\delta x_i,$$
$$\lambda_i+\delta\lambda_i,$$
$$u_j+\delta u_j,$$

and $$t_f+\delta t_f$$

where $i=1,2,\dots,n,\ j=1,2,\dots,m,$ and $\delta x_i(a)=0$ by the assumed boundary conditions in Eq.17, the variation of J* should disappear ($\delta J^*=0$), and conclude that:

$$0=\int_a^{t_f}\left[\delta\mathbf{H}-\delta\left(\lambda^T(t)\left\{\mathcal{A}\begin{bmatrix}x_1'(t)\\ x_2'(t)\\ \vdots\\ x_n'(t)\end{bmatrix}+\mathcal{B}\begin{bmatrix}{}^{CK}_{a}D_t^{\alpha,\rho}x_1(t)\\ {}^{CK}_{a}D_t^{\alpha,\rho}x_2(t)\\ \vdots\\ {}^{CK}_{a}D_t^{\alpha,\rho}x_n(t)\end{bmatrix}\right\}\right)\right]dt$$
$$+\delta t_f\left[\mathbf{H}-\lambda^T(t)\left\{\mathcal{A}\begin{bmatrix}x_1'(t)\\ x_2'(t)\\ \vdots\\ x_n'(t)\end{bmatrix}+\mathcal{B}\begin{bmatrix}{}^{CK}_{a}D_t^{\alpha,\rho}x_1(t)\\ {}^{CK}_{a}D_t^{\alpha,\rho}x_2(t)\\ \vdots\\ {}^{CK}_{a}D_t^{\alpha,\rho}x_n(t)\end{bmatrix}\right\}\right]_{t=t_f}+\delta\psi\left(t_f,x(t_f)\right), \quad 20$$

Using the chain rule of **H** evaluated at$(x(t),u(t),\lambda(t),t)$and the property of $\delta(fg)=(\delta f)g+f(\delta g)$, to get

$$=\int_a^{t_f}\begin{bmatrix}\sum_{i=1}^{n}\frac{\partial\mathbf{H}}{\partial x_i}\delta x_i+\sum_{j=1}^{m}\frac{\partial\mathbf{H}}{\partial u_j}\delta u_j+\sum_{i=1}^{n}\frac{\partial\mathbf{H}}{\partial\lambda_i}\delta\lambda_i\\ -\delta\lambda^T(t)\left\{\mathcal{A}\begin{bmatrix}x_1'(t)\\ x_2'(t)\\ \vdots\\ x_n'(t)\end{bmatrix}+\mathcal{B}\begin{bmatrix}{}^{CK}_{a}D_t^{\alpha,\rho}x_1(t)\\ {}^{CK}_{a}D_t^{\alpha,\rho}x_2(t)\\ \vdots\\ {}^{CK}_{a}D_t^{\alpha,\rho}x_n(t)\end{bmatrix}\right\}\\ -\lambda^T(t)\mathcal{A}\delta'\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}-\lambda^T(t)\mathcal{B}\begin{bmatrix}{}^{CK}_{a}D_t^{\alpha,\rho}\delta x_1(t)\\ {}^{CK}_{a}D_t^{\alpha,\rho}\delta x_2(t)\\ \vdots\\ {}^{CK}_{a}D_t^{\alpha,\rho}\delta x_n(t)\end{bmatrix}\end{bmatrix}dt$$
$$+\delta t_f\left[\mathbf{H}-\lambda^T(t)\left\{\mathcal{A}\begin{bmatrix}x_1'(t)\\ x_2'(t)\\ \vdots\\ x_n'(t)\end{bmatrix}+\mathcal{B}\begin{bmatrix}{}^{CK}_{a}D_t^{\alpha,\rho}x_1(t)\\ {}^{CK}_{a}D_t^{\alpha,\rho}x_2(t)\\ \vdots\\ {}^{CK}_{a}D_t^{\alpha,\rho}x_n(t)\end{bmatrix}\right\}\right]_{t=t_f}$$
$$+\frac{\partial\psi}{\partial t}\left(t_f,x(t_f)\right)\delta t_f+\frac{\partial\psi}{\partial x}\left(t_f,x(t_f)\right)\left(x'(t_f)\delta t_f+\delta x(t_f)\right), \quad 21$$

Now, Integration by parts gives the relations, to get Firstly,

$$\int_a^{t_f}\lambda_i(t)\delta' x_i(t)dt=-\int_a^{t_f}\delta x_i(t)\lambda_i'(t)dt+\delta x_i(t_f)\lambda_i(t_f),\ i=1,2,\dots,n. \quad 22$$

and

$$\int_a^{t_f}\lambda^T(t)\begin{bmatrix}{}^{CK}_{a}D_t^{\alpha,\rho}\delta x_1(t)\\ {}^{CK}_{a}D_t^{\alpha,\rho}\delta x_2(t)\\ \vdots\\ {}^{CK}_{a}D_t^{\alpha,\rho}\delta x_n(t)\end{bmatrix}dt=$$

$$\int_a^{t_f}[\lambda_1(t)\quad\lambda_2(t)\quad\cdots\quad\lambda_n(t)]\begin{bmatrix}{}^{CK}_{a}D_t^{\alpha,\rho}\delta x_1(t)\\ {}^{CK}_{a}D_t^{\alpha,\rho}\delta x_2(t)\\ \vdots\\ {}^{CK}_{a}D_t^{\alpha,\rho}\delta x_n(t)\end{bmatrix}dt$$

$$=\int_a^{t_f}\lambda_i(t)\,{}^{CK}_{a}D_t^{\alpha,\rho}\delta x_i(t)dt\qquad i=1,2,\dots,n.$$

Using Theorem 1, to get

$$\int_a^{t_f}\lambda_i(t)\,{}^{CK}_{a}D_t^{\alpha,\rho}\delta x_i(t)dt$$
$$=\int_a^{t_f}\left(\delta x_i(t)t^{\rho-1}\right){}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda_i(t)\right)dt$$

$$+\left[\delta x_i(t)\,{}^{RK}_{t}D^{-(1-\alpha,\rho)}_{t_f}\left(t^{1-\rho}\lambda_i(t)\right)\right]_{t=a}^{t=t_f}$$

$$\int_a^{t_f}\lambda_i(t)\,{}^{CK}_{a}D^{\alpha,\rho}_{t}\delta x_i(t)dt = \int_a^{t_f}\left(\delta x_i(t)t^{\rho-1}\right){}^{RK}_{t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\lambda_i(t)\right)dt + \delta x_i(t_f)\,{}^{RK}_{t}D^{-(1-\alpha,\rho)}_{t_f}\left({t_f}^{1-\rho}\lambda_i(t_f)\right) - \delta x_i(a)\,{}^{RK}_{t}D^{-(1-\alpha,\rho)}_{t_f}\left(a^{1-\rho}\lambda_i(a)\right)$$

since $(\delta x_i(a)=0)$, to get

$$\int_a^{t_f}\lambda_i(t)\,{}^{CK}_{a}D^{\alpha,\rho}_{t}\delta x_i(t)dt = \int_a^{t_f}\delta x_i(t)t^{\rho-1}\,{}^{RK}_{t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\lambda_i(t)\right)dt + \delta x_i(t_f)\,{}^{RK}_{t}D^{-(1-\alpha,\rho)}_{t_f}\left({t_f}^{1-\rho}\lambda_i(t_f)\right), \quad 23$$

Substitute the results of Eq.22 and 23, into Eq.21, as follows

$$0 = \int_a^{t_f}\left[\begin{matrix}\delta x^T(t)\left(\frac{\partial \mathbf{H}}{\partial x} + \mathcal{A}\begin{bmatrix}\lambda_1'(t)\\ \lambda_2'(t)\\ \vdots \\ \lambda_n'(t)\end{bmatrix} - \mathcal{B}t^{\rho-1}\begin{bmatrix}{}^{RK}_{t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\lambda_1(t)\right)\\ {}^{RK}_{t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\lambda_2(t)\right)\\ \vdots \\ {}^{RK}_{t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\lambda_n(t)\right)\end{bmatrix}\right)\\ +\delta u^T(t)\frac{\partial \mathbf{H}}{\partial u}\\ +\delta\lambda^T(t)\left(\frac{\partial \mathbf{H}}{\partial \lambda} - \mathcal{A}\begin{bmatrix}x_1'(t)\\ x_2'(t)\\ \vdots \\ x_n'(t)\end{bmatrix} - \mathcal{B}\begin{bmatrix}{}^{CK}_{a}D^{\alpha,\rho}_{t}x_1(t)\\ {}^{CK}_{a}D^{\alpha,\rho}_{t}x_2(t)\\ \vdots \\ {}^{CK}_{a}D^{\alpha,\rho}_{t}x_n(t)\end{bmatrix}\right)\end{matrix}\right]dt$$

$$-\delta x^T(t)\left[\begin{matrix}\mathcal{A}\begin{bmatrix}\lambda_1(t)\\ \lambda_2(t)\\ \vdots \\ \lambda_n(t)\end{bmatrix} + \mathcal{B}\begin{bmatrix}{}^{RK}_{t}D^{-(1-\alpha,\rho)}_{t_f}\left(t^{1-\rho}\lambda_1(t)\right)\\ {}^{RK}_{t}D^{-(1-\alpha,\rho)}_{t_f}\left(t^{1-\rho}\lambda_2(t)\right)\\ \vdots \\ {}^{RK}_{t}D^{-(1-\alpha,\rho)}_{t_f}\left(t^{1-\rho}\lambda_n(t)\right)\end{bmatrix}\\ -\frac{\partial \Psi}{\partial x}(t,x(t))\end{matrix}\right]_{t=t_f}$$

$$+\delta t_f\left[\begin{matrix}\mathbf{H} - \lambda^T(t)\left\{\mathcal{A}\begin{bmatrix}x_1'(t)\\ x_2'(t)\\ \vdots \\ x_n'(t)\end{bmatrix} + \mathcal{B}\begin{bmatrix}{}^{CK}_{a}D^{\alpha,\rho}_{t}x_1(t)\\ {}^{CK}_{a}D^{\alpha,\rho}_{t}x_2(t)\\ \vdots \\ {}^{CK}_{a}D^{\alpha,\rho}_{t}x_n(t)\end{bmatrix}\right\}\\ +\frac{\partial \Psi}{\partial t}(t,x(t)) + \frac{\partial \Psi}{\partial x}(t,x(t))x'(t_f)\end{matrix}\right]_{t=t_f} \quad 24$$

Now, rewrite the transversality conditions in Eq.24 by using the Taylor series for $f=(x+\delta x)$ about the point $x=t_f$ can be given as:

$$\delta x(t_f) = \delta x_{t_f} - x'(t_f)\delta t_f + O\left(\delta {t_f}^2\right) \quad 25$$

where $\delta x_{t_f} = (x+\delta x)(t_f+\delta t_f) - x(t_f)$,

and $\lim\limits_{\gamma\to\infty}\frac{O(\gamma)}{\gamma}$ is finite.

Thus, to obtain

$$0 = \int_a^{t_f}\left[\begin{matrix}\delta x^T(t)\left(\frac{\partial \mathbf{H}}{\partial x} + \mathcal{A}\begin{bmatrix}\lambda_1'(t)\\ \lambda_2'(t)\\ \vdots \\ \lambda_n'(t)\end{bmatrix} - \mathcal{B}t^{\rho-1}\begin{bmatrix}{}^{RK}_{t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\lambda_1(t)\right)\\ {}^{RK}_{t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\lambda_2(t)\right)\\ \vdots \\ {}^{RK}_{t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\lambda_n(t)\right)\end{bmatrix}\right)\\ +\delta u^T(t)\frac{\partial \mathbf{H}}{\partial u}\\ +\delta\lambda^T(t)\left(\frac{\partial \mathbf{H}}{\partial \lambda} - \mathcal{A}\begin{bmatrix}x_1'(t)\\ x_2'(t)\\ \vdots \\ x_n'(t)\end{bmatrix} - \mathcal{B}\begin{bmatrix}{}^{CK}_{a}D^{\alpha,\rho}_{t}x_1(t)\\ {}^{CK}_{a}D^{\alpha,\rho}_{t}x_2(t)\\ \vdots \\ {}^{CK}_{a}D^{\alpha,\rho}_{t}x_n(t)\end{bmatrix}\right)\end{matrix}\right]dt$$

$$+\delta t_f\left[\begin{matrix}\mathbf{H} - \left(\mathcal{B}\lambda(t_f)\right)^T\begin{bmatrix}{}^{CK}_{a}D^{\alpha,\rho}_{t}x_1(t_f)\\ {}^{CK}_{a}D^{\alpha,\rho}_{t}x_2(t_f)\\ \vdots \\ {}^{CK}_{a}D^{\alpha,\rho}_{t}x_n(t_f)\end{bmatrix}\\ +\left(\mathcal{B}x'(t_f)\right)^T\begin{bmatrix}{}^{RK}_{t}D^{-(1-\alpha,\rho)}_{t_f}\left({t_f}^{1-\rho}\lambda_1(t_f)\right)\\ {}^{RK}_{t}D^{-(1-\alpha,\rho)}_{t_f}\left({t_f}^{1-\rho}\lambda_2(t_f)\right)\\ \vdots \\ {}^{RK}_{t}D^{-(1-\alpha,\rho)}_{t_f}\left({t_f}^{1-\rho}\lambda_n(t_f)\right)\end{bmatrix} + \frac{\partial \Psi}{\partial t}\left(t_f, x(t_f)\right)\end{matrix}\right]$$

$$-\delta x_{t_f}{}^T\left[\mathcal{A}\begin{bmatrix}\lambda_1(t_f)\\ \lambda_2(t_f)\\ \vdots\\ \lambda_n(t_f)\end{bmatrix}+\mathcal{B}\begin{bmatrix}{}^{RK}_{\;t}D_{t_f}^{-(1-\alpha,\rho)}\left(t_f{}^{1-\rho}\lambda_1(t_f)\right)\\ {}^{RK}_{\;t}D_{t_f}^{-(1-\alpha,\rho)}\left(t_f{}^{1-\rho}\lambda_2(t_f)\right)\\ \vdots\\ {}^{RK}_{\;t}D_{t_f}^{-(1-\alpha,\rho)}\left(t_f{}^{1-\rho}\lambda_n(t_f)\right)\end{bmatrix}-\frac{\partial\Psi}{\partial x}\left(t_f,x(t_f)\right)\right]+O(\delta t_f{}^2)=0. \blacksquare$$

since the variation functions were chosen arbitrarily, the necessary optimality conditions for a system Caputo–Katugampola FOCPs were obtained and explained in the following theorem.

Theorem 4: If (x,u,t_f) is a minimizer of Eq.15 under the dynamic constraint in Eq.16, as follows

$$\mathcal{A}\begin{bmatrix}x_1'(t)\\ x_2'(t)\\ \vdots\\ x_n'(t)\end{bmatrix}+\mathcal{B}\begin{bmatrix}{}^{CK}_{\;a}D_t^{\alpha,\rho}x_1(t)\\ {}^{CK}_{\;a}D_t^{\alpha,\rho}x_2(t)\\ \vdots\\ {}^{CK}_{\;a}D_t^{\alpha,\rho}x_n(t)\end{bmatrix}=g(x(t),u(t),t)$$

and the boundary conditions in Eq.17, then $\exists\ \lambda(t)\in R^n$, for which (x,u,λ) satisfies:

The Hamiltonian system

$$\left\{\begin{array}{l}\mathcal{A}\begin{bmatrix}\lambda_1'(t)\\ \lambda_2'(t)\\ \vdots\\ \lambda_n'(t)\end{bmatrix}-\mathcal{B}t^{\rho-1}\begin{bmatrix}{}^{RK}_{\;t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda_1(t)\right)\\ {}^{RK}_{\;t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda_2(t)\right)\\ \vdots\\ {}^{RK}_{\;t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda_n(t)\right)\end{bmatrix}=-\frac{\partial\mathbf{H}}{\partial x}(x(t),u(t),\lambda(t),t),\\ \mathcal{A}\begin{bmatrix}x_1'(t)\\ x_2'(t)\\ \vdots\\ x_n'(t)\end{bmatrix}+\mathcal{B}\begin{bmatrix}{}^{CK}_{\;a}D_t^{\alpha,\rho}x_1(t)\\ {}^{CK}_{\;a}D_t^{\alpha,\rho}x_2(t)\\ \vdots\\ {}^{CK}_{\;a}D_t^{\alpha,\rho}x_n(t)\end{bmatrix}=\frac{\partial\mathbf{H}}{\partial\lambda}(x(t),u(t),\lambda(t),t),\end{array}\right\} \qquad 26$$

where

$$\frac{\partial\mathbf{H}}{\partial x}=\begin{bmatrix}\frac{\partial\mathbf{H}}{\partial x_1}(t)\\ \frac{\partial\mathbf{H}}{\partial x_2}(t)\\ \vdots\\ \frac{\partial\mathbf{H}}{\partial x_n}(t)\end{bmatrix},\frac{\partial H}{\partial\lambda}=\begin{bmatrix}\frac{\partial\mathbf{H}}{\partial\lambda_1}(t)\\ \frac{\partial\mathbf{H}}{\partial\lambda_2}(t)\\ \vdots\\ \frac{\partial\mathbf{H}}{\partial\lambda_n}(t)\end{bmatrix},\text{for all } t\in[a,t_f];$$

The stationary condition

$$\frac{\partial\mathbf{H}}{\partial u}(x(t),u(t),\lambda(t),t)=0,$$

$$\frac{\partial\mathbf{H}}{\partial u}=\begin{bmatrix}\frac{\partial\mathbf{H}}{\partial u_1}(t)\\ \frac{\partial\mathbf{H}}{\partial u_2}(t)\\ \vdots\\ \frac{\partial\mathbf{H}}{\partial u_m}(t)\end{bmatrix},\text{for all } t\in[a,t_f]; \qquad 27$$

The transversality conditions

$$0=\left[\mathbf{H}-(\mathcal{B}\lambda(t))^T\begin{bmatrix}{}^{CK}_{\;a}D_t^{\alpha,\rho}x_1(t)\\ {}^{CK}_{\;a}D_t^{\alpha,\rho}x_2(t)\\ \vdots\\ {}^{CK}_{\;a}D_t^{\alpha,\rho}x_n(t)\end{bmatrix}+(\mathcal{B}x'(t))^T\begin{bmatrix}{}^{RK}_{\;t}D_{t_f}^{-(1-\alpha,\rho)}\left(t^{1-\rho}\lambda_1(t)\right)\\ {}^{RK}_{\;t}D_{t_f}^{-(1-\alpha,\rho)}\left(t^{1-\rho}\lambda_2(t)\right)\\ \vdots\\ {}^{RK}_{\;t}D_{t_f}^{-(1-\alpha,\rho)}\left(t^{1-\rho}\lambda_n(t)\right)\end{bmatrix}+\frac{\partial\Psi}{\partial t}(t,x(t))\right]_{t=t_f} \qquad 28$$

and

$$\left[\mathcal{A}\begin{bmatrix}\lambda_1(t)\\ \lambda_2(t)\\ \vdots\\ \lambda_n(t)\end{bmatrix}+\mathcal{B}\begin{bmatrix}{}^{RK}_{\;t}D_{t_f}^{-(1-\alpha,\rho)}\left(t^{1-\rho}\lambda_1(t)\right)\\ {}^{RK}_{\;t}D_{t_f}^{-(1-\alpha,\rho)}\left(t^{1-\rho}\lambda_2(t)\right)\\ \vdots\\ {}^{RK}_{\;t}D_{t_f}^{-(1-\alpha,\rho)}\left(t^{1-\rho}\lambda_n(t)\right)\end{bmatrix}-\frac{\partial\Psi}{\partial x}(t,x(t))\right]_{t=t_f} \qquad 29$$

where the Hamiltonian is defined by Eq.18 and

$$\frac{\partial\Psi}{\partial x}=\begin{bmatrix}\frac{\partial\Psi}{\partial x_1}(t)\\ \frac{\partial\Psi}{\partial x_2}(t)\\ \vdots\\ \frac{\partial\Psi}{\partial x_n}(t)\end{bmatrix}. \quad\blacksquare$$

To study three basic special cases on the end time t_f or on $x(t_f)$ in both cases when they are fixed and free in Corollary 1.

Corollary 1: Let (x,u) be a minimizer of Eq.15 subject to the dynamic constraint in Eq.16 and the boundary condition in Eq.17, then

1) The transversality conditions aren't used in Theorem 4 if both t_f and $x(t_f)$ are fixed.
2) Only the transversality condition in Eq.29 is used in Theorem 4, if t_f is fixed and $x(t_f)$ is free.

3) Only the transversality condition in Eq.28 is used in Theorem 4, if t_f is free and $x(t_f)$ is fixed.

Remark 1: If t_f is fixed and $x(t_f)$ is free and

Let $\mathcal{A}=\begin{bmatrix}1&1&\cdots&1\\1&1&\cdots&1\\\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&1\end{bmatrix}$,

$$\mathcal{B}=\begin{bmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{bmatrix},\psi\left(t_f,x(t_f)\right)\equiv 0,$$

and $\alpha=\rho=1$. Then conclude

The Hamiltonian system

$$\left\{\begin{array}{l}\begin{bmatrix}\lambda_1'(t)\\\lambda_2'(t)\\\vdots\\\lambda_n'(t)\end{bmatrix}=-\dfrac{\partial\mathbf{H}}{\partial x}(x(t),u(t),\lambda(t),t),\\ \begin{bmatrix}x_1'(t)\\x_2'(t)\\\vdots\\x_n'(t)\end{bmatrix}=\dfrac{\partial\mathbf{H}}{\partial\lambda}(x(t),u(t),\lambda(t),t),\end{array}\right\}$$

The stationary condition

$$\frac{\partial\mathbf{H}}{\partial u}(x(t),u(t),\lambda(t),t)=0,$$
for all $t\in[a,t_f]$;

The transversality condition $\begin{bmatrix}\lambda_1(t_f)\\\lambda_2(t_f)\\\vdots\\\lambda_n(t_f)\end{bmatrix}=0.$

Studying the Sufficient Optimality Conditions for a Class of System Caputo–Katugampola FOCPs

Under some convexity assumptions sufficient conditions are studied for a class of system Caputo–Katugampola FOCPs in Theorem 5, as follows

Theorem 5: Let $\bar{x}=\begin{bmatrix}\overline{x_1}\\\overline{x_2}\\\vdots\\\overline{x_n}\end{bmatrix},\bar{u}=\begin{bmatrix}\overline{u_1}\\\overline{u_2}\\\vdots\\\overline{u_m}\end{bmatrix}$ and $\bar{\lambda}=\begin{bmatrix}\overline{\lambda_1}\\\overline{\lambda_2}\\\vdots\\\overline{\lambda_n}\end{bmatrix}$ satisfying conditions (26) – (29) of Theorem 4, and assume that

1. f and g are convex in x and u, and Ψ is convex in x.
2. t_f is fixed.
3. $\begin{bmatrix}\overline{\lambda_1}\\\overline{\lambda_2}\\\vdots\\\overline{\lambda_n}\end{bmatrix}\geq 0$ for all $t\in[a,t_f]$ or g is linear in x and u.

Then $\left(\begin{bmatrix}\overline{x_1}\\\overline{x_2}\\\vdots\\\overline{x_n}\end{bmatrix},\begin{bmatrix}\overline{u_1}\\\overline{u_2}\\\vdots\\\overline{u_m}\end{bmatrix}\right)$ is an optimal solution to the problem (15) – (17).

Proof:

Deriving the Hamiltonian in Eq.18 relative to $x(t)$ and replaced in the Hamiltonian system of Theorem 4, to deduce

$$\frac{\partial f}{\partial x}(\overline{x}(t),\overline{u}(t),t)=-\mathcal{A}\begin{bmatrix}\overline{\lambda_1}'(t)\\\overline{\lambda_2}'(t)\\\vdots\\\overline{\lambda_n}'(t)\end{bmatrix}+\mathcal{B}t^{\rho-1}\begin{bmatrix}{}^{RK}_{\ t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\overline{\lambda_1}(t)\right)\\{}^{RK}_{\ t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\overline{\lambda_2}(t)\right)\\\vdots\\{}^{RK}_{\ t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\overline{\lambda_n}(t)\right)\end{bmatrix}-\frac{\partial g}{\partial x}(\overline{x}(t),\overline{u}(t),t)\overline{\lambda}(t), \quad 30$$

where

$$\frac{\partial f}{\partial x}=\begin{bmatrix}\frac{\partial f}{\partial x_1}(t)\\\frac{\partial f}{\partial x_2}(t)\\\vdots\\\frac{\partial f}{\partial x_n}(t)\end{bmatrix},\quad\left(\frac{\partial g}{\partial x}\right)_{n\times n}=\begin{bmatrix}\frac{\partial g_1}{\partial x_1}&\cdots&\frac{\partial g_1}{\partial x_n}\\\vdots&\ddots&\vdots\\\frac{\partial g_n}{\partial x_1}&\cdots&\frac{\partial g_n}{\partial x_n}\end{bmatrix}.$$

Now, deriving the Hamiltonian relative to $u(t)$ and replaced in the stationary condition of Theorem 4, as follows:

$$\frac{\partial\mathbf{H}}{\partial u}(\overline{x}(t),\overline{u}(t),t)=\frac{\partial f}{\partial u}(\overline{x}(t),\overline{u}(t),t)+\frac{\partial g}{\partial u}(\overline{x}(t),\overline{u}(t),t)\overline{\lambda}(t)=0,$$

$$\frac{\partial f}{\partial u}(\overline{x}(t),\overline{u}(t),t)=-\frac{\partial g}{\partial u}(\overline{x}(t),\overline{u}(t),t)\overline{\lambda}(t), \quad 31$$

where

$$\frac{\partial f}{\partial u}=\begin{bmatrix}\frac{\partial f}{\partial u_1}(t)\\\frac{\partial f}{\partial u_2}(t)\\\vdots\\\frac{\partial f}{\partial u_m}(t)\end{bmatrix},\quad\left(\frac{\partial g}{\partial u}\right)_{m\times n}=\begin{bmatrix}\frac{\partial g_1}{\partial u_1}&\cdots&\frac{\partial g_1}{\partial u_m}\\\vdots&\ddots&\vdots\\\frac{\partial g_n}{\partial u_1}&\cdots&\frac{\partial g_n}{\partial u_m}\end{bmatrix}.$$

since t_f is fixed, then by the case (2) of Corollary 1 has the only transversality condition.

$$\left[\mathcal{A}\begin{bmatrix}\overline{\lambda_1}(t)\\ \overline{\lambda_2}(t)\\ \vdots\\ \overline{\lambda_n}(t)\end{bmatrix}+\mathcal{B}\begin{bmatrix}{}^{RK}_{\ t}D^{-(1-\alpha,\rho)}_{t_f}\left(t^{1-\rho}\overline{\lambda_1}(t)\right)\\ {}^{RK}_{\ t}D^{-(1-\alpha,\rho)}_{t_f}\left(t^{1-\rho}\overline{\lambda_2}(t)\right)\\ \vdots\\ {}^{RK}_{\ t}D^{-(1-\alpha,\rho)}_{t_f}\left(t^{1-\rho}\overline{\lambda_n}(t)\right)\end{bmatrix}-\frac{\partial\psi}{\partial x}\left(t,\overline{x}(t)\right)\right]_{t=t_f}=0, \quad 32$$

Let (x,u) be admissible, i.e., let Eq.16 and Eq.17 be satisfied for (x,u). In this case,

$$J(x,u)-J(\overline{x},\overline{u})=\int_a^{t_f}\left[f(x(t),u(t),t)-f(\overline{x}(t),\overline{u}(t),t)\right]dt+\psi\left(t_f,x(t_f)\right)-\psi\left(t_f,\overline{x}(t_f)\right),$$

since f is convex on x and u, and ψ is convex in x from assumption 1, then by Theorem 1, to obtain

$$\geq\int_a^{t_f}\left[\left(x(t)-\overline{x}(t)\right)^T\frac{\partial f}{\partial x}(\overline{x}(t),\overline{u}(t),t)+\left(u(t)-\overline{u}(t)\right)^T\frac{\partial f}{\partial u}(\overline{x}(t),\overline{u}(t),t)\right]dt+\left(x(t_f)-\overline{x}(t_f)\right)^T\frac{\partial\psi}{\partial x}\left(t_f,\overline{x}(t_f)\right),$$

By using Eq. 30 and Eq. 31, to get

$$=\int_a^{t_f}\begin{bmatrix}-\left(x(t)-\overline{x}(t)\right)^T\mathcal{A}\begin{bmatrix}\overline{\lambda_1}'(t)\\ \overline{\lambda_2}'(t)\\ \vdots\\ \overline{\lambda_n}'(t)\end{bmatrix}\\ +\left(x(t)-\overline{x}(t)\right)^T\mathcal{B}t^{\rho-1}\begin{bmatrix}{}^{RK}_{\ t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\overline{\lambda_1}(t)\right)\\ {}^{RK}_{\ t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\overline{\lambda_2}(t)\right)\\ \vdots\\ {}^{RK}_{\ t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\overline{\lambda_n}(t)\right)\end{bmatrix}\\ -\left(\overline{\lambda}(t)\right)^T\frac{\partial g}{\partial x}(\overline{x}(t),\overline{u}(t),t)\left(x(t)-\overline{x}(t)\right)\\ -\left(u(t)-\overline{u}(t)\right)^T\frac{\partial g}{\partial u}(\overline{x}(t),\overline{u}(t),t)\overline{\lambda}(t)\end{bmatrix}dt$$

$$+\left(x(t_f)-\overline{x}(t_f)\right)^T\frac{\partial\psi}{\partial x}\left(t_f,\overline{x}(t_f)\right), \quad 33$$

Integrating by parts, with a note that $x(a)=\overline{x}(a)$, and using Eq.32, to obtain

$$\geq\int_a^{t_f}\begin{bmatrix}\left(\overline{\lambda}(t)\right)^T\left(\mathcal{A}\begin{bmatrix}x_1'(t)\\ x_2'(t)\\ \vdots\\ x_n'(t)\end{bmatrix}+\mathcal{B}\,{}^{CK}_{\ a}D^{\alpha,\rho}_{t}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}\right)\\ -\left(\overline{\lambda}(t)\right)^T\left(\mathcal{A}\begin{bmatrix}\overline{x_1}'(t)\\ \overline{x_2}'(t)\\ \vdots\\ \overline{x_n}'(t)\end{bmatrix}+\mathcal{B}\,{}^{CK}_{\ a}D^{\alpha,\rho}_{t}\begin{bmatrix}\overline{x_1}(t)\\ \overline{x_2}(t)\\ \vdots\\ \overline{x_n}(t)\end{bmatrix}\right)\\ -\left(\overline{\lambda}(t)\right)^T\frac{\partial g}{\partial x}(\overline{x}(t),\overline{u}(t),t)\left(x(t)-\overline{x}(t)\right)\\ -\left(u(t)-\overline{u}(t)\right)^T\frac{\partial g}{\partial u}(\overline{x}(t),\overline{u}(t),t)\overline{\lambda}(t)\end{bmatrix}dt,$$

By using the dynamic control system in Eq.16, and since g is convex on x and u, from assumption 1, to get

$$\int_a^{t_f}\begin{bmatrix}\left(\overline{\lambda}(t)\right)^T\frac{\partial g}{\partial x}(\overline{x}(t),\overline{u}(t),t)\left(x(t)-\overline{x}(t)\right)\\ +\left(u(t)-\overline{u}(t)\right)^T\frac{\partial g}{\partial u}(\overline{x}(t),\overline{u}(t),t)\overline{\lambda}(t)\\ -\left(\overline{\lambda}(t)\right)^T\frac{\partial g}{\partial x}(\overline{x}(t),\overline{u}(t),t)\left(x(t)-\overline{x}(t)\right)\\ -\left(u(t)-\overline{u}(t)\right)^T\frac{\partial g}{\partial u}(\overline{x}(t),\overline{u}(t),t)\overline{\lambda}(t)\end{bmatrix}dt=0.\ \blacksquare$$

Conclusions:

In this paper, a new system of FOCPs with Caputo–Katugampola FDs ${}^{CK}_{\ a}D^{\alpha,\rho}_{t}x_i(t), i=1,2,\dots,n,$ has been studied. we are assuming that the end time t_f free and a Lagrange multiplier vector $\lambda(t)\in R^n$.The necessary optimality conditions for the system are obtained when $\alpha\in(0,1),\rho>0$ and $a\in R$ and consist of a Hamiltonian system, stationary condition, and transversality conditions which contributes to solving non–linear dynamical control systems with FDs to obtain approximate solutions for state and control variables with the help of the proposed numerical methods. A special case was deduced to study the system of FOCPs if both the final time and the final state are fixed, then in this case the optimality conditions obtained are applied without the transversality conditions. Also, the necessary optimality conditions have been proven to be sufficient for a system of FOCPs.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Mustansiriyah.

Authors' contributions statement:

M.A. H. and S.Q.H. conceived of the presented idea. M.A. H. developed the theory, the design and implementation of the research, the analysis of the results, and the writing of the manuscript. Encouraged S.Q.H. to investigate the results of all theorems, and revise and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

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الشروط المثلى الضرورية والكافية لنظام من مسائل السيطرة المثلى ذات الرتب الكسرية مع مشتقات كابوتو-كاتوجامبولا

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قسم الرياضيات، كلية التربية، الجامعة المستنصرية، بغداد ، العراق.

الخلاصة:

تمت دراسة الشروط المثلى الضرورية مع مضاعفات لاجرانج $\lambda(t) \in R^n$ واشتقاقها لصف جديد يتضمن نظام من مشتقات كابوتو–كاتوجامبولا ذات الرتب الكسرية لمسائل السيطرة المثلى مع افتراض الوقت النهائي مجانا. تم اثبات صيغة التكامل بالاجزاء لمشتقات كابوتو–كاتوجامبولا الكسرية اليسرى والتي تساهم في ايجاد واشتقاق الشروط المثلى الضرورية . ايضاً، تم الحصول على ثلاث حالات خاصة من ضمنها دراسة الشروط المثلى الضرورية في الحالة التي يكون فيها كلا من الوقت النهائي t_f والحل النهائي $x(t_f)$ ثابتة. وفقًا لافتراضات التحدب نثبت الشروط المثلى الضرورية أنها شروط مثلى كافية.

الكلمات المفتاحية: حساب التباينات، مشتقة كابوتو–كاتوجامبولا الكسرية، نظام هاميلتوني، الشروط المثلى الضرورية والكافية، السيطرة المثلى.