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Almost Projective Semimodules

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Abstract:

The basis of this paper is to study the concept of almost projective semimodules as a generalization of projective semimodules. Some of its characteristics have been discussed, as well as some results have been generalized from projective semimodules.

Keywords: Almost projective semimodules, Injective semimodules, Projective semimodules, Semimodules, Semirings.

Introduction:

Module theory is one of the important branches of algebra. In recent years, interest in semimodules has appeared as a generalization and extension of the module. The Projective module plays a useful role in the study of the category of modules. Some authors generalize projective modules for semimodule. In this paper projective semimodule has been expanded to almost projective semimodule. Every module over the ring is semimodule but the converse is not true. The major objective of this work is to represent and investigate the dual notion of almost injective semimodules. In [1], the authors study the concept of an almost projective module and they discussed the concept of almost injective modules as a dual of almost projective modules. In this paper, almost projective modules have been expanded for semimodules taking into account the differences between modules and semimodules, which are mainly derived from their definitions. Almost projective semimodule which is a dual of almost Injective semimodule studied by K. Aljebory and A. Alhossaini [2]. If Ṃ and Ɲ are two semimodule, a semimodule Ṃ is said to be almost Ɲ-projective, if for each epimorphism ϵ: Ɲ ⟶ X where X is any semimodule and every homomorphism δ: Ṃ ⟶ X, either there is a homomorphism ψ: Ṃ ⟶ Ɲ such that $\epsilon\psi = \delta$, or, there is a homomorphism γ: Y ⟶ Ṃ where Y is a nonzero direct summand of Ɲ such that $\delta\gamma = \epsilon\lambda_Y$ where λ_Y is the injection map from Y into Ɲ. A semimodule Ṃ is called almost-projective if it is almost Ɲ-projective for every finitely generated Ȓ-semimodule Ɲ. Some characterizations of this notion have been discussed. Throughout this work, Ȓ will be semiring with identity and Ṃ be a unitary Ȓ-semimodule. A semiring Ȓ is a nonempty set with both binary operations addition and multiplication such that; (Ȓ, +) is an abelian monoid with identity element 0; (Ȓ,·) is a monoid with identity 1; the multiplication distributives over the addition and 0 ɽ = ɽ 0, ∀ɽ ∈ Ȓ [3]. Clearly, every ring is a semiring but the converse is not true, a trivial example of a semiring that is not a ring is (ℕ, +, ·). A left Ȓ-semimodule Ṃ is a commutative monoid (Ṃ,+) together with operation Ȓ × Ṃ → Ṃ; defined by (ȿɱ)⟼ȿɱ such that ∀ȿ, ɽ ∈ Ȓ and ɱ, ɳ∈Ṃ, ȿ(ɱ + ɳ) = ȿɱ + ȿɳ; (ȿ + ɽ)ɱ = ȿɱ + ɽɱ; (ȿɽ)ɱ = ȿ(ɽɱ); $0_{Ȓ}$ɱ = $0_{Ṃ}$ = ȿ$0_{Ṃ}$ and $1_{Ȓ}$ɱ = ɱ if there is $1_{Ȓ}$, Ṃ is called unitary semimodule [3] (in the same way the right semimodule is defined). A nonempty subset Z of a semimodule Ṃ is called a sub semi-module if Z is closed under addition and scalar multiplication. An Ȓ-subsemimodule Ӿ of Ṃ is called subtractive if, ∀ƀ, ȿ ∈ Ṃ, ƀ + ȿ ∈ Ӿ, ƀ ∈ Ӿ implies ȿ ∈ Ӿ. An Ȓ-semimodule Ṃ is said to be subtractive if it has only subtractive subsemimodules[3]. An Ȓ-semimodule Ṃ is called semisubtractive if for any ɱ, ɳ ∈ Ṃ, there is ȿ ∈ Ṃ such that ɱ + ȿ = ɳ, or some ƀ ∈ Ṃ that ɳ + ƀ = ɱ [4]. This research branches into two parts, in part 1, some definitions and remarks that have needed in the paper and found in some literature have been represented. Part 2, deals with this topic with some of its qualities and linked it to other concepts.

Preliminaries: Concepts related to this work are defined in this section.

Definition 1:[5]An element ɱ of semimodule Ɱ is said to be cancellable if $ɱ + ɳ = ɱ + ʂ$, then $ɳ = ʂ$. If each element in Ɱ is cancellable, then Ɱ is called a cancellative semimodule.

Definition 2: [6] Let Ӿ and ¥ be two subsemimodules of Ȑ-semimodule Ɱ. Ɱ is called the direct sum of Ӿ and ¥, if each ɱ ∈Ɱ has uniquely written as $ɱ = x + y$ where $x \in$ Ӿ, $y \in$ ¥, denoted by Ɱ = Ӿ⨁¥, each Ӿ and ¥ is said to be a direct summand of Ɱ.

Remark 1: If Ɱ is a direct sum of sub semi-modules Ӿ and ¥, then Ӿ ∩ ¥ =0 and Ɱ = Ӿ + ¥, but the converse is not true. For K_4 ={0, 1, a, b}, (K_4 ,+) is monoid with addition defined by; ǩ + ǩ = ǩ , ǩ + ɦ =1 and for all ǩ , ɦ ∈ K_4 , K_4 is 𝔹-semimodule where 𝔹={0,1}with 1+1=1. (𝔹,+,·) is a semiring and it is called Boolean semiring. Take a subsemimodule Ӿ = {0, 1, a}, then Ӿ = {0, 1} + {0, a} and {0,1} ∩ {0, a} = {0} but 1 can be written by, 1 = 1 + 0 and 1 = 1 + a this means 1 has no uniquely representation. The converse is true if the semimodule Ɱ has the conditions; semi-sub tractive and cancellative [6]. Knowing that it is always satisfied in the module.

Remark 2:[7] An Ȑ-semimodule T is called free if and only if T is isomorphic to a direct sum of copies of a semiring Ȑ.

Remark 3: The Ȑ-semimodule Ȑ is free (by Remark 2).

Definition 3:[6] An Ȑ-semimodule Ɱ is said to be an Ɲ-projective semimodule, if for each Ȑ-epimorphism ∅: Ɲ → Ӿ and each Ȑ-homomorphism γ: Ɱ → Ӿ, there is a homomorphism θ: Ɱ → Ɲ such that ∅θ= γ. A semimodule Ɱ is called projective if it is projective relative to every Ȑ-semimodule.

Proposition 1:[7] Every free semimodule is projective. In particular, every semiring with identity is projective over itself.

Definition 4:[8] A subsemimodule Ӿ of Ɱ is said to be fully invariant if for each endomorphism ∅ of Ɱ, then ∅(Ӿ) ⊆ Ӿ.

Remark 4:[6] The subtractive subsemimodules of ℕ over itself are of the form k ℕ, $k \in$ ℕ. Where ℕ is the set of natural numbers.

Example 1: Every subtractive subsemimodule of ℕ as ℕ-semimodule is fully invariant. Since the subtractive subsemimodules of ℕ are of the form kℕ, k∈ ℕ let ∅: ℕ → ℕ, then ∅(kℕ) = k∅(ℕ) ⊆ kℕ.

Definition 5:[9] A semimodule Ɱ is called to be indecomposable if the direct summands of it are only {0} and Ɱ.

Almost projective semimodules

In this section, a new generalization of projective semimodules is introduced and some characterizations of this notion are discussed.

Definition 6: A semimodule Ɱ is said to be almost Ɲ-projective, if for each epimorphism α: Ɲ ⟶ Ӿ and every homomorphism δ: Ɱ ⟶ Ӿ, either there is ψ: Ɱ ⟶ Ɲ such that αψ = δ, or there is a homomorphism γ: ¥ ⟶ Ɱ where ¥ is a nonzero direct summand of Ɲ such that $\delta\gamma$ = α$\lambda_{¥}$, where $\lambda_{¥}$: ¥ ⟶ Ɲ is the injection map. This means either Fig.1a or Fig.1b commutes.

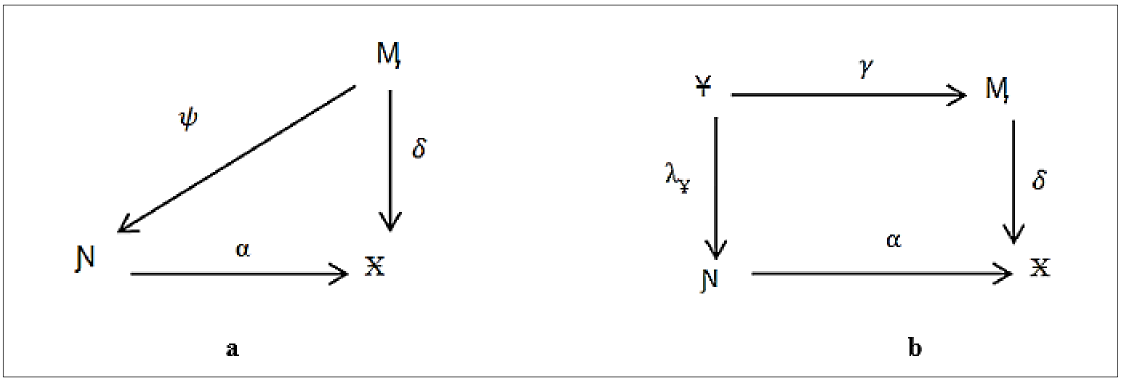

Figure 1. Ɱ **is almost** Ɲ**-projective**

A semimodule Ɱ is called almost-projective if it is almost Ɲ-projective for every finitely generated Ȑ-semimodule Ɲ.

Remarks 5:

(1) If Ɱ is an Ɲ-projective semimodule, then Ɱ is an almost Ɲ-projective semimodule.

(2) Every projective semimodule is almost-projective.

Remark 6: It is well known that every projective module is a direct summand of a free module, this is not true in semimodule as Example (2.3) in[10].

Remark 7: Every semiring R̈ with identity is almost-projective R̈-semimodule.
Proof: Since every projective semimodule is almost-projective, by Remark 5 and from Proposition 1 every semiring with identity is projective over itself. For example, the set of natural numbers ℕ which is a semiring, ℕ over itself is almost -projective semimodule, and (𝔹, +, ·) is a Boolean semiring with identity 1, 𝔹 as 𝔹-semimodule is almost-projective

Some basic characterizations of almost-projective semimodule will be investigated in this section.
Proposition 2: Let Ɱ be almost Ɲ-projective semimodule and ¥ is a direct summand of Ɱ, then ¥ is almost Ɲ-projective again.
Proof: Assume that ¥ is a direct summand of Ɱ and Ɱ is almost a Ɲ-projective semimodule. Consider Fig.2a and Fig.2b.

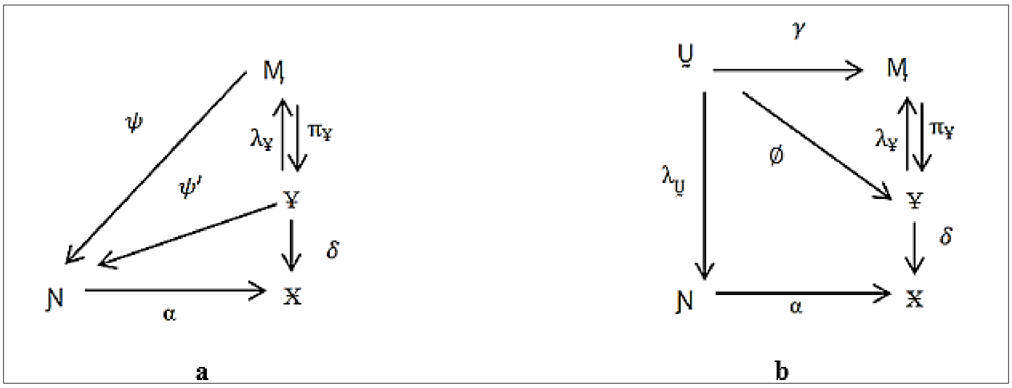

Figure 2. Summand of Ɱ

Where $\pi_¥$ and $\lambda_¥$ are the projection and injection maps respectively and Ӿ is any semimodule. Since Ɱ is almost Ɲ- projective semimodule, then either there is ψ:Ɱ ⟶ Ɲ such that $\alpha\,\psi = \delta\ \pi_¥$ Where α:Ɲ ⟶ Ӿ is an epimorphism, or there is γ: Ṵ ⟶ Ɱ such that $\delta\ \pi_¥\gamma = \alpha\lambda_Ṵ$ where Ṵ is a nonzero direct summand of Ɲ, in the first diagram define $\psi' = \psi\lambda_¥$, then $\alpha\ \psi' = \alpha\,\psi\lambda_¥ = \delta\ \pi_¥\lambda_¥ = \delta$. Now from the second diagram, set $\phi = \pi_¥\gamma$, then $\delta\phi = \delta\pi_¥\gamma = \alpha\lambda_Ṵ$.Thus ¥ is almost Ɲ-projective semimodule.
Proposition 3: If Ɱ is almost Ɲ-projective semimodule and Ṵ fully invariant direct summand of Ɲ, then Ɱ is almost Ṵ-projective.
Proof: Let Ɱ be almost Ɲ-projective and Ṵ be fully invariant direct summand of Ɲ. By Fig.3a and Fig.3b

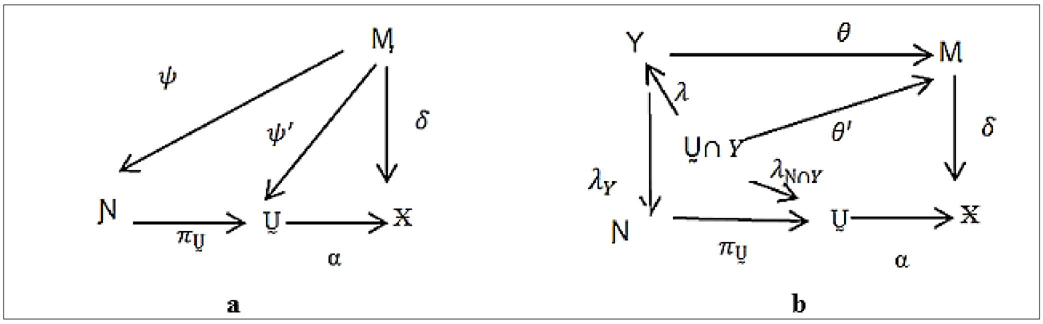

Figure 3. Direct summand of Ɲ

where Ɱ is almost Ɲ-projective, then either there is ψ: Ɱ ⟶ Ɲ such that $\alpha\pi_Ṵ\psi = \delta$, or there is $\theta: Y$ ⟶ Ɱ such that $\delta\theta = \alpha\pi_Ṵ\lambda_Y$ where $0 \neq Y \leq_\oplus$ Ɲ. In the first diagram put $\psi' = \pi_Ṵ\psi$, then $\alpha\psi' = \alpha\pi_Ṵ\psi = \delta$. In the second diagram, define θ': Ṵ ∩ Y ⟶ Ɱ where $0 \neq$ Ṵ ∩ $Y \leq_\oplus$ Ṵ (since Ṵ is a fully invariant direct summand of Ɲ, then Ṵ ∩ Y is a direct summand of Ṵ) such that $\theta' = \theta\lambda$, then $\delta\theta' = \delta\theta\lambda = \alpha\ \lambda_{Ɲ\cap Y}$. Hence Ɱ is almost a Ṵ-projective semimodule.
Proposition 4: Let $Ɱ_1$, $Ɱ_2$ and Ɱ be semimodules, then Ɱ = $Ɱ_1 \oplus Ɱ_2$ is almost Ɲ-projective if and only if $Ɱ_i$ (i=1, 2) is almost a Ɲ-projective semimodule.
Proof: Suppose $Ɱ_i$ ($i = 1, 2$) are almost Ɲ-projective and Ɱ=$Ɱ_1 \oplus Ɱ_2$, consider the following Fig.4a and Fig.4b:

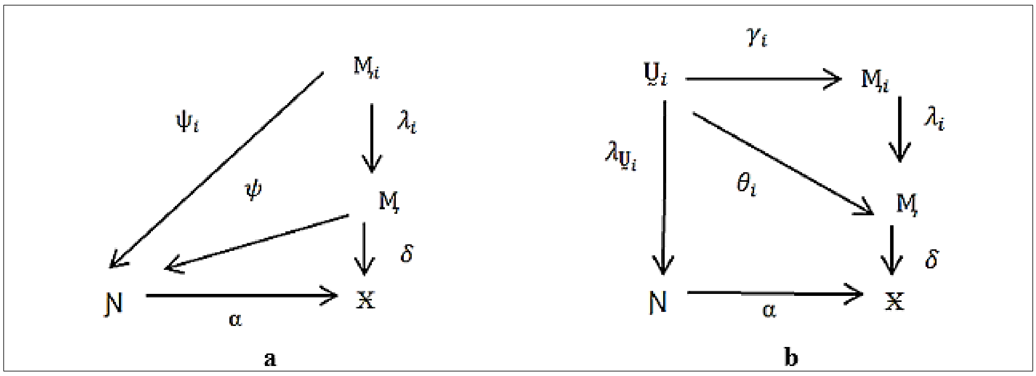

Figure 4. Direct sum of M_i

There are four cases: **Case1/** If M_1 and M_2 satisfy the first diagram, then there is ψ_1: $M_1 \longrightarrow Ɲ$ and ψ_2: $M_2 \longrightarrow Ɲ$ such that $\alpha\psi_1$=$\delta\lambda_1$ and $\alpha\psi_2 = \delta\lambda_2$. Define ψ: $M \longrightarrow Ɲ$ by $\psi = \psi_1 + \psi_1$, then $\alpha\,\psi = \alpha(\psi_1 + \psi_1) = \alpha\psi_1 + \alpha\psi_1 = \delta\lambda_1 + \delta\lambda_2 = \delta(\lambda_1 + \lambda_2) = \delta$. **Case2/** If M_1 and M_2 satisfy the second diagram, this means there is γ_1: $Ṵ_1 \longrightarrow M_1$ and γ_2: $Ṵ_2 \longrightarrow M_2$ where $Ṵ_1$ and $Ṵ_2$ are nonzero direct summands of Ɲ such that $\delta\lambda_1\gamma_1 = \alpha\lambda_{Ṵ_1}$ and $\delta\lambda_2\gamma_2 = \alpha\lambda_{Ṵ_2}$. Define θ_2: $Ṵ_2 \longrightarrow M$, by $\theta_2 = \lambda_2\gamma_2$, then $\delta\theta_2 = \delta\lambda_2\gamma_2 = \alpha\lambda_{Ṵ2}$. Also, it can be considered θ_1=$\lambda_1\gamma_1$: $Ṵ_1 \longrightarrow M$ and get a similar conclusion. **Case3/** If M_1 satisfies the first diagram and M_2 satisfies the second diagram, this means there is ψ_1: $M_1 \longrightarrow Ɲ$ such that $\alpha\psi_1 = \delta\lambda_1$ and there is γ_2: $Ṵ_2 \longrightarrow M_2$ such that $\delta\lambda_2\gamma_2 = \alpha\lambda_V$ where $Ṵ_2$ is a nonzero direct summand of Ɲ, define θ_2: $Ṵ_2 \longrightarrow M$ by $\theta_2 = \lambda_2\,\gamma_2$, then $\delta\theta_2 = \delta\lambda_2\gamma_2 = \alpha\lambda_{Ṵ2}$. **Case4/** If M_2 satisfies the first diagram and M_1 satisfies the second diagram, similar to the case3, by taking θ_1: $Ṵ_1 \longrightarrow M$, then $\delta\theta_1 = \delta\lambda_1\gamma_1 = \alpha\lambda_{Ṵ1}$. From the previous four cases either the first diagram is satisfied as in case 1 or the second as in the other three cases, therefore M is almost Ɲ-projective. Conversely, suppose that M is almost Ɲ-projective. From Proposition 2 both of M_1 and M_2 are almost Ɲ-projective.

Corollary 1: Let $\{M_k\}_{k=1}^{n}$ be a family of semimodules, then M_k where $1 \leq k \leq n$ are almost Ɲ-projective semimodules if and only if $\oplus_k M_k$, is almost Ɲ-projective for fixed semimodule Ɲ.

Proof: From Proposition 6 and by induction, the result is gotten.

Remark 8: [6] Let α: Ḝ $\longrightarrow$ Ḓ be an Ȑ-homomorphism, then:

1. ker(α)= {ẽ ∈ Ḝ | α(ẽ) = 0} = $\alpha^{-1}\{0\}$ which is subtractive subsemimodule of Ḝ.
2. α(Ḝ) ={ α(ẽ) | ẽ ∈ Ḝ}
3. Im(α)={ ʂ ∈ Ḓ | ʂ + α(ẽ) = α(ẽ′) for some ẽ, ẽ′ ∈ Ḝ}which is subtractive subsemimodule of Ḓ. It is clear that α(Ḝ)⊆ Im(α) and if α(Ḝ) is a subtractive subsemimodule of Ḓ, then α(Ḝ)= Im(α).

Definition 7:[11] The sequence of Ȑ-semimodules …⟶$Ӿ_i \xrightarrow{\alpha_i} Ӿ_{i+1} \xrightarrow{\alpha_{i+1}} Ӿ_{i+2}$ ⟶…is called:

1. Exact, if Im α_i =ker α_{i+1}, ∀i∈ I
2. Proper exact, if α_i($Ӿ_i$)= ker α_{i+1},∀i∈ I.

Remarks 9: Let α: Ḝ ⟶ Ḓ be an Ȑ-homomorphism,

1. It is well known in the category of modules ker α={0} if and only if α is monic, this is not held in general in the class of semimodules, take ε: $K_4 \longrightarrow$ Ӿ defined by $\varepsilon(0)$ = 0, $\varepsilon(a) = a$ and $\varepsilon(b) = \varepsilon(1) = 1$ where $K_4 = \{0, 1, a, b\}$ is 𝔹-semimodule(the scalar multiplication of 𝔹 on K_4 is defined by 0. ʂ = 0 and 1. ʂ = ʂ, ∀ʂ ∈ K_4) and its subsemimodule Ӿ = {0, 1, a}, then ker ε = {0} but ε is not monic [12] .If Ḝ is semisubtractive and Ḓ is cancellative then ker α = {0} if and only if α is monic[6].
2. For module always, α(Ḝ) =Im(α) but for semimodule not always true, take the inclusion map of the function in (1) i: Ӿ ⟶ K_4 since i(Ӿ) = Ӿ and Im(i) = K_4. α(Ḝ) =Im(α) if and only if α(Ḝ) is subtractive subsemimodule of Ḓ [6] .

Proposition 5: Let M be almost Ɲ-projective semimodule and Ɲ indecomposable semimodule, for any diagram of Ȑ-semimodules and Ȑ-homomorphisms of the form in Fig.5:

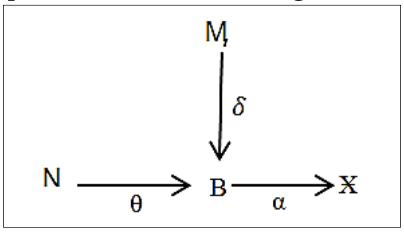

Figure 5. Ȑ**-semimodules and** Ȑ**-homomorphisms**

Such that the row is exact and $\alpha\delta$= 0, either there is $\emptyset$: M ⟶ Ɲ such that $\theta\emptyset = \delta$, or there is γ: Ɲ ⟶ M such that $\delta\gamma = \theta$.

Proof: Since $\alpha\delta$= 0, then Im(δ)⊆ ker(α)= Im(θ). Let θ':Ɲ ⟶ Im(θ) where θ'(ɳ) = θ(ɳ), for every ɳ in N̦, thus θ' is onto and $\delta(\text{ɱ}) = \delta'(\text{ɱ}), \forall \text{ɱ} \in$ M̧ take Fig.6:

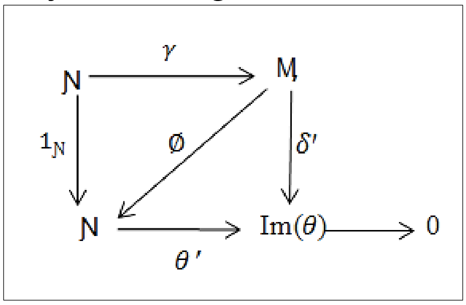

Figure 6. Exact row

In which the rows are exact, since M̧ is almost N̦-projective, then either there is $\emptyset$: M̧ $\longrightarrow$ N̦ such that $\theta'\emptyset = \delta'$ implies that $i\theta'\emptyset = i\,\delta'$ where i: Im$(\theta) \longrightarrow$ N̦ is the inclusion map, then $\theta\emptyset = \delta$, or there is γ: N̦ $\longrightarrow$ M̧ such that $\delta'\gamma = \theta' 1_N = \theta'$, implies $i\delta'\gamma = i\theta'$, then $\delta\gamma = \theta$.

Proposition 6: If M̧ is an almost N̦-projective semimodule and N̦ an indecomposable semimodule, $\emptyset \in$ Ṡ= End(N̦), then either Hom(M̧, $\emptyset$(N̦)) ⊆ $\emptyset$Hom(M̧, N̦) or Hom(N̦, $\emptyset$(N̦)) ⊆ $\emptyset$Ṡ.

Proof: It is clear that $\emptyset$Hom(M̧, N̦)⊆ Hom(M̧, $\emptyset$(N̦)), let $\delta \in$ Hom(M̧, $\emptyset$(N̦)) since M̧ is almost N̦-projective, then either $\emptyset f = \delta$ for some f ∈Hom(M̧, N̦) $\Longrightarrow \delta \in \emptyset$Hom(M̧, N̦), or there is γ: N̦ $\longrightarrow$ M̧ such that $\delta\gamma = \emptyset 1_N$, let $\alpha \in$ Hom(N̦, $\emptyset$(N̦)) such that $\alpha = \delta\gamma = \emptyset 1_N$(since N̦ an indecomposable semimodule, then N̦ is the only nonzero direct summand of itself) $\Longrightarrow \alpha \in \emptyset$Ṡ, as Fig.7.

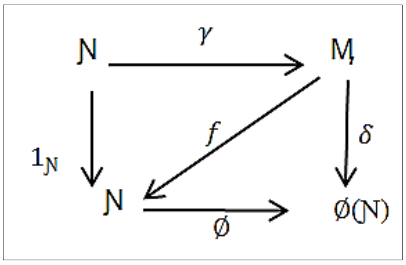

Figure 7. Hom(M̧,∅(N̦)) ⊆ ∅Hom(M̧,N̦) or Hom(N̦, ∅(N̦)) ⊆ ∅Ṡ

Definition 8:[13] A short exact sequence $0 \longrightarrow$ Ӿ $\overset{\alpha}{\rightarrow}$ ¥ $\overset{\beta}{\rightarrow}$ U̮ $\longrightarrow 0$ of Ȑ-semimodules is called split exact sequence if, there is map $\emptyset$: U̮ $\longrightarrow$ ¥ such that $\beta\emptyset = 1_U$.

Example 2: Consider the sequence of Ȑ-semimodules $0 \longrightarrow$ U̮ $\overset{i}{\rightarrow}$ U̮⊕D̯ $\overset{\pi}{\rightarrow}$ D̯ $\longrightarrow 0$ where i is the inclusion map and π is the projection map defined by $\pi(u, d) = d$, for all $(u, d) \in$ U̮⊕D̯ with ker (π)= U̮, this sequence is split exact.

Proposition 7: Let $0 \longrightarrow$ Ӿ $\overset{\alpha}{\rightarrow}$ ¥ $\overset{\beta}{\rightarrow}$ U̮ $\longrightarrow 0$ be a short exact sequence of semimodules such that U̮ is almost ¥-projective, then either this sequence is split, or there is $\gamma\colon Z \longrightarrow$ U̮ such that $\gamma = \beta\lambda_Z$, where Z is a nonzero direct summand of ¥.

Proof: Since U̮ is almost ¥-projective, then there is $\emptyset$: U̮ $\longrightarrow$ ¥ such that $\beta\,\emptyset = 1_U$ where β: ¥ $\longrightarrow$ U̮ and hence this sequence splits, or there is $\gamma\colon Z \longrightarrow$ U̮ where $0 \neq Z \leq_{\oplus}$ ¥ such that $\gamma = \beta\lambda_Z$.

Proposition 8: Let M̧ be Ȑ-semimodule and $0 \longrightarrow$ Ӿ $\overset{\alpha}{\rightarrow}$ ¥ $\overset{\beta}{\rightarrow}$ U̮ $\longrightarrow 0$ be a short exact sequence of Ȑ-semimodules with ¥ is almost M̧-projective, then U̮ is almost M̧-projective.

Proof: Consider the following Fig. 8a and Fig.8b:

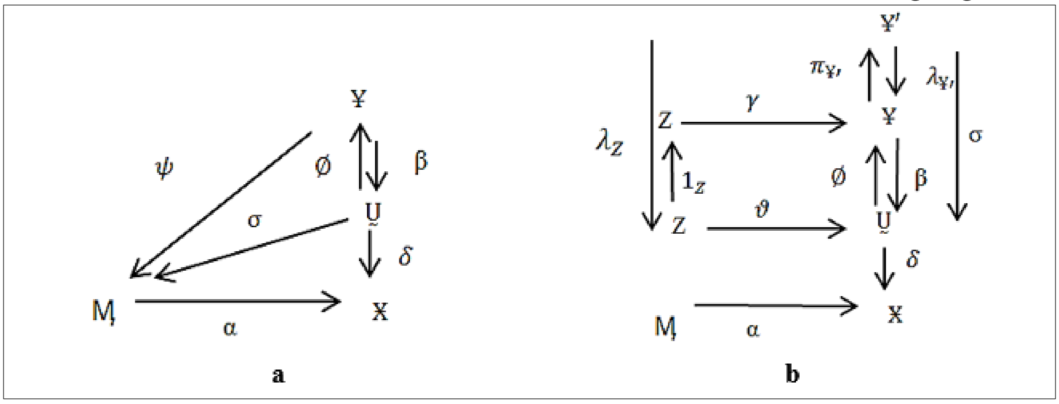

Figure 8. Exact sequence

Where α: M̧ $\longrightarrow$ Ӿ is an Ȑ-epimorphism and δ: U̮ $\longrightarrow$ Ӿ is any Ȑ-homomorphism, since ¥ is almost M̧-projective, then either $\exists\psi$: ¥ $\longrightarrow$ M̧ such that $\alpha\psi = \delta\,\beta$, or there is $\gamma\colon Z \longrightarrow$ ¥ where $0 \neq Z \leq_{\oplus}$ M̧, such that $\delta\beta\gamma = \alpha\lambda_Z$. By Proposition 7 either this sequence splits that is there exists $\emptyset$: U̮ $\longrightarrow$ ¥ such that $\beta\,\emptyset = 1_U$, then in the first diagram define σ: U̮ $\longrightarrow$ M̧ by $\sigma = \psi\emptyset$ hence $\alpha\ \sigma = \alpha\psi\emptyset = \delta\beta\emptyset = \delta$, or there is σ: ¥′ $\longrightarrow$ U̮ where ¥′ is a nonzero direct summand of ¥ such that $\sigma = \beta\lambda_{¥'}$, define $\vartheta\colon Z \longrightarrow$ U̮ by $\vartheta = \beta\lambda_{¥'}\pi_{¥'}\gamma 1_Z$. Then

$\delta\vartheta = \delta\beta(\lambda_{Ұ'}\pi_{Ұ'})\gamma = \delta\beta\gamma = \alpha\lambda_Z$. Therefore Ṵ is almost Ṃ-projective.

Proposition 9: If Ṃ is an Ȑ-semimodule and 0⟶ Ӿ $\xrightarrow{\alpha}$ Ұ $\xrightarrow{\beta}$ Ṵ ⟶0 be the short exact sequence of Ȑ-semimodules such that Ұ is almost Ṃ-projective semimodule, then Ұ⊕Ṵ is almost Ṃ-projective.

Proof: From Proposition 10, Ṵ is almost Ṃ-projective, and by Proposition 4, then Ұ⊕Ṵ is almost Ṃ-projective.

Definition 9:[14] A semimodule Ṃ is said to be Ɲ-injective if, for every subsemimodule Ṵ of Ɲ, any homomorphism from Ṵ to Ṃ can be extended from Ṃ to Ɲ.

Proposition 10: If Ṃ and Ɲ are two Ȑ-semimodules, then:

(1) If Ɲ is Ṃ-injective and every subsemimodule of Ṃ is Ɲ-projective, then every factor of Ɲ is Ṃ-injective semimodule.

(2) If every subsemimodule of Ṃ is an Ɲ-projective semimodule and every factor of Ɲ is Ṃ-injective, then Ɲ is Ṃ-injective.

Proof: (1) Assume that Ɲ is Ṃ-injective and Ḝ is subsemimodule of Ṃ, Ӿ is subsemimodule of Ɲ and β:Ḝ ⟶ Ɲ/Ӿ is a homomorphism. Let i:Ḝ⟶ Ṃ and ν:Ɲ ⟶ Ɲ/Ӿ be the injection map and natural epimorphism respectively. Since Ḝ is Ɲ-projective ∃ψ:Ḝ ⟶ Ɲ such that $\beta=\nu\,\psi$. But Ɲ is Ṃ-injective semimodule, so, there is θ: Ṃ ⟶ Ɲ such that $\theta i = \psi$, if $\omega = \nu\theta$, then $\omega i = \nu\theta i = \nu\psi = \beta$, therefore Ɲ/Ӿ is Ṃ-injective. As the Fig 9:

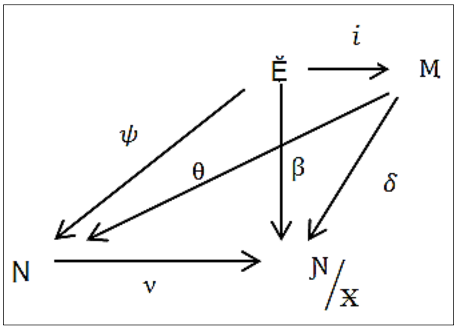

Figure 9. Factor of Ɲ is Ṃ-injective semimodule

(2)Assume that Ḝ is a subsemimodule of Ṃ and Ḝ is a Ɲ-projective semimodule, let ν:Ɲ ⟶ Ɲ/Ӿ is the natural epimorphism, and ψ: Ḝ ⟶ Ɲ be a homomorphism, β:Ḝ ⟶ Ɲ/Ӿ be Ȑ-homomorphism, since Ɲ/Ӿ is Ṃ-injective semimodule, there is δ:Ṃ ⟶ Ɲ/Ӿ such that $\beta = \delta i$ where i: Ḝ ⟶ Ṃ is the inclusion map. But Ḝ is Ɲ-projective, then $\nu\psi=\beta$, put $\psi = \theta i$ then $\nu\psi = \nu\theta i = \beta = \delta i$, then Ɲ is Ṃ-injective semimodule. As Fig.9.

Open problem: Under what conditions, ‘‘Ɲ-projective’’ in proposition 12, can be replaced by an almost Ɲ-projective semimodule, such that it is valid?

Conclusion:

In this work, a generalization of projective semimodule was presented. This concept is studied in the module differently from what was dealt with in this work. Some basic properties of this notion were discussed. Since every module is semimodule but the converse is not true, in some results, some conditions were added to achieve them.

Authors' Declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Babylon.

Authors' Contributions Statement:

Kh. S. and A. M. conceived of the presented idea. Kh. S. developed the theory and performed the computations. Khitam Sahib and Asaad Mohammed verified the analytical methods. A. encouraged Kh. S. to investigate this notion and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

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شبه المقاسات الاسقاطية تقريبا

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الخلاصة:
الهدف من هذا البحث هو دراسة مفهوم شبه المقاس الاسقاطي تقريبا كأعمام الى شبه المقاس الاسقاطي. بعض الخصائص الاساسية لهذا المفهوم تم مناقشتها, كذلك بعض النتائج تم اعمامها الى صنف شبه المقاسات الاسقاطية تقريبا.

الكلمات المفتاحية: شبه المقاسات الاسقاطية تقريبا، شبه المقاسات الاغمارية، شبه المقاسات الاسقاطية، شبه المقاسات، شبه الحلقات.