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Delay differential equation of the 2nd order and it's an oscillation yardstick

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Abstract

This study focuses on studying an oscillation of a second-order delay differential equation. Start work, the equation is introduced here with adequate provisions. All the previous is braced by theorems and examplesthat interpret the applicability and the firmness of the acquired provisions.

Keywords: Delay differential equations, Eventually positive,Increasing function, Oscillatory,Solution.

Introduction

Nowadays, one of the most dominant connotations in mathematics is the delay differential equations (are denoted here by DDE's), which have received a lot of attention from the authorsthree decades ago. For instance, some interested authors are in the resources [1-4]. Furthermore, this field of science is rivetedalso in various scientific disciplines as in [5]. A new discussion on this subject is in [6].Moreover, the authors in[7,8]have studied the oscillation of third and fourth-order delay DDE's. Where the author in[9] is ascertain the oscillation of the 1st order DDE's with constant delay, in addition to the integral conditions are provided. While thesolutions behavior is discussed actually in [10]. Besides, the motif oscillatory solution is in [11].

While in [12] is discussed the oscillation of a 2nd order mixed and multiple delay differential equations under a canonical operator. This work is divided into four sections, where section one is called the introduction, while section two is named after materials and methods which concludes introducing a DDE withadequate provisions. Moreover, the results are braced by theorems. And examplesinterpret the applicability and the firmness of the acquired provisions in the third section. Where the last section is the conclusion.

Materials and Methods

Onset sheds light on studying the specific solutions, called oscillation, of the below DDE of the 2nd order.

Definition. 1: Let the DDE

$$ty''(t)-\sigma\rho(t)y\big(t-r(t)\big)=0\,,t\geq T_0 \quad \mathbf{1}$$

Where T_0 is a positive real number, a constant$\sigma\in R$, $\rho,r\in C([T_0\,,\infty),(0,\infty)), 0<r(t)<t$.

Now, here is an account of some important theorems with their proof that illustrates the purpose of work in oscillations.

Theorem. 1: If$y(t)$ is a solution of Eq.1, which is eventually positive, then for sufficiently large $t_0>T_0$

$$\sigma\int_{t-\tau}^{t}p(s)\frac{y\big(s-r(s)\big)}{y(s)}ds \leq 2\tau y(t)-2\int_{t-\tau}^{t}y(s)ds\,,\quad t>t_0+3\tau$$

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Proof:

Firstly, $\left(t-r(t)\right)>0,\ t>t_0+\tau$. So, dividing Eq.1 by $y(t)$ for $t>t_0+2\tau$ and integrating from $t-\tau$ to t, plainly

$$\int_{t-\tau}^{t} t\frac{y''(s)}{y(s)}ds-\sigma\int_{t-\tau}^{t} p(s)\frac{y(s-r(s))}{y(s)}ds=0,\ \ t>t_0+3\tau \qquad \mathbf{2}$$

The first integral is solved by supposing $y(t)=\tan\theta$, and after a couple of steps, which leads to:

$$\int_{t-\tau}^{t} t\frac{y''(s)}{y(s)}ds=2t[y(t)-y(t-\tau)]-2(t-\tau)[y(t)-y(t-\tau)]-\int_{t-\tau}^{t}2[y(s)-y(s-\tau)]ds, t>t_0+3\tau.$$

Actually, because of y is an increasing function, so $y(t)-y(t-\tau)\le y(t)$, which is giving that

$$\int_{t-\tau}^{t} t\frac{y''(s)}{y(s)}ds\le 2ty(t)-2(t-\tau)y(t)-2\int_{t-\tau}^{t}y(s)ds=2\tau y(t)-2\int_{t-\tau}^{t}y(s)ds, t>t_0+3\tau.$$

Over here, from Eq.2

$$\sigma\int_{t-\tau}^{t}p(s)\frac{y\left(s-r(s)\right)}{y(s)}ds=\int_{t-\tau}^{t}t\frac{y''(s)}{y(s)}ds\le 2\tau y(t)-2\int_{t-\tau}^{t}y(s)ds,\qquad t>t_0+3.$$

Hence

$$\sigma\int_{t-\tau}^{t}p(s)\frac{y\left(s-r(s)\right)}{y(s)}ds\le 2\tau y(t)-2\int_{t-\tau}^{t}y(s)ds,\qquad t>t_0+3\tau.$$

Theorem. 2: Let $y(t)$ be an eventually positive solution of Eq.1, then

$$\sigma\int_{t-\tau}^{t}p(s)\frac{y(s-r(s))}{y(s)}ds\le\sigma\tau p(t)-\sigma\int_{t-\tau}^{t}\tau p'(s)ds,\qquad t\ge t_0+3\tau \qquad \mathbf{3}$$

Proof:

Here starting proof by supposing $u=p(s), dv=\frac{y(s-r(s))}{y(s)}$ and continue substituting, do not forget the fact that $y(t)$ is an increasing function to get the following result

$$\sigma\int_{t-\tau}^{t}p(s)\frac{y\left(s-r(s)\right)}{y(s)}ds\le\sigma\tau p(t)-\sigma\int_{t-\tau}^{t}\tau p'(s)ds,\quad t\ge t_0+3\tau\,.$$

Definition. **2:**Let $\text{Ꞓ}(t)=\sigma\tau p(t)-\sigma\int_{t-\tau}^{t}\tau p'(s)ds,\ \ t\ge t_0+3\tau$. As well as, suppose that the following assumptions are verified.

(Ꞓ_1) $\rho\in C^1([T_0,\infty),(0,\infty))$

(Ꞓ_2) $\exists T_n\in\left((n-1)\tau,n\tau\right), n\in N,\ s.t.\ \rho'(t)>0,$ when $t\in(T_n-\tau,T_n)$, and $\rho'(t)<0$, when $t\in(T_n,n\tau]$

(Ꞓ_3) $\sup\{\sigma\tau p(t)-\sigma\int_{t-\tau}^{t}\tau p'(s)ds\}>0$

(Ꞓ_4) $\rho\left((n-1)\tau\right)=0,\ n\epsilon N$

Theorem.3: If the four cases of Definition2 are satisfied, then

$$\sup\{\text{Ꞓ}(n\tau)\}>0, n\in N$$

Proof:

$$\text{Ꞓ}(n\tau)=\sigma\tau p(n\tau)-\sigma\int_{(n-1)\tau}^{n\tau}\tau p'(s)ds$$

$$\sup\{\text{Ꞓ}(n\tau)\}=\sup\left\{\sigma\tau p(n\tau)-\sigma\int_{(n-1)\tau}^{n\tau}\tau p'(s)ds\right\}.$$

Visibly, $\sup\{\text{Ꞓ}(n\tau),\ n\in N\}>0$ by Ꞓ_3

Theorem. 4: If the assumptions of Definition1 are hold here, $\rho(t)$ is periodic with period $n\tau, n\in N$ and

$$\rho'(t-\tau)>0,\quad t\in((T_n,n\tau),n\in N \qquad \mathbf{4}$$

$$\lim_{t\to\infty}\inf\sigma\int_{t-\tau}^{t}\rho(s)ds>0 \qquad \mathbf{5}$$

then, Eq.1 has only an oscillatory solution.

Proof

One basically uses contradiction in this proof by supposing that Eq.1 has eventually positive solution $y(t)$, so

$$A'(t) = \sigma\tau\rho'(t) - \sigma\tau\rho'(t) + \sigma\tau\rho'(t-\tau), \quad t \geq t_0 + 3\tau$$

$$A'(t) = \sigma\tau\rho'(t-\tau), \quad t \geq t_0 + 3.$$

Under A_2, which is giving that $A'(t) > 0$, $t \in (T_n - \tau, T_n), n \in N$

Obviously, $A(t)$ is an increasing function on $(T_n, n\tau], n \in N$.

To discuss the other cases, that is; $A(T_n) < 0$, when $\rho(t_n - \tau) < 0, t_n \in (T_n, n\tau], n \in N$. Add to that $A(T_n) = 0$, when $\rho(t_n - \tau) = 0, t_n \in (T_n, n\tau], n \in N$.

Again A_3 gives right here $\sup\{n\tau - T_n, n \in N\} > 0$.

Moreover, from putting $\beta(t) = \rho(t-\tau), t \in (T_n, n\tau], n \in N$, then by A_2

$$\beta'(t) = \rho'(t-\tau) < 0, t \in (T_n, n\tau], n \in N.$$

Besides, A_4 informs that $\beta(n\tau) = \rho(n\tau - \tau) = \rho\big((n-1)\tau\big) = 0, , n \in N$.

It is going obviously from A_3 that

$$\rho(t-\tau) > 0 \qquad \textbf{6}$$

So let's go by assuming that $t_n \leq x_n = n\tau - \varepsilon, \inf\{0 < \varepsilon < n\tau - T_n\}, n \in N$.

Then Eq.3 helps to write the following step

$$\sigma \int_{x_n-\tau}^{x_n} p(s)\frac{y\big(s-r(s)\big)}{y(s)} ds \leq \sigma\tau p(x_n) - \sigma \int_{x_n-\tau}^{x_n} \tau p'(s) ds, \quad x_n \geq t_0 + 3\tau, \quad n \in N.$$

Around here Eq.5 shows that $\frac{y(t-r(t))}{y(t)}$ is bounded.

Because of $\frac{y(t-r(t))}{y(t)} \geq \frac{y(t-\tau)}{y(t)}$.

Hence there exist a positive constant M such that $\frac{y(t-\tau)}{y(t)} \leq \frac{y(t-r(t))}{y(t)} \leq M, \quad t \geq T \geq t_0 + 2\tau$

$$\sigma \int_{x_n-\tau}^{x_n} p(s)\frac{y(s-r(s))}{y(s)} ds \leq \sigma \int_{x_n-\tau}^{x_n} p(s) M ds, \quad x_n \geq T, n \in N \qquad \textbf{7}$$

By Eq.6 and the periodicity of $\rho(t)$, which leads to

$$\sigma \int_{x_n-\tau}^{x_n} p(s)\frac{y\big(s-r(s)\big)}{y(s)} ds > \sigma \int_{x_n-\tau}^{x_n} p(s) M ds, \quad x_n \geq T, n \in N.$$

Unfortunately, it is a contradiction with Eq.7

Let $\{t_n\}$ be a sequence s.t. $t_n \to n\tau \text{ as } n \to \infty$, $t_n \in (T_n, n\tau], n \in N$. Then

$$A(n\tau) = \sigma\tau p(n\tau) - \sigma \int_{(n-1)\tau}^{n\tau} \tau p'(s) ds$$

$$A(n\tau) = \sigma\tau p(n\tau) - \sigma\Big[\int_{(n-1)\tau}^{T_n} \tau p'(s) ds + \int_{T_n}^{n\tau} \tau p'(s) ds\Big]$$

$$= \sigma\tau p(n\tau) - \sigma\Big[\int_{t_n-\tau}^{T_n} \tau p'(s) ds - \int_{t_n-\tau}^{(n-1)\tau} \tau p'(s) ds + \int_{T_n}^{t_n} \tau p'(s) ds + \int_{t_n}^{n\tau} \tau p'(s) ds\Big]$$

$$= \sigma\tau p(n\tau) - \sigma\Big[\int_{t_n-\tau}^{t_n} \tau p'(s) ds - \int_{t_n-\tau}^{(n-1)\tau} \tau p'(s) ds + \int_{t_n}^{n\tau} \tau p'(s) ds\Big]$$

$$= \sigma\tau p(n\tau) - \sigma \int_{t_n-\tau}^{t_n} \tau p'(s) ds + \sigma \int_{t_n-\tau}^{(n-1)\tau} \tau p'(s) ds - \sigma \int_{t_n}^{n\tau} \tau p'(s) ds.$$

One can easily check when $t_n \to n\tau \text{ as } n \to \infty$, then $\sigma \int_{t_n-\tau}^{(n-1)\tau} \tau p'(s) ds = 0$ and $\sigma \int_{t_n}^{n\tau} \tau p'(s) ds = 0$, where the first term is also equal zero after some simplified steps and using the assumption A_4 . From all of these, the result is $A(n\tau) = 0$, which contradicts Theorem 3.

Finely, this ends the proof.

Theorem 5: Ifall presumptions of def.1are hold here, $\rho(t)$ is periodic with period $n\tau, n \in N$ and

$$\rho'(t-\tau) > 0, \quad t \in ((T_n, n\tau), n \in N$$

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and

$$\lim_{t\to\infty} \sup \sigma \int_{t-\tau}^{t} \rho(s)ds > 1 \quad \mathbf{8}$$

then, Eq.1 has only an oscillatory solutions.
The proof of the above theorem is clear by tracking the same steps in the proof of Theorem 4.

Implementation:

1● Let the following DDE

$$ty''(t) - (cost)y\left(t - \frac{\pi}{2}\right) = 0, \quad t \geq 0 \quad \mathbf{9}$$

Here, when compare the above equation with Eq.1 to get the corresponding results $T_0 = 0$, $\rho(t) = \cos t$, $\sigma = 1$, and $r(t) = \tau = \frac{\pi}{2}$

Clear that assumption A_1 is hold, where to satisfy A_2 in some few steps

$$T_n = \left(n - \frac{1}{2}\right)\frac{\pi}{2}$$

Where

$$\rho'(t) = -sint > 0, t \in \left((n - \frac{3}{2})\frac{\pi}{2}, \left(n - \frac{1}{2}\right)\frac{\pi}{2}\right), n \in N.$$

Taking the angle located in the third and fourth quadrants and ignoring the case of the other two quadrants. The opposite of this case applies to the following:

$$\rho'(t) < 0, t \in \left(\left(n - \frac{1}{2}\right)\frac{\pi}{2}, \frac{n\pi}{2}\right], n \in N.$$

To verify A_3

$$\sup\left\{\frac{\pi}{2}\cos t - \int_{t-\frac{\pi}{2}}^{t} \frac{\pi}{2}(-\sin s)ds\right\}$$
$$= \sup\left\{\frac{\pi}{2}\cos t - \frac{\pi}{2}\cos t + \frac{\pi}{2}\cos\left(t - \frac{\pi}{2}\right)\right\}$$
$$= \sup\left\{\frac{\pi}{2}\left(\cos t \cos\frac{\pi}{2} + \sin t \sin\frac{\pi}{2}\right)\right\}$$
$$= \sup\left\{\frac{\pi}{2}(\sin t)\right\} > 0.$$

Finally, to check A_4

$$\rho\big((n-1)\tau\big) = \cos\left((n-1)\frac{\pi}{2}\right) = 0, \ n \in N \text{is even.}$$

Clear to verify Eq.4

$$\rho'(t-\tau) = -\sin\left(t - \frac{\pi}{2}\right) = \cos t > 0, \quad t \in ((T_n, (2n+1)\tau), n \in N, n\ is\ even,$$

and

$$\lim_{t\to\infty} \inf \int_{t-\frac{\pi}{2}}^{t} \rho(s)ds = \lim_{t\to\infty} \inf \int_{t-\frac{\pi}{2}}^{t} \cos(s)\, ds$$
$$= \lim_{t\to\infty} \inf\left\{\sin t - \sin\left(t - \frac{\pi}{2}\right)\right\}$$
$$= \lim_{t\to\infty} \inf\{\sin t + \cos t\} < 0.$$

So, condition in Eq.5 is not verified.
While

$$\lim_{t\to\infty} \sup \int_{t-\frac{\pi}{2}}^{t} \rho(s)ds$$
$$= \lim_{t\to\infty} \sup \int_{t-\frac{\pi}{2}}^{t} \cos(s)\, ds$$
$$= \lim_{t\to\infty} \sup\left\{sin\, t - sin\left(t - \frac{\pi}{2}\right)\right\} > 1.$$

Therefore condition in Eq.8 is satisfied here.
Hence Eq.9 in its entirety solutions are oscillatory by Theorem 5.

2● Suppose that the DDE

$$ty''(t) - \left(\frac{1}{\pi e} + \delta sint\right)y\left(t - \frac{\pi}{2}\right) = 0 \quad t > 0 \quad \mathbf{10}$$

Comparing this DDE with Eq.1. Inasmuch the corresponding values are $T_0 = 0$, $\rho(t) = \frac{1}{\pi e} + \delta sint$, $\delta \in (0, \frac{1}{\pi e}]$, $\sigma = 1$, and $\tau = \frac{\pi}{2}$.

Here A_1 is hold, where to satisfy A_2 below

$$T_n = (2n + 0.5)\frac{\pi}{2}$$
$$\rho'(t) = \delta cost > 0, t \in \left((2n - 0.5)\frac{\pi}{2}, (2n + 0.5)\frac{\pi}{2}\right), n \in N, n\ is\ even.$$

$$\rho'(t) < 0, \quad t \in \left((2n + 0.5)\frac{\pi}{2}, n\frac{\pi}{2}\right), n \in N, n\ is\ odd.$$

Now to satisfy A_3

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$$\sup\left\{\frac{\pi}{2}\left(\frac{1}{\pi e}+sint\right)-\delta\int_{t-\frac{\pi}{2}}^{t}\frac{\pi}{2}(coss)ds\right\}$$
$$=\sup\left\{\frac{\pi}{2}\left(\frac{1}{\pi e}+sint\right)-\delta\frac{\pi}{2}sint+\delta\frac{\pi}{2}\sin\left(t-\frac{\pi}{2}\right)\right\}$$
$$=\sup\left\{\frac{\pi}{2}\left(\frac{1}{\pi e}-\delta cost\right)\right\}>0.$$

At last, to check A_4

$$\rho\big((n-1)\tau\big)=\frac{1}{\pi e}+\delta\sin(n-1)\frac{\pi}{2}=0,$$
$$where\ n=2n\in N, and\ n \text{ is even}, \delta=\frac{1}{\pi e}\ .$$

The next step is to verify Eq.4

$$\rho'(t-\tau)=\delta cos\left(t-\frac{\pi}{2}\right)=\delta sint>0,\quad t\in((T_n,(2n+1)\tau),n\in N, n\ is\ odd,$$

and

$$\liminf_{t\to\infty}\int_{t-\frac{\pi}{2}}^{t}\rho(s)ds=\liminf_{t\to\infty}\int_{t-\frac{\pi}{2}}^{t}\left(\frac{1}{\pi e}+\delta\sin(s)\right)ds$$
$$=\liminf_{t\to\infty}\left\{\frac{1}{\pi e}\left(t-t+\frac{\pi}{2}\right)-\delta\left(cos\,t-cos\left(t-\frac{\pi}{2}\right)\right)\right\}$$
$$=\liminf_{t\to\infty}\left\{\frac{1}{2e}-\delta(cos\,t-\sin t)\right\}<0.$$

Hence, the condition in Eq.5 is not verified as you have seen above.

But

$$\limsup_{t\to\infty}\int_{t-\frac{\pi}{2}}^{t}\rho(s)ds$$
$$=\limsup_{t\to\infty}\int_{t-\frac{\pi}{2}}^{t}\left(\frac{1}{\pi e}+\delta\sin(s)\right)ds$$
$$=\limsup_{t\to\infty}\left\{\frac{1}{2e}-\delta(cos\,t-\sin t)\right\}>1$$

Over here the condition in Eq.8 is satisfied. Moreoverby Theorem. 5, notice that in its entirety the solutions ofEq.10 are oscillatory.

Conclusion

Recently, many authors are interested in studying the oscillation of DDE. In this paper, are kept going on studying the oscillation solution behavior of DDE of second order. Furthermore, we have put theorems of course with proof to support this work. It is basically, the implementation which is represented by examples, is given tostrengthen this task and make it more acceptable.

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Author's Contribution Statement

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in Mustansiriyah University.

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المعادلة التفاضلية التباطؤية من الرتبة الثانية ومعيار تذبذبها

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الخلاصة

تركز هذه الدراسة على دراسة تذبذب المعادلة التفاضلية التباطؤية من الدرجة الثانية . بدء العمل, تم تقديم المعادلة هنا مع الشروط المناسبة. كل ما سبق مدعوم بالنظريات والأمثلة التي تفسر قابلية التطبيق للشروط المستنتجة.

ألكلمات المفتاحية: المعادلات التفاضلية التباطؤية، اللانهاية الموجبة، الدالة المتزايدة، التذبذب،الحل.