2024, 21(4): 1363-1370
https://doi.org/10.21123/bsj.2023.7706
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Dismountable and non-Dismountable Spaces via Proximity Space

***Raghad Almohammed** *[1], **L .A.A.Jabar** [2]*

Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Babylon, Iraq.
*Corresponding Author.

Received 09/09/2022, Revised 24/03/2023, Accepted 26/03/2023, Published Online First 20/09/2023, Published 01/04/2024

Abstract

Most problems studied by researchers were converted into mathematical formulas in the proximity space because this space is easily employed to solve the different problems of life. Therefore, the current study attempts to find a new property for proximity spaces using the proximity and cluster properties and linking them to obtain new cumulative points, it is called the *Bushy Points* and the set of all these points is the *Bushy set*. They form a coherent mathematical foundation for the topological formula that is looking for, which can be called *dismountable* space. If the space does not have this property, it can be called a *non-dismountable* space.

Keywords: Attached space, Bushy set, Scant set, Sparse set, Strongly non-dismountable.

Introduction

The essential focus in building the mathematical axis in this work is the proximity space, first blocks were laid by Reazi [1]. Then, the focus that emerged from it is the cluster family, which was built on the properties of proximity founded by Lodato [1]. Then the events followed and intersected in order to highlight the properties of these spaces and their basic features, among them, for example, Efremovic, Leader, Smirnov, Fenstad, and Njastad. The last focus that relied on is the end points, which are considered an important applied mathematical starting point in the course of engineering and life problems experienced.

Researchers played an essential role in supporting this aspect with many mathematical concepts within the proximity space. Hence, some researchers used the proximity space and linked it with the ideal concept in order to find a new type of end point [2-5]. Others tried to employ the proximity space in another direction, which is creating new sets and constructing set theory ($\sigma - Algebra$) and then a topological space based on those sets[6-9]. Other contemporary researchers tried to enrich this space with many interesting concepts[10-13].

The current study presents special sets, studies their properties and use them to find concepts to solve and obtain topological properties based on these sets.

Results and Discussion

The following are the basic definitions related to the proximity spaces and clusters and some theorems needed in this work:

Definition 1:[1] The ordered pair (X, δ) represents a proximity space if the following intuitive is satisfied the axioms

2024, 21(4): 1363-1370
https://doi.org/10.21123/bsj.2023.7706
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

P1) If $A\delta B$, then $B\delta A$;

P2) $A\delta(B \cup H)$ if and only if $A\delta B$ or $A\delta H$;

P3) $A\delta B$, then $A \neq \emptyset$ and $B \neq \emptyset$;

P4) If $A \cap B$, then $A\delta B$;

P5) If $A\bar{\delta}B$, then there exists $E \in \mathcal{P}(X)$ such that $A\bar{\delta}E$ and $X - E\bar{\delta}B$; (Efremovich axiom)

P6) $A\delta B$ and $\{b\}\delta H$ for every $b \in B$, then $A\delta H$ (Lodato axiom).

Definition 2:[1] Subfamily of the proximity space is called a cluster denoted by σ if and only if it satisfies the following three conditions:

C1) For all $A, B \in \sigma \Longrightarrow A\delta B$;

C2) $(A \cup B) \in \sigma \Leftrightarrow A \in \sigma$ or $B \in \sigma$;

C3) $A\delta B$ for each $B \in \sigma \Rightarrow A \in \sigma$.

Proposition 1:[1] Let (X, δ) be a proximity space. A, B nonempty subsets of X, then

1) For any $E \subset X$ either $E \in \sigma$ *or* $E^c \in \sigma$.

2) If $A \in \sigma$ *and* $A \subset B$, then $B \in \sigma$.

3) $\sigma_x = \{A \subset X;\ A\delta x\}$ is cluster point;

4) If $B \subseteq H$ *and* $A\delta B \Longrightarrow A\delta H$;

Theorem 1: [1] Let (X, δ_x) and (Y, δ_y) be two proximity spaces then a function $f: X \longrightarrow Y$ is $\delta-$ continuous if and only if $A\delta_x B \Rightarrow f(A)\delta_y f(B)$ for each $A, B \subseteq X$.

Theorem 2: [1] Let f be a proximity mapping from (X, δ_X) to (Y, δ_Y). Then for each cluster σ_X in X, there corresponds a cluster σ_Y in Y such that $f(\sigma_X) = \{A \subset Y;\ A\delta_Y f(B)$ for every B *in* $\sigma_X\}$.

Definition 3: Let σ a cluster in a proximity space (X, δ), the binary relation $\approx_\sigma$ defined on (X, δ) as follows:

$A \approx_\sigma B \Leftrightarrow (A - B) \cup (B - A) \in \sigma$ or
$A \approx_\sigma B \Leftrightarrow (A \cup B) - (B \cap A) \in \sigma$.

Definition 4: Let (X, δ) be a proximity space, σ is cluster define on X. then σ −Topological Proximity denoted by τ_σ is the family nonempty subsets of X satisfies the conditions:

1) $X, \emptyset \in \tau_\sigma$;

2) For every sub collection $\mathcal{U}$ of τ_σ there exists $\mathcal{V} \in \tau_\sigma$ such that $\cup\mathcal{U} \approx_\sigma \mathcal{V}$;

3) For every $\mathcal{U}, \mathcal{V} \in \tau_\sigma$ there exists $\mathcal{W} \in \tau_\sigma$ such that $(\mathcal{U} \cap \mathcal{V}) \approx_\sigma \mathcal{W}$;

4) $\tau_\sigma \cap \sigma = X$.

The pair (X, δ) is called proximity space, that is, if (X, δ) is proximity space, and τ is any topology defined on X and σ is a cluster defined on (X, δ), then the quadruple $(X, \tau, \delta, \sigma)$ is called proximity cluster topological space. So that, the quadruple pair $(X, \delta, \sigma, \tau_\sigma)$ is denoted of $\sigma-$ Topological Proximity Space.

Definition 5: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. A point $x \in X$ is said to be *follower point* of a subset A of topological space (X, τ) , if and only if for every $\mathcal{U} \in \tau(x)$, and every $C \in \sigma$ such that $(\mathcal{U} \cap A)\delta C$, where $\tau(x)$ is the set of all open neighborhood of point x. All the follower points of a set A is denoted by A_{f_σ} .

Definition 6: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. A point $x \in X$ is said to be *takeoff point* of a subset A of topological space (X, τ) , if and only if there exist $\mathcal{U} \in \tau(x)$, such that $(\mathcal{U} \cap A^c)\bar{\delta}C$ for some $C \in \sigma$. All the takeoff points of a set A is denoted by A_{t_σ}.

Example 1: Let $X = \{1,2,3\}$, δ is discrete proximity ($A\,\delta B \Leftrightarrow A \cap B \neq \emptyset$ (See [1]), $\sigma = \{\{2\}, \{1,2\}, \{2,3\}, X\}$. Let $\tau = \{X, \emptyset, \{3\}, \{1\}, \{1,3\}\}$, and $A = \{2,3\}, B = \{1\}, H = X$.

$1 \notin A_{f_\sigma}$ because $\{1\} \in \tau(1)$, and $(\{1\} \cap \{2,3\})\bar{\delta}C$ for every $C \in \sigma$ since $\{1\} \cap \{2,3\} = \emptyset$ and $\emptyset\bar{\delta}C\ \forall C \in \sigma$. Also, $3 \notin A_{f_\sigma}$ because $\{3\} \in \tau(3)$, and $\{2\} \in \sigma$ but $(\{3\} \cap \{2,3\})\bar{\delta}\ \{2\}$. Thus $A_{f_\sigma} = \{2\}$ because for every $\mathcal{U} \in \tau(2)$, and every $C \in \sigma$ we have $(\mathcal{U} \cap \{2,3\})\delta C$. Similar of B and H, that is, $B_{f_\sigma} = \emptyset$ and $H_{f_\sigma} = \{2\}$. So that $A_{t_\sigma} = X$ and $H_{t_\sigma} = X$, because $\mathcal{U}, \mathcal{V} \in \tau(x)$, such that $(\mathcal{U} \cap A^c)\bar{\delta}C$ and $(\mathcal{V} \cap H^c)\bar{\delta}C$ for some $C \in \sigma$.

2024, 21(4): 1363-1370
https://doi.org/10.21123/bsj.2023.7706
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

But $B_{t_\sigma} = \{1,3\}$ this means $2 \notin B_{t_\sigma}$ because for every $\mathcal{U} \in \tau(2), (\mathcal{U} \cap \{2,3\})\delta C$ for every $C \in \sigma$.

Bushy Set and Bushy Spaces

The topological space is known to depend on the neighborhood of the point, and therefore these points play an important role in finding important topological properties and features within this space. Researchers in this work are trying to introduce points characterized by the fact that all points of space, and all of their neighborhoods, are cluster elements.

Definition 7: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. A be a nonempty subset of X is called *Bushy set* denoted by $\mathscr{b}(A)$ if and only if for every $x \in X$ *and* every $\mathcal{U} \in \tau(x), \mathcal{U} \cap A\delta C$ for every $C \in \sigma$, and these points are called *bushy points*.

The family of all bushy set on X denoted by $\mathcal{B}(X, \delta)$. Moreover, if the δ define of X is discrete proximity ($A\,\delta B \Leftrightarrow A \cap B \neq \emptyset$ (See [1]), then $\cap\, \mathcal{B}(X, \delta) \neq \emptyset$ since there exists $x \in C$ for every $C \in \sigma$.

Example 2: Let $X = \{1,2,3\}$, δ be a discrete proximity, $\sigma = \{\{1\}, \{1,2\}, \{1,3\}, X\}$, and let $\tau = \{X, \emptyset, \{1\}, \{1,2\}\}$. Let $A = \{1,3\}$ *and* $B = \{3\}$. Then A is a bushy set because every $x \in X$ *and* every $\mathcal{U} \in \tau(x), \mathcal{U} \cap A\delta C$ for every $C \in \sigma$. But B is not bushy set since there exists $1 \in X$ *and* $\{1,2\} \in \tau(1)$ such that $\{1,2\} \cap \{3\}\overline{\delta}C$ for every $C \in \sigma$.

Proposition 2: Let$(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. Then:

1-Every super set of the bushy set is a bushy set.

2- Finite union of bushy sets is also a bushy set.

3-Every bushy set of X is member of cluster. This mean $\mathcal{B}(X, \delta)$ is subset of cluster.

It is to be noted that, according to earlier results, X is not always be a bushy set, but if the space has at least one subset of X is a bushy set, then X is a bushy set.

Proposition 3: Let $(X, \delta, \sigma, \tau_\sigma)$ be a σ −Topological Proximity space. Then

$$\mathcal{B}(X, \delta) = \begin{cases} \mathcal{P}(X) / \emptyset & if\, \delta \text{ is indiscrete} \\ \emptyset & otherwise \end{cases}$$

Proof: Case one**:** If δ is indiscrete proximity, then $\tau_\sigma = \{X, \emptyset\}$, and $\sigma = \{A \subseteq X; A \neq \emptyset\}$. Then only nonempty $\tau_\sigma - open$ set is X for every nonempty subset A of X, that is $(X \cap A)\delta C$ for every $C \in \sigma$. Hence $\mathcal{B}(X, \delta) = \mathcal{P}(X) / \emptyset$.

Case two: If possible there exists nonempty subset A of X is *bushy* set, then every proper $\mathcal{U} \in \tau_\sigma$, $(\mathcal{U} \cap A)\delta C$ for every $C \in \sigma$, by axiom [C3] $\mathcal{U} \cap A \in \sigma$, thus $\mathcal{U} \in \sigma$ which is a contradiction with $\tau_\sigma \cap \sigma = \{X\}$. Hence $\mathcal{B}(X, \delta) = \emptyset$.

Remark 1: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. If A is a bushy set, then A is a dense set.

Proof: Let A be a bushy set. Then for every $x \in X$ *and* every $\mathcal{U} \in \tau(x), \mathcal{U} \cap A\delta C$ for every $C \in \sigma$, that is, $\mathcal{U} \cap A \neq \emptyset$ for every $\mathcal{U} \in \tau(x)$. Hence A is a dense set.

Remark 2: Let $(X, \tau_1, \delta, \sigma)$ and $(X, \tau_2, \delta, \sigma)$be a two proximity cluster topological spaces. If $\tau_1 \subseteq \tau_2$ and A be $\tau_2 - bushy$ set, then A is $\tau_1 - bushy$ set.

Proof: Let A is $\tau_2 - bushy$ set. Then for every $x \in X$ and every $\mathcal{U} \in \tau_2(x), \mathcal{U} \cap A\delta C$ for every $C \in \sigma$. Since $\tau_1 \subseteq \tau_2$, $\mathcal{V} \cap A\delta C$ for every $\mathcal{V} \in \tau_1$. Hence A is $\tau_1 - bushy$ set.

Definition 8: The proximity cluster topological space is called a *Bushy space* if and only if every bushy subset of X is an open.

If we take the indiscrete proximity ($A\delta B$ if and only if $A \neq \emptyset$ and $B \neq \emptyset$) and τ any topology define on X and cluster σ defined on (X, δ), then X is *Bushy space*. As evidenced by Definition 8, the complement of every bushy set in *Bushy space* is a closed set.

Proposition 4: If $(X, \tau, \delta, \sigma)$ is a *Bushy space* has at least one subset of X is a bushy set, then every member of cluster σ is a *Bushy* relative subspace.

Proof: Assuming that X is a *Bushy space*, $Y \subseteq X$, and $Y \in \sigma$. Let $A \subseteq Y$ be a bushy set in Y. Then for every $\mathcal{V} \in \tau_Y$, $(\mathcal{V} \cap A)\delta C$ for every $C \in \sigma_Y$, that is,

2024, 21(4): 1363-1370
https://doi.org/10.21123/bsj.2023.7706
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

$A_{f_\sigma} = Y$. It is sufficient to show that A is τ_Y −open such that $A = \mathcal{U} \cap Y$ for some $\mathcal{U} \in \tau$.
Let $\mathcal{U} = A \cup (X - Y) \Rightarrow \mathcal{U}_{f_\sigma} = (A \cup (X - Y))_{f_\sigma} = A_{f_\sigma} \cup (X - Y)_{f_\sigma}$. According to the space has at least one bushy set we have that $X = X_{f_\sigma} = (Y \cup (X - Y))_{f_\sigma} = Y_{f_\sigma} \cup (X - Y)_{f_\sigma} = (A_{f_\sigma})_{f_\sigma} \cup (X - Y)_{f_\sigma} \subseteq A_{f_\sigma} \cup (X - Y)_{f_\sigma} = \mathcal{U}_{f_\sigma}$, hence $\mathcal{U}$ is bushy set in X, and from X is *Bushy space* we have that $\mathcal{U}$ is open set in X.
So, $\mathcal{U} \cap Y = \left(A \cup (X - Y)\right) \cap Y = A \cap Y = A$ that is, A is τ_Y −open in Y i.e., Y is *Bushy relative subspace*.

The converse don't always be true, because by Example 2 if we take $A = \{1,3\}, then, \tau_A = \{A, \emptyset, \{1\}\}$,$\{1\} \subset A$ is τ_A −bushy set and τ_A −open , $A = \{1,3\}$ is also τ_A −bushy and τ_A −open ,hence τ_A is *Bushy space*. $\{1,3\} \subset X$ is τ − bushy set but not is τ −open set, hence X don't *Bushy space*.

Proposition 5: If every member of cluster is open, then X is a *Bushy space*.

Proof: Let us suppose that A is bushy set in X. Then for every $x \in X$ and every $\mathcal{U} \in \tau(x)$, $\mathcal{U} \cap A\delta C$ for every $C \in \sigma$, and so $\mathcal{U} \cap A \in \sigma$ by Proposition 1 part 2, $A \in \sigma$. Since every member of cluster is open it follows that A is open set and so X is Bushy space.

Remark 3: Let $(X, \tau, \delta, \sigma)$ has F. I. P. (finite intersection property) for every nonempty open sets and every member of cluster is open. A is an open set if and only if A is a bushy set.

Proof: Let us suppose that A is an open set by F. I. P. we have that $\mathcal{U} \cap A \neq \emptyset$ for every $\mathcal{U} \in \tau(x)$. Since every $C \in \sigma$ is open, $(\mathcal{U} \cap A) \cap C \neq \emptyset$ by Proposition 1 part 5 $(\mathcal{U} \cap A)\delta C$ for every $C \in \sigma$ this is true for every $x \in X$, hence A is bushy set. The Converse is the same as the one given in Proposition 5.

Proposition 6: The intersection of any finite family of bushy sets in *Bushy space* is bushy set.

Proof: Let $\{A_i\}_{i=1}^{n}$ be family of bushy set. Since X is *Bushy space* then A_i is open set for every $i \in \Delta$. We prove that by Induction Law. First, to prove the result is true when $i = 2$. Let A_1 and A_2 are bushy sets. Then for every $x \in X$ and every $\mathcal{U} \in \tau(x)$, $\mathcal{U} \cap A_1\delta C$ for every $C \in \sigma$. Since X is *Bushy space*, $A_1 \in \tau$, that is $\mathcal{U} \cap A_1$ is nonempty open set. Since A_2 is bushy set then for every $x \in X$ and every $\mathcal{U} \in \tau(x)$, $\mathcal{U} \cap (A_1 \cap A_2)\delta C$ for every $C \in \sigma$, hence $A_1 \cap A_2$ is bushy set. Second, suppose the result is true for $i = n - 1$. Finally, to prove the result is true for $i = n$. Let A_n is bushy set. Then for every nonempty $\mathcal{U} \in \tau$, $\mathcal{U} \cap A_n\delta C$ for every $C \in \sigma$. But $\{\cap A_i\}_{i=1}^{n-1}$ is nonempty open set this implies that $\mathcal{U} \cap (\{\cap A_i\}_{i=1}^{n-1} \cap A_n)\delta C$ for every $C \in \sigma$. Hence $\{\cap A_i\}_{i=1}^{n}$ is a bushy set.

Corollary 1: The intersection of finite family of dense sets in *Bushy space* is a dense set.

Proof: By Remark 1 every bushy set is dense set. Then by Proposition 6 the intersection of finite family of dense sets in *Bushy space* is a dense set.

Proposition 7: If A is bushy set in *Bushy space* X, then A builds a chain of open and dense subsets of X.

Proof: This is an immediate consequence of Proposition 2 part 1, Remark 1and Definition 8.

Proposition 8: If X is *Bushy space* and $\{A_i \subseteq X, i \in N\}$ is any finite closed cover of X, then $int(A_k) \neq \emptyset$ for some $k \in N$.

Proof: Suppose $X = \bigcup_{i=1}^{n}(A_i)$. By De-Morgan's Law, $\emptyset = \bigcap_{i=1}^{n}(X - A_i)$. Since X is a *Bushy space* and $X - A_i$ is an open set for every i, clear that by Corollary 1 not every $X - A_i$ is dense set, hence there exist at least one $X - A_k$ is not dense set. Thus there exists nonempty open set $\mathcal{U}$ such that $\mathcal{U} \cap (X - A_k) = \emptyset$ that is $\mathcal{U} \subseteq A_k$ for at least one k, hence $int(A_k) \neq \emptyset$.

Proposition 9: Let f be an open and δ −continuous proximity mapping from $(X, \tau, \delta_x, \sigma)$ into $\left(Y, \rho, \delta_y, \sigma\right)$. If $A$is bushy set in X, then $f(A)$is bushy set in Y.

Proof: Suppose A is bushy set in X. Then every $x \in X$ and every $\mathcal{U} \in \tau(x)$, $(\mathcal{U} \cap A)\delta_x C$ for every $C \in \sigma$, since f is δ −continuous this implies $f(\mathcal{U} \cap A)\delta_y f(C)$ for every $f(x) \in Y$ and every$f(\mathcal{U})$ is neighborhood of $f(x)$, $f(\mathcal{U} \cap A) \subseteq (f(\mathcal{U}) \cap f(A))\delta_y f(C)$ for every$f(C) \in f(\sigma)$. Since f is open and δ −continuous , $f(\mathcal{U}) \in \rho$ and $f(\sigma)$ is cluster in Y . Thus $f(A)$ is bushy set in Y.

2024, 21(4): 1363-1370
https://doi.org/10.21123/bsj.2023.7706
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Proposition 10: Let f be an injective open $\delta-$continuous mapping from $(\mathrm{X}, \tau, \delta_x, \sigma)$ into $(\mathrm{Y}, \rho, \delta_y, \sigma)$. If Y is $Bushy\ space$, then X is also $Bushy\ space$.

Proof: Let $A \subseteq X$ is bushy set in X. By Proposition 9, $f(A)$ is also bushy set in Y. Since Y is $Bushy\ space$, $f(A)\ is\ \rho-$open. Further, f is injective continuous this implies $f^{-1}(f(A)) = A$ is $\tau-$open, hence X is $Bushy\ space$.

Definition 9: The proximity cluster topological space is called $Attached\ space$ if and only if for every nonempty open subset of X is bushy subset.

Example 3: Let $X = \{1,2,3\}$, δ is a discrete proximity, $\sigma = \{\{3\},\{1,3\},\{2,3\},X\}$. Let $\tau = \{X, \emptyset, \{3\}, \{2,3\}\}$. Then $\{2,3\}, \{3\}$ are open and bushy sets, hence X is attached space.

Note that, if τ is a discrete topology, then X is not attached space since every singleton set is an open but not bushy set. Also can be note that, attached space cannot be separated by two nonempty open subsets because if possible X can be separated by two nonempty open subsets A and B such that $A \cap B = \emptyset$, $A \cup B = X$, that is, $A = X - B\ and\ B = X - A$. Since A is bushy set, A is dense set hence $cl(A) = X$ and so $B = int(B) = X - cl(X - B) = X - cl(A) = X - X = \emptyset$, which is an contradiction.

In addition, it concludes from the above the following proposition:

Proposition 11: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space, the following statements are equivalent:

1- Every nonempty open subset is a bushy set.

2- The cluster σ defines on (X, δ) containing every nonempty open subset.

Proof: 1 ⇒ 2 Let us assume that every nonempty open subset is a bushy set. Let us consider G is bushy open set. Then every $x \in X$ and every $\mathcal{U} \in \tau(x)$, $\mathcal{U} \cap G \delta C$ for every $C \in \sigma$ hence $G \in \sigma$.

2 ⇒ 1 If possible $G \in \tau$ is not a bushy set. Then there exists $\mathcal{U} \in \tau(x)$, $\mathcal{U} \cap G \bar{\delta} C$ for some $C \in \sigma$ hence $\mathcal{U} \cap G \notin \sigma$ this is a contradiction with hypothesis. Thus G is bushy set.

Corollary 2: If the cluster σ containing every nonempty open subset, then the space is attached space.

Proof: This is an immediate consequence of Proposition 11and Definition 9.

Thus we can see that if the space satisfies Remark 3, then the space is $Bushy\ space$ and the Attached space together.

Definition 10: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space, a subset A of X is called $scant\ set$ if and only if $(A_f)_t = \emptyset$.

By Example 2, the sets $\{2, 3\}$, $\{2\}$, $\{3\}$ and $\emptyset$ are scant set. Evident, if $\emptyset_t = \emptyset$, then $\emptyset$ is scant set and if A is bushy set then $X - A$ is scant for every $X - A \notin \sigma$. Because $(X - A)_f = \emptyset$ and $((X - A)_f)_t = X - (X - (X - A)_f)_f = X - (X - \emptyset)_f = X - X_f = X - X = \emptyset$.

It is also clear that the takeoff set in the proximity cluster topology space is an open set. Therefore, this set becomes dense when the space carries an "attached" property.

Definition 11: $(\mathrm{X}, \tau, \delta, \sigma)$ is non-dismountable space if and only if X cannot contains two disjoint bushy subsets. Otherwise X is dismountable space.

It is easy to see that $(\mathrm{X}, \delta, \sigma, \tau_\sigma)$ is a non-dismountable space when δ is not indiscrete proximity because it doesn't have any bushy subset in this space. Also, every dismountable space is a resolvable space. Because if X is a dismountable space, there exist two disjoint bushy subsets, that is, there exist two disjoint dense subsets, hence X is a resolvable space.

There is a relationship between dense, attached, non-dismountable, and irresolvable space, which is embodied in the following proposition,

Proposition 12: Let $(\mathrm{X}, \delta, \tau, \sigma)$ is attached space and submaximal. X is a non-dismountable space if and only if X is irresolvable space.

Proof: If suppose that X is resolvable. Then there exist two disjoint dense sets A, B such that $A \cap B = \emptyset, A \cup B = X$. Since every dense set is open and X is attached space, $A\ and\ B$ are bushy sets, that is, X is

2024, 21(4): 1363-1370
https://doi.org/10.21123/bsj.2023.7706
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

dismountable. But this is a contradiction with hypothesis. Conversely, If suppose that X is dismountable. Then there exist two disjoint bushy sets A, B such that $A \cap B = \emptyset, A \cup B = X$. Since every bushy set is dense set, A and B are dense sets, hence X is resolvable, this is a contradiction. Thus X is a non-dismountable space.

Proposition 13: Let τ_1 and τ_2 be two topological defined on X, such that $\tau_1 \subseteq \tau_2$. If $(X, \tau_1, \delta, \sigma)$ is non-dismountable space, then $(X, \tau_2, \delta, \sigma)$ is non-dismountable space.

Proof: Let us suppose that τ_2 is dismountable space. Then there exist two disjoint τ_2 −bushy sets A, B such that $A \cap B = \emptyset, A \cup B = X$. By Remark 2 every τ_2 −bushy set is τ_1 −bushy set, that is τ_1 is dismountable space this contradiction, hence $(X, \delta, \tau_2, \sigma)$ is non-dismountable space.

The above proposition can be generalized to any topology that is finer than the non-dismountable topology defined on the same cluster.

Definition 12: The proximity cluster topological space $(X, \tau, \delta, \sigma)$ is called,

1- Hereditarily non-dismountable space if and only if every subspace is non-dismountable.

2- Strongly non-dismountable space if and only if each open subspace is non-dismountable.

For example, if we take a special type proximity, the discrete proximity, then the space $(X, \tau, \delta, \sigma)$:

- Is non-dismountable space.
- The subspace of X is also non-dismountable space.
- Is hereditarily non-dismountable space.
- Is strongly non-dismountable Space.

Proposition 14: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. Then every *Bushy space* is non-dismountable space.

Proof: If possible that X is a dismountable space. Then there exist two disjoint bushy sets A, B such that $A \cap B = \emptyset, A \cup B = X$, that is $A = X - B$ and $B = X - A$. Since A is bushy set, A is dense set, so that $cl(A) = X$ but $B = int(B) = X - cl(X - B) = X - cl(A) = X - X = \emptyset$, this is a contradiction with hypothesis. Thus X is non-dismountable space.

Proposition 15 Let $(X, \tau, \delta, \sigma)$ be a non-dismountable space and for every subset A of X satisfies $A \subseteq A_{f_\sigma}$. The following statement is equivalent:

1- Every open subspace of X is non-dismountable space.

2- Every takeoff set of bushy set is also bushy set.

3- Every complement of bushy subset of X is scant set.

4- Every closed subset of X is the union of open set and scant set.

Proof: 1 ⇒ 2 Suppose every open is non-dismountable. Let A is bushy subset of X, that means $A_{f_\sigma} = X$. If possible $(A_{t_\sigma})_{f\sigma} \neq X$,and $G = X - (A_{t_\sigma})_{f\sigma} \neq \emptyset$. Since $(A_{t_\sigma})_{f\sigma}$ is closed set, G is open set, and so $G \cap A_{f\sigma} \subseteq (G \cap A)_{f\sigma}$ hence $G = G \cap (G \cap A)_{f\sigma}$, that is, $(A \cap G)$ is bushy in G. By hypothesis $A_{t_\sigma} \subseteq \left(A_{t_\sigma}\right)_{f\sigma}$ thus $G = X - \left(A_{t_\sigma}\right)_{f\sigma} \subseteq X - A_{t_\sigma} = (X - A)_{f\sigma}$. Since G is open set, $G = G \cap (X - A)_{f\sigma} \subseteq (G \cap X - A)_{f\sigma}$ this implies to $G = G \cap (G - A)_{f\sigma}$ this means $(G - A)$ is bushy set in G. Further, $G = (G \cap A) \cup (G - A)$ and $(G \cap A) \cap (G - A) = \emptyset$, so $(G \cap A), (G - A)$ are bushy set in G hence G is dismountable subspace, which is a contradiction with hypothesis.

2 ⇒ 3 Suppose $X - A$ is bushy set. By part 2, $\left((X - A)_{t_\sigma}\right)_{f\sigma} = X$ by De-Morgan's law we have $\emptyset = X - \left((X - A)_{t_\sigma}\right)_{f\sigma} = \left(A_{f_\sigma}\right)_{t\sigma}$, hence A is scant set.

3 ⇒ 4 Let $int(A) \in \tau$. Since $int(A) \subseteq A_{t_\sigma}$, $A_{t_\sigma} = \emptyset$ or $A_{t_\sigma} \neq \emptyset$. In case of $A_{t_\sigma} = \emptyset$ this implies $\emptyset = A_{t_\sigma} = X - (X - A)_{f_\sigma} \Rightarrow (X - A)_{f_\sigma} = X$, that is, $(X - A)$ is bushy set. By part 3, A is scant set, hence $A = \emptyset \cup A$ where $\emptyset \in \tau$ and A is scant set. In case of $A_{t_\sigma} \neq \emptyset$. $(A - A_{t_\sigma})_{t_\sigma} = (A \cap X - A_{t_\sigma})_{t_\sigma} = A_{t_\sigma} \cap \left(X - A_{t_\sigma}\right)_{t_\sigma} \subseteq (A_{t_\sigma})_{t_\sigma} \cap (X - A_{t_\sigma})_{t_\sigma} = (A_{t_\sigma} \cap X - A_{t_\sigma})_{t_\sigma} = \emptyset_{t_\sigma} = \emptyset$. Thus $\emptyset = X - (X - A - A_{t_\sigma})_{f_\sigma}$by De-Morgan's law we

have$(X - A - A_{t_\sigma})_{f_\sigma} = X$ this means $(X - A - A_{t_\sigma})$ is bushy set $\Rightarrow (A - A_{t_\sigma})$ is scant set, and so $A = A_{t_\sigma} \cup (A - A_{t_\sigma})$where A_{t_σ} is open and $(A - A_{t_\sigma})$ is scant set.

$4 \Rightarrow 1$ If possible G is nonempty dismountable, there exist A, B subsets of G such that $G = A \cup B$, $A \cap B = \emptyset$ and $G \subseteq A_{f_\sigma}$, $G \subseteq B_{f_\sigma}$, Since $A_{f_\sigma}$is closed set , $A_{f_\sigma} = \mathcal{U} \cup C$ where $\mathcal{U}$ is open set and C is scant set, so that $\mathcal{U} \neq \emptyset$ otherwise $A_{f_\sigma} = C$, this mean A_{f_σ} is scant set $\Rightarrow \left(A_{f_\sigma f_\sigma}\right)_{t\sigma} = \emptyset$ since $A \subseteq A_{f_\sigma} \Rightarrow \left(A_{f_\sigma f_\sigma}\right)_{t\sigma} = \left(A_{f_\sigma}\right)_{t\sigma}$ but $G \subseteq A_{f_\sigma}$ this implies to $G \subseteq G_{t_\sigma} \subseteq \left(A_{f_\sigma}\right)_{t\sigma} = \emptyset \Rightarrow G = \emptyset$ this is an contradiction , hence $\mathcal{U} \neq \emptyset$. Clear that $int(\mathcal{U}) = \mathcal{U}$ and $int(\mathcal{U}) \subseteq int(A)$. Since A is bushy set in G, $int(A) \cap B \neq \emptyset \Rightarrow A \cap B \neq \emptyset$ this is a contradiction. Thus G is non-dismountable.

Proposition 16: Let $(X, \delta, \tau, \sigma)$ be a *Bushy space* and $X = X_{f_\sigma}$, then X is hereditary non-dismountable.

Proof: Let Y is nonempty *Bushy subspace* of $X$and there exist nonempty A, B are subsets of y such that $Y = A \cup B$, $A \cap B = \emptyset$ and $Y \subseteq A_{f_\sigma}$, $Y \subseteq B_{f_\sigma}$, that is, $X - A_{f_\sigma} \subseteq X - Y \Rightarrow (X - A_{f_\sigma})_{f_\sigma} \subseteq (X - Y)_{f_\sigma}$. $X = X_{f_\sigma} = ((X - A_{f_\sigma}) \cup A_{f_\sigma})_{f_\sigma} = \left(X - A_{f_\sigma}\right)_{f_\sigma} \cup (A_{f_\sigma})_{f_\sigma} \subseteq \left(X - A_{f_\sigma}\right)_{f_\sigma} \cup A_{f_\sigma} \subseteq (X - Y)_{f_\sigma} \cup A_{f_\sigma}$=$(X - B)_{f_\sigma}$, this mean $(X - B)_{f_\sigma} = X$, hence $X - B$ is bushy set in X. By hypothesis B is closed hence $Y \subseteq B_{f_\sigma} \subseteq B$ that is, $y = B$ this implies that $A = \emptyset$ which is a contradiction with hypothesis.

Conclusion

One problem that has drawn the attention of researchers is the possibility of finding resolvable and non-resolvable spaces, as the concept of density was used as a criterion in defining these spaces. But the question arises, is it possible to find another criterion parallel to the concept of density, through which it is possible to determine the resolvability of spaces.

In this study, sets were presented to study the possibility of dismountable the defined spaces on the proximity space and building other spaces that could not be dismountable, while studying the most important characteristics of those concepts and their relationship to the concepts of resolvable and irresolvable that are recognized in the general topological space.

Authors' Declaration

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Babylon.

Authors' Contribution Statement

A. and J. conceived of the presented idea. A. developed the theory and performed the computations. A. and J. verified the analytical methods. J. encouraged A. to investigate this notion and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

References

1. Somashekhar Naimpally. Proximity Approach to Problems in Topology and Analysis. Canada: Gruyter Oldenbourgsee; 2009. Chapter 2, The Lodato Proximity; p. 3-21. https://www.degruyter.com/document/doi/10.1524/9783486598605/pdf .
2. AL Talkany YK, Al Swidi LA. ψ −Operator Proximity in i-Topological Space. Tur Com Mat. 2021; 12(1S): 679-684. https://turcomat.org/index.php/turkbilmat/article/download/1973/1704/3706 .
3. AL Talkany YK, Al Swidi LA. New Concepts of Dense set in i-Topological space and Proximity Space. Tur Com Mat. 2021; 12: 685-690. https://turcomat.org/index.php/turkbilmat/article/view/1974 .
4. Hadi MH, Hadi HA, Al Swidi LA. Development of local function. J Interdiscip Math. 2022; 25: 2503-2509.

https://www.tandfonline.com/doi/abs/10.1080/09720502.2022.2040849 .

5. Qahtan GA, Al Swidi LA. Shrink central continuous function. J Interdiscip Math. 2022; 25: 2617-2622. https://www.tandfonline.com/doi/abs/10.1080/09720502.2022.2057051 .
6. Abdulsada DA, Al-Swidi LAA. Center Set Theory of Proximity Space. *J Discret Math. Sci Cryptogr*. 2021; 1804 (1): 012130. https://iopscience.iop.org/article/10.1088/1742-6596/1804/1/012130 .
7. Hadi MH, Al Swidi LA. The Neutrosophic Axial Set theory. Neutrosophic Sets Syst. 2022; 51:1-9. http://fs.unm.edu/NSS/NeutrosophicAxial18.pdf .
8. Al Swidi LA, Regadh DA, Hadi MH. About the dual soft theory. *J Discret Math. Sci Cryptogr*. 2022; 25(8): 2717-2722. https://doi.org/10.1080/09720529.2022.2060917 .
9. Regadh DA, Al Swidi LA, Hadi MH. On fuzzy Intense separation axioms in fuzzy ideal topological space. JIM. 2022; 25(2): 1357-1363. https://www.tandfonline.com/doi/abs/10.1080/09720502.2022.2040855.
10. Almohammed R , Al Swidi LA. New concepts of fuzzy local function .Baghdad Sci J. 2020; 17: 515-522. https://www.iasj.net/iasj/pdf/630852e69e1b32a9.
11. Al Swidi LA, Awad FSS. On Soft Turning Points. Baghdad Sci J. 2018; 15(3): 352-360. https://www.bsj.uobaghdad.edu.iq/index.php/BSJ/article/view/2484/2415.
12. Ayat T, Amal M. Some New Fixed Point Theorems in Weak Partial Metric Spaces. Baghdad Sci J. 2020; 18: 175-180. https://doi.org/10.21123/bsj.2022.6724.
13. Hadeel H, Salwa A. Fixed Point Theorems in General Metric Space with an Application. Baghdad Sci J. 2021; 18(1): 812-815. https://www.iasj.net/iasj/pdf/69cdf244e05ed03a.

الفضاءات القابلة للتفكك والفضاءات غير القابلة للتفكك عبر فضاء القرب

رغد حامد المحمد، لؤي عبد الهاني جبار

قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة بابل، بابل، العراق.

الخلاصة

اهتم كثيرا من الباحثين في دراسة اغلب المشكلات بعد تحويلها الى صيغ رياضية داخل فضاء القرب لسهولة توظيف هذا الفضاء مع المشكلات الحياتية المختلفة التي تواجهنا. لذلك يصب اهتمام البحث في ايجاد صفة جديدة الى فضاءات القرب باستخدام خواص القرب وخواص العناقيد وربطها معا للحصول على نقاط تراكمية جديدة اطلقنا عليها اسم نقاط كثه ومجموعة كل هذه النقاط اطلقنا عليها اسم مجموعة كثه حيث تم تكوين اساس رياضي متماسك الى الصيغة التبلوجية التي نبحث عنها والتي اطلقنا عليها التفكك واذا كان الفضاء لا يحمل هذه الصفة نطلق عليه فضاء غير قابل للتفكك.

الكلمات المفتاحية: فضاء ملتصق، مجموعة كثه، مجموعة ضئيلة، مجموعة متناثرة، فضاء لا يمكن تفكيكه بشدة.