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Numerical Solutions for the Nonlinear PDEs of Fractional Order by Using a New Double Integral Transform with Variational Iteration Method

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Abstract

This paper considers a new Double Integral transform called Double Sumudu-Elzaki transform DSET. The combining of the DSET with a semi-analytical method, namely the variational iteration method DSETVIM, to arrive numerical solution of nonlinear PDEs of Fractional Order derivatives. The proposed dual method property decreases the number of calculations required, so combining these two methods leads to calculating the solution's speed. The suggested technique is tested on four problems. The results demonstrated that solving these types of equations using the DSETVIM was more advantageous and efficient.

Keywords: Double Sumudu-Elzaki transform, Fractional Calculus, Fractional nonlinear PDEs, Numerical Solution, Variational Iteration Method.

Introduction

Based on the idea of fractional calculus, which originated more than three decades ago. The study and use of arbitrary order integrals and derivatives using real or complex number powers of the differential and integral operators are the subjects of the mathematical analysis branch known as fractional calculus. Models of real-world problems may be more accurately represented using fractional derivatives than integer-order derivatives[1-3].

Integral transform methods are essential for the solution of many different varieties of problems. Multiple integral transforms, including the Laplace, Sumudu, Fourier, Natural, Mellin, and Elzaki, have been used for the solution of PDEs[4-7], as a result of the rapid developments in research and engineering. Therefore, notice that several academics are attempting to create new methods that allow us to solve this form of problem. These attempts, which are still continuing, have resulted in the promotion of these studies in numerous ways, including the Homotopy analysis method (HAM), Adomian decomposition method (ADM), and Variational iteration method (VIM)[8-10], which have become well-known among a significant number of researchers in this field. A new approach has just been developed, which combines the Laplace transform, Sumudu transform, Natural transform, or Elzaki transform, with these techniques[11-14].

The properties and theories of double integrals, such as[15-17], are novel. Some authors have used these transforms in conjunction with other mathematical techniques, such as the HAM, ADM, and VIM[18-21], to solve linear and nonlinear fractional differential equations.

In all applied science and engineering. Partial Differential Equations (PDEs) of fractional order are utilized to explain various situations. Finding exact or approximate solutions to these kinds of equations has received a lot of attention in recent research[22-24].

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Many nonlinear phenomena are major parts of applied research and engineering[25-37]. Nonlinear equations of fractional order have been found in a variety of real-world problems. Different phenomena may be described with the help of nonlinear PDEs of fractional order. Nonlinear PDEs of fractional-order derivatives computed with unknown functions of two variables are challenging to solve, such equations are more difficult to solve than linear PDEs. The fact that these equations are so widely used has made mathematicians aware of them. Nonetheless, solving these mathematical problems is neither numerically nor conceptually simple.

In this paper, the DSETVIM has been used to solve nonlinear time-fractional derivatives NT-FDPDEs. Our current article has been structured as follows: Definitions of the Sumudu transform and the ELzaki transforms in the context of fractional calculus are presented in Section 2. Our proposed analysis of the revised approach with the convergence theorem will be presented in Section 3. There are four examples of how this method was employed were provided in Section 4. The last part is the conclusion.

Basic definitions:

With the use of the Sumudu and ELzaki Transform, the fundamental ideas and features of the fractional calculus theory are given in this part.

Definition 1:[1] A real function $\Phi(t), t > 0$, is said to be in the space $C_\vartheta, \vartheta \in R$, if there exists a real number $q, (q > \vartheta)$, such that $\Phi(t) = t^q \Phi_1(t)$, where $\Phi_1(t) \in C[0, \infty)$, and it is said to be in the space C_ϑ^m if $\Phi^{(m)} \in C_\vartheta, m \in N$.

Definition 2:[2] The Riemann-Liouville fractional integral $I^\alpha f$ of order $(\alpha \geq 0)$ of a function $\Phi(t)$ is defined as:

$$(I^\alpha \Phi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\Phi(\xi) d\xi}{(t-\xi)^{1-\alpha}}, t > 0, \alpha > 0,$$

$$(I^0 \Phi)(t) = \Phi(t), \alpha = 0.$$

Additionally, the Riemann-Liouville fractional integral has the following property:

$$I^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma},$$

Where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,\ z > 0$, is called the gamma function.

Theorem 3:[3] In the sense of Caputo meaning, the fractional derivative of $\Phi(t)$, is as follows:

$$({}^cD^\alpha \Phi)(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\Phi^{(m)}(\xi) d\xi}{(t-\xi)^{\alpha-m+1}},$$

For $m - 1 < \alpha < m, m \in \mathbb{N}, t > 0, \Phi \in C_{-1}^m$.

The operator ${}^cD^\alpha$ has the following fundamental characteristics:

$${}^cD^\alpha\ {}^cD^\sigma \Phi(t) = {}^cD^{\alpha+\sigma} \Phi(t).$$

$${}^cD^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(1-\alpha+\gamma)} t^{\gamma-\alpha}.$$

$${}^cD^\alpha I^\alpha \Phi(t) = \Phi(t).$$

$$I^\alpha\ {}^cD^\alpha \phi(t) = \Phi(t) - \sum_{k=0}^{m-1} \Phi^{(k)}(0) \frac{t^k}{k!}.$$

Definition 4:[38] The Sumudu Transform ST of the function $\Phi(z)$ for all $z \geq 0$ is defined as:

$$\mathrm{S_z}(\Phi(\mathrm{z})) = \frac{1}{\rho} \int_0^\infty \Phi(\mathrm{z})\, \mathrm{e}^{-\left(\frac{z}{\rho}\right)} \mathrm{dx} = \overline{\Phi}(\rho),\ \rho \in$$

(p_1, p_2), where the operator $\mathrm{S_z}$ is called the Sumudu transform operator.

Definition 5:[39] The Elzaki Transform ET of the function $\Phi(t)$ for all $t \geq 0$ is defined as:

$$\mathrm{E_t}(\Phi(t)) = \tau \int_0^\infty \Phi(t)\, \mathrm{e}^{-\left(\frac{t}{\tau}\right)} dt = \overline{\Phi}(\tau), \qquad \tau \in$$

(p_1, p_2), where the operator $\mathrm{E_t}$ is called the Elzaki transform operator.

These functions are of exponential order, and they take into consideration functions in the set G described by:

$$G = \left\{ f(\tau) \colon \exists Q, p_1, p_2 > 0, |f(\tau)| < Q e^{\frac{|\tau|}{p_i}}, if\ \tau \in (-1)^i \times [0, \infty) \right\}.$$

Definition 6:[40] The DSET of $S_z E_t[\psi(z, t)] = \overline{\psi}(\gamma, \delta)$ is defined as:

$$S_z E_t[\psi(z, t)] = \overline{\psi}(\gamma, \delta) = \frac{\delta}{\gamma} \int_0^\infty \int_0^\infty \psi(z, t) e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)} dz dt.$$

The linearity of the DSET may be shown very clearly in the following relationship, which can be seen below:

$$S_z E_t[\rho\psi(z, t)] + \tau\chi(z, t)] = \frac{\delta}{\gamma} \int_0^\infty \int_0^\infty e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)} [\rho\psi(z, t) + \tau\chi(z, t)] dz dt,$$

$$= \frac{\rho\delta}{\gamma} \int_0^\infty \int_0^\infty e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)} \psi(z, t) dz dt + \frac{\tau\delta}{\gamma} \int_0^\infty \int_0^\infty e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)} \chi(z, t) dz dt,$$

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$\rho S_zE_t[\psi(z,t)] + \tau S_zE_t[\chi(z,t)] = \rho\overline{\psi}(\gamma,\delta) + \tau\overline{\chi}(\gamma,\delta)$.

Definition 7:[27] The inverse of DSET, i.e. IDSET $S_zE_t^{-1}[\overline{\psi}(\gamma,\delta)] = \psi(z,t)$ is defined by:

$$S_zE_t^{-1}[\overline{\psi}(\gamma,\delta)] = \psi(z,t) = \frac{1}{2\pi i}\int_{\zeta-i\infty}^{\zeta+i\infty}\frac{1}{\gamma}e^{-\frac{z}{\gamma}}d\gamma\,.\,\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\delta e^{-\frac{t}{\delta}}\overline{\psi}(\gamma,\delta)d\delta.$$

Basic derivative properties of the DSET[40]**:**

$$S_zE_t\left[\frac{\partial\psi(z,t)}{\partial z}\right] = \frac{1}{\gamma}\overline{\psi}(\gamma,\delta) - \frac{1}{\gamma}E_t\big(\psi(0,t)\big),$$

$$S_zE_t\left[\frac{\partial^2\psi(z,t)}{\partial z^2}\right] = \frac{1}{\gamma^2}\overline{\psi}(\gamma,\delta) - \frac{1}{\gamma^2}E_t\big(\psi(0,t)\big) - \frac{1}{\gamma}E_t\left(\frac{\partial\psi(0,t)}{\partial z}\right),$$

$$S_zE_t\left[\frac{\partial\psi(z,t)}{\partial t}\right] = \frac{1}{\delta}\overline{\psi}(\gamma,\delta) - \delta S_z\big(\psi(z,0)\big),$$

$$S_zE_t\left[\frac{\partial^2\psi(z,t)}{\partial t^2}\right] = \frac{1}{\delta^2}\overline{\psi}(\gamma,\delta) - S_z\big(\psi(z,0)\big) - \delta S_z\left(\frac{\partial\psi(z,0)}{\partial t}\right),$$

$$S_zE_t\left[\frac{\partial^2\psi(z,t)}{\partial z\,\partial t}\right] = \frac{1}{\gamma\delta}\overline{\psi}(\gamma,\delta) - \frac{1}{\gamma\delta}E_t\big(\psi(z,0)\big) - \delta S_z\left(\frac{\partial\psi(z,0)}{\partial z}\right),$$

$$S_zE_t\left(\frac{\partial^m\psi(z,t)}{\partial z^m}\right) = \gamma^{-m}\overline{\psi}(\gamma,\delta) - \sum_{k=0}^{m-1}\gamma^{-m+k}E_t\left(\frac{\partial^k\psi(0,t)}{\partial z^k}\right),$$

$$S_zE_t\left(\frac{\partial^n\psi(z,t)}{\partial t^n}\right) = \delta^{-n}\overline{\psi}(\gamma,\delta) - \sum_{j=0}^{n-1}\delta^{-n+j+2}S_z\left(\frac{\partial^j\psi(z,0)}{\partial t^j}\right),$$

$$S_zE_t\left[\frac{\partial^v\psi(z,t)}{\partial x^v}\right] = \gamma^{-v}\overline{\psi}(\gamma,\delta) - \sum_{k=0}^{m-1}\gamma^{-v+k}E_t\left(\frac{\partial^k\psi(0,t)}{\partial z^k}\right),$$

$$S_zE_t\left[\frac{\partial^\mu\psi(z,t)}{\partial t^\mu}\right] = \delta^{-\mu}\overline{\psi}(\gamma,\delta) - \sum_{j=0}^{n-1}\delta^{-\mu+j+2}S_z\left(\frac{\partial^j\psi(z,0)}{\partial t^j}\right).$$

For the Existence condition and the properties of DSET see [40].

Principle of the DSETVIM:

This paragraph will use the suggested method DSETVIM for solving NT-FDPDEs $\mu, (n-1 < \mu \le n, n = 1,2,\ldots)$.

$$^{c}D_t^{\mu}\varphi(z,t) + R\varphi(z,t) + N\varphi(z,t) = f(z,t), \qquad 1$$

Depending on the initial conditions (I.Cs):

$$\left[\frac{\partial^{n-1}\varphi(z,t)}{\partial t^{n-1}}\right]_{t=0} = g_{n-1}(z), \qquad 2$$

where $f(z,t)$ is the source term, R denotes the linear differential operator, N stands for the generic nonlinear differential operator, and $^{c}D_t^{\mu} = \frac{\partial^\mu\varphi(z,t)}{\partial t^\mu}$ is the Caputo fractional derivative.

Applying the DSET on both sides of Eq.1,

$$S_zE_t[\,^{c}D_t^{\mu}\varphi(z,t)] + S_zE_t[R\varphi(z,t)] + S_zE_t[N\varphi(z,t)] = S_zE_t[f(z,t)]. \qquad 3$$

Depending on the derivative properties of DSET, the Eq.3 becomes

$$\overline{\Phi}(\gamma,\delta) = \sum_{j=0}^{n-1}\delta^{j+2}S_z\left(\frac{\partial^j\psi(z,0)}{\partial t^j}\right) + \delta^\mu S_zE_t[f(z,t)] - \delta^\mu S_zE_t[R\varphi(z,t) + N\varphi(z,t)]. \qquad 4$$

Where S_z is a single ST.

The results of this calculation, which uses the IDSET on both sides of Eq.4, are as follows:

$$\varphi(z,t) = \Lambda(z,t) - S_zE_t^{-1}\left[\delta^\mu S_zE_t[R\varphi(z,t) + N\varphi(z,t)]\right], \qquad 5$$

where $\Lambda(z,t) = \sum_{j=0}^{n-1}\delta^{j+2}S_z\left(\frac{\partial^j\psi(z,0)}{\partial t^j}\right) + \delta^\mu S_zE_t[f(z,t)]$.

Now, by putting $\frac{\partial}{\partial t}$ including both sides of Eq.5

$$\frac{\partial\varphi(z,t)}{\partial t} + \frac{\partial}{\partial t}S_zE_t^{-1}\left[\delta^\mu S_zE_t[R\varphi(z,t) + N\varphi(z,t)]\right] - \frac{\partial\Lambda(z,t)}{\partial t} = 0.$$

By applying the variational iteration technique[8], which can then be used to create the correct functional, as shown below:

$$\varphi_{m+1}(z,t) = \varphi_m(z,t) - \int_0^t\left[\frac{\partial\varphi_m(z,t)}{\partial\delta} + \frac{\partial}{\partial\delta}S_zE_t^{-1}\left[\delta^\mu S_zE_t[R\varphi_m + N\varphi_m]\right] - \frac{\partial\Lambda}{\partial\delta}\right]d\delta. \qquad 6$$

Or alternately

$$\varphi_{m+1}(z,t) = \Lambda(z,t) - S_zE_t^{-1}\left[\delta^\mu S_zE_t[R\varphi_m + N\varphi_m]\right]. \qquad 7$$

Recall that $\varphi(z,t) = \lim_{m\to\infty}\varphi_m(z,t)$.

The limit stated above will determine whether the equation under consideration has an exact solution ES or an approximate solution AS.

The Convergence Theorem

The convergence theorem of DSET is shown in this section.

Theorem 8: If the integral $\delta\int_0^\infty e^{-\frac{t}{\delta}}\psi(z,t)dt$, converges at $\delta = \delta_0$, then the integral converges for $\delta < \delta_0$.

Proof: for the proof see [41].

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Theorem 9: If the integral $h(z,\delta)=\delta\int_0^\infty e^{-\frac{t}{\delta}}\psi(z,t)dt$, converges at $\delta<\delta_0$, and the integral $\frac{1}{\gamma}\int_0^\infty h(z,\delta)e^{-\frac{z}{\gamma}}dz$ converges at $\gamma=\gamma_0$, then the integral $\frac{1}{\gamma}\int_0^\infty h(z,\delta)e^{-\frac{z}{\gamma}}dz$ converges for $\gamma<\gamma_0$.
Proof: for the proof see [42].

Theorem 10: Let the function $\psi(z,t)$ is continuous in the positive quadrant of the zt-plane. If the integral $\frac{\delta}{\gamma}\int_0^\infty\int_0^\infty \psi(z,t)e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)}dtdz$ converges at $\gamma=\gamma_0,\delta=\delta_0$, then the integral, $\frac{\delta}{\gamma}\int_0^\infty\int_0^\infty \psi(z,t)e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)}dtdz$ converges for $\gamma<\gamma_0,\delta<\delta_0$.
Proof:

$$\frac{\delta}{\gamma}\int_0^\infty\int_0^\infty \psi(z,t)e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)}dtdz=$$
$$\frac{1}{\gamma}\int_0^\infty e^{-\frac{z}{\gamma}}\left(\delta\int_0^\infty e^{-\frac{t}{\delta}}\psi(z,t)dt\right)dz,$$
$$\frac{\delta}{\gamma}\int_0^\infty\int_0^\infty \psi(z,t)e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)}dtdz=\frac{1}{\gamma}\int_0^\infty e^{-\frac{z}{\gamma}}h(z,t)dz,$$

Where $h(z,\delta)=\delta\int_0^\infty e^{-\frac{t}{\delta}}\psi(z,t)dt$,
By using Theorem 8 the integral $\delta\int_0^\infty e^{-\frac{t}{\delta}}\psi(z,t)dt$ converges for $\delta<\delta_0$,
and by using Theorem 9 the integral $\frac{1}{\gamma}\int_0^\infty e^{-\frac{z}{\gamma}}h(z,\delta)dz$ converge for $\gamma<\gamma_0$,
we see the integral $\frac{1}{\gamma}\int_0^\infty e^{-\frac{z}{\gamma}}h(z,t)dz$ is converges for $\gamma<\gamma_0,\ \delta<\delta_0$, hence the integral
$\frac{\delta}{\gamma}\int_0^\infty\int_0^\infty \psi(z,t)e^{-\left(\frac{z}{\gamma}+\frac{t}{\delta}\right)}dtdz$ converge for $\gamma<\gamma_0$, $\delta<\delta_0$.

Applications:
The technique mentioned in the preceding paragraph will be used to solve the following NT-FDPDEs in the following cases:
Example 1: Starting with the NT-FDPDE[43].

$$ {}^{c}D_t^{\mu}\varphi(z,t)+\varphi(z,t)\varphi_z(z,t)-\varphi_z(z,t)=0,\ 0<\mu\le 1. \quad 8$$

Depending on the I.C: $\varphi(z,0)=z+1$.
If $\mu=1$, Eq.8 becomes:
$\varphi_t(z,t)+\varphi(z,y)\varphi_z(z,t)-\varphi_z(z,t)=0, 0<\mu\le 1$.
The ES of Eq.8 is
$\varphi(z,t)=1+\frac{z}{1+t}$.

Applying DSET, including both sides of Eq.8:

$$\mathrm{S_z E}_t\left[\frac{\partial^\mu\varphi(z,t)}{\partial t^\mu}\right]+\mathrm{S_zE}_t[\varphi(z,t)\varphi_z(z,t)]-\mathrm{S_zE}_t[\varphi_z(z,t)]=0, \quad 9$$

Depending on the derivative properties of DSET, Eq.9 becomes:

$$\delta^{-\mu}\overline{\Phi}(\gamma,\delta)-\sum_{j=0}^{n-1}\delta^{-\mu+j+2}\mathrm{S_z}\left(\frac{\partial^j\varphi(z,0)}{\partial t^j}\right)+\mathrm{S_zE_y}[\varphi(z,y)\varphi_z(z,y)]-\mathrm{S_zE_y}[\varphi_z(z,y)]=0,$$

where $\mathrm{S_z}$ is a single ST.

$$\overline{\Phi}(\gamma,\delta)-\delta^2\mathrm{S_z}(\varphi(z,0))+\delta^\mu\mathrm{S_zE}_t[\varphi(z,t)\varphi_z(z,t)]-\delta^\mu\mathrm{S_zE}_t[\varphi_z(z,t)]=0,$$
$$\overline{\Phi}(\gamma,\delta)=\delta^2+\delta^2\gamma+\delta^\mu\mathrm{S_zE}_t[\varphi(z,t)\varphi_z(z,t)]-\delta^\mu\mathrm{S_zE}_t[\varphi_z(z,t)]=0. \quad 10$$

Now, taking IDSET on each side of Eq.10:

$$S_zE_t^{-1}\left(\overline{\Phi}(\gamma,\delta)\right)=S_zE_t^{-1}(\delta^2+\delta^2\gamma)-S_zE_t^{-1}(\delta^\mu\mathrm{S_zE}_t[(\varphi(z,t)-1)\varphi_z(z,t)]),$$

The formula shown below may be created using Eq.7:

$$\varphi_{m+1}(z,t)=z+1-S_zE_t^{-1}\left(\delta^\mu\mathrm{S_zE}_t\left[(\varphi_m(z,t)-1)\left(\varphi_m(z,t)\right)_z\right]\right), \quad 11$$

Using the iteration formula, Eq.11 becomes:

$$\varphi_0(z,t)=z+1,$$
$$\varphi_1(z,t)=z+1-S_zE_t^{-1}(\gamma\delta^{\mu+2})=z+1-z\frac{t^\mu}{\Gamma(\mu+1)},$$
$$\varphi_2(z,t)=z+1-z\frac{t^\mu}{\Gamma(\mu+1)}-S_zE_t^{-1}\left(\left(2\gamma\delta^{2\mu+2}-\frac{\gamma\delta^{3\mu+2}\Gamma(2\mu+1)}{\Gamma(\mu+1)^2}\right)\right),$$
$$\varphi_2(z,t)=z+1-z\frac{t^\mu}{\Gamma(\mu+1)}+2z\frac{t^{2\mu}}{\Gamma(2\mu+1)}-z\frac{\Gamma(2\mu+1)}{\Gamma^2(\mu+1)}\frac{t^{3\mu}}{\Gamma(3\mu+1)},$$
$$\varphi_3(z,t)=z+1-z\frac{t^\mu}{\Gamma(\mu+1)}+2z\frac{t^{2\mu}}{\Gamma(2\mu+1)}-zt^{3\mu}\left(\frac{4}{\Gamma(2\mu+1)}+\frac{1}{\Gamma(\mu+1)^2}\right)\frac{\Gamma(2\mu+1)}{\Gamma(3\mu+1)}+zt^{4\mu}\left(\frac{4}{\Gamma(2\mu+1)\Gamma(\mu+1)}+\frac{2\Gamma(2\mu+1)}{\Gamma(3\mu+1)\Gamma(\mu+1)^2}\right)\frac{\Gamma(3\mu+1)}{\Gamma(4\mu+1)}-zt^{5\mu}\left(\frac{2\Gamma(2\mu+1)}{\Gamma(3\mu+1)\Gamma(\mu+1)^3}+\frac{4}{\Gamma(2\mu+1)^2}\right)\frac{\Gamma(4\mu+1)}{\Gamma(5\mu+1)}+zt^{6\mu}\left(\frac{4}{\Gamma(3\mu+1)\Gamma(\mu+1)^2}\right)\frac{\Gamma(5\mu+1)}{\Gamma(6\mu+1)}-zt^{7\mu}\left(\frac{\Gamma(2\mu+1)^2}{\Gamma(3\mu+1)^2\Gamma(\mu+1)^4}\right)\frac{\Gamma(6\mu+1)}{\Gamma(7\mu+1)}, \quad 12$$
$$\vdots$$

From Eq.12, the AS of Eq.8 is

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$$\varphi(z,t) = z + 1 - z\frac{t^{\mu}}{\Gamma(\mu+1)} + 2z\frac{t^{2\mu}}{\Gamma(2\mu+1)} - zt^{3\mu}\left(\frac{4}{\Gamma(2\mu+1)} + \frac{1}{\Gamma(\mu+1)^2}\right)\frac{\Gamma(2\mu+1)}{\Gamma(3\mu+1)} + zt^{4\mu}\left(\frac{4}{\Gamma(2\mu+1)\Gamma(\mu+1)} + \frac{2\Gamma(2\mu+1)}{\Gamma(3\mu+1)\Gamma(\mu+1)^2}\right)\frac{\Gamma(3\mu+1)}{\Gamma(4\mu+1)} - zt^{5\mu}\left(\frac{2\Gamma(2\mu+1)}{\Gamma(3\mu+1)\Gamma(\mu+1)^3} + \frac{4}{\Gamma(2\mu+1)^2}\right)\frac{\Gamma(4\mu+1)}{\Gamma(5\mu+1)} + zt^{6\mu}\left(\frac{4}{\Gamma(3\mu+1)\Gamma(\mu+1)^2}\right)\frac{\Gamma(5\mu+1)}{\Gamma(6\mu+1)} - zt^{7\mu}\left(\frac{\Gamma(2\mu+1)^2}{\Gamma(3\mu+1)^2\Gamma(\mu+1)^4}\right)\frac{\Gamma(6\mu+1)}{\Gamma(7\mu+1)} + \cdots,$$

And in the special case $\mu = 1$, Eq.12, becomes:

$$\varphi(z,t) = 1 + z\left(1 - t + t^2 - t^3 + \frac{2}{3}t^4 - \frac{1}{3}t^5 + \frac{1}{9}t^6 - \frac{1}{63}t^7 + \cdots\right).$$

Recall that the ES of Eq.12 is calculated by

$$\varphi(z,t) = \lim_{m\to\infty} \varphi_m(z,t).$$

Then,

$\varphi(z,t) = 1 + \frac{z}{1+t}$, $|t| < 1$. Fig. 1 shows the Numerical solution: (a) the ES and (b) the AS of Eq.8 in case $\mu = 1$, while Fig. 2 illustrated the absolute error between the ES and AS.

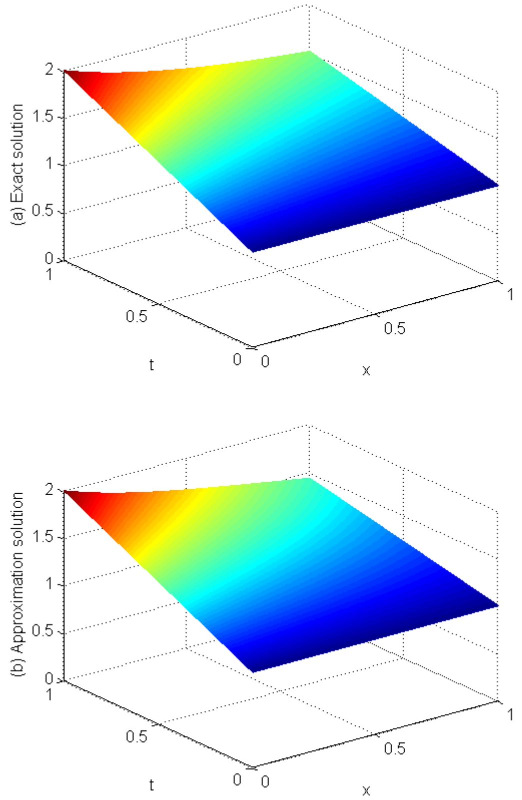

Figure 1. In the case $\mu = 1$, (a) the ES and (b) the AS

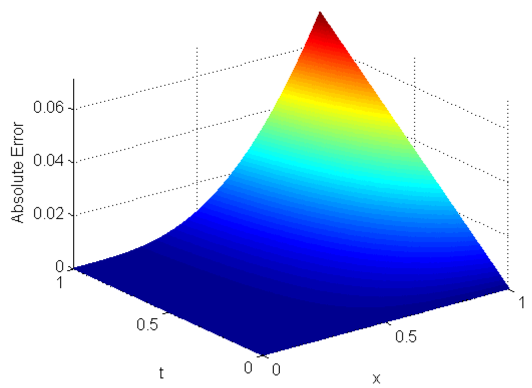

Figure 2. The Absolute Error when $\mu = 1$

Example 2: Consider the following NT-FDPDE[43].

$$^{c}D_t^{\mu}\varphi(z,t) - \frac{3}{8}((\varphi_{zz}(z,t))^2)_z = \frac{3}{2}t,\ 2 < \mu \le 3. \quad 13$$

Depending on the I.C:

$$\varphi(z,0) = -\frac{1}{2}z^2, \varphi_t(z,0) = \frac{1}{3}z^3, \varphi_{tt}(z,0) = 0.$$

The following result is obtained by employing the differentiation property and applying DSET, including both sides of Eq.13:

$$S_zE_t\left[\frac{\partial^{\mu}\varphi(z,t)}{\partial t^{\mu}}\right] - S_zE_t\left[\frac{3}{8}\left(\left(\varphi_{zz}(z,t)\right)^2\right)_z\right] = S_zE_t\left[\frac{3}{2}t\right], \quad 14$$

Depending on the derivative properties of DSET, Eq.14 becomes

$$\delta^{-\mu}\overline{\Phi}(\gamma,\delta) - \sum_{j=0}^{n-1}\delta^{-\mu+j+2}S_z\left(\frac{\partial^j\varphi(z,0)}{\partial t^j}\right) - S_zE_t\left[\frac{3}{8}\left(\left(\varphi_{zz}(z,t)\right)^2\right)_z\right] = S_zE_t\left[\frac{3}{2}t\right],$$

where S_z is a single ST.

$$\overline{\Phi}(\gamma,\delta) - \delta^2S_z(\varphi(z,0)) - \delta^3S_z\left(\varphi_y(z,0)\right) - \delta^4S_z(\varphi_{tt}(z,0)) - \delta^{\mu}S_zE_y\left[\frac{3}{8}\left(\left(\varphi_{zz}(z,t)\right)^2\right)_z\right] = S_zE_t\left[\frac{3}{2}t\right],$$

$$\overline{\Phi}(\gamma,\delta) + \delta^{\mu}S_zE_t\left[\frac{3}{8}\left(\left(\varphi_{zz}(z,t)\right)^2\right)_z\right] = S_zE_t\left[\frac{3}{2}t\right],$$

$$\overline{\Phi}(\gamma,\delta) = -\delta^2\gamma^2 + 2\delta^3\gamma^3 + \delta^{\mu}S_zE_t\left[\left[\frac{3}{8}\left(\left(\varphi_{zz}(z,t)\right)^2\right)_z\right]\right] + \frac{3}{2}\delta^{3+\mu}, \quad 15$$

Now, taking IDSET on each side of the Eq.15:

$$S_zE_t^{-1}\left(\overline{\Phi}(\gamma,\delta)\right) = S_zE_t^{-1}\left(-\delta^2\gamma^2 + 2\delta^3\gamma^3 + \frac{3}{2}\delta^{3+\mu}\right) + S_zE_t^{-1}\left(\delta^{\mu}S_zE_t\left[\left[\frac{3}{8}\left(\left(\varphi_{zz}(z,t)\right)^2\right)_z\right]\right]\right).$$

The formula shown below may be created using Eq.7:

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$$\varphi_{m+1}(z,t) = -\frac{1}{2}z^2 + \frac{1}{3}z^3 t + \frac{3}{2}\frac{t^{\mu+1}}{\Gamma(\mu+2)} + S_z E_t^{-1}\left(\delta^{\mu} S_z E_t\left[\left[\frac{3}{8}\left(\left(\left(\varphi_m(z,t)\right)_{zz}\right)^2\right)_z\right]\right]\right). \quad 16$$

Using the iteration formula Eq.16 becomes:

$$\varphi_0(z,t) = -\frac{1}{2}z^2 + \frac{1}{3}z^3 t,$$

$$\varphi_1(z,t) = -\frac{1}{2}z^2 + \frac{1}{3}z^3 t + S_z E_y^{-1}(6\gamma\delta^{\mu+4}) = -\frac{1}{2}z^2 + \frac{1}{3}z^3 t + 6z\frac{t^{\mu+2}}{\Gamma(\mu+3)},$$

$$\varphi_2(z,t) = -\frac{1}{2}z^2 + \frac{1}{3}z^3 t + 6z\frac{t^{\mu+2}}{\Gamma(\mu+3)},$$

$$\varphi_3(z,t) = -\frac{1}{2}z^2 + \frac{1}{3}z^3 t + 6z\frac{t^{\mu+2}}{\Gamma(\mu+3)}, \quad 17$$

$\vdots$

From Eq.17, the AS of Eq.13, is

$$\varphi(z,t) = -\frac{1}{2}z^2 + \frac{1}{3}z^3 t + 6z\frac{t^{\mu+2}}{\Gamma(\mu+3)}.$$

And in the special case $\mu = 3$, the ES of Eq.13 is:

$$\varphi(z,t) = \frac{1}{20}zt^5 + \frac{1}{3}z^3 t - \frac{1}{2}z^2.$$

The AS of some of 4-order approximate solutions for Eq.13 for different values of μ are included in Table 1.

Table 1. The AS of Example 2 uses four terms DSETVIM.

Z	t	$\mu = 2.92$	$\mu = 2.95$	$\mu = 2.98$	Exact
0.2	0.2	-0.0194	-0.0194	-0.0194	-0.0194
0.5		-0.1166	-0.1166	-0.1166	-0.1166
0.7		-0.2221	-0.2221	-0.2221	-0.2221
1.0		-0.4333	-0.4333	-0.4333	-0.4333
0.2	0.4	-0.0188	-0.0188	-0.0188	-0.0188
0.5		-0.1080	-0.1080	-0.1080	-0.1080
0.7		-0.1988	-0.1988	-0.1988	-0.1989
1.0		-0.3660	-0.3660	-0.3661	-0.3661
0.2	0.6	-0.0174	-0.0175	-0.0175	-0.0176
0.5		-0.0976	-0.0978	-0.0979	-0.0980
0.7		-0.1731	-0.1733	-0.1735	-0.1736
1.0		-0.2953	-0.2956	-0.2959	-0.2961
0.2	0.8	-0.0140	-0.0142	-0.0144	-0.0145
0.5		-0.0821	-0.0826	-0.0831	-0.0834
0.7		-0.1401	-0.1409	-0.1416	-0.1420
1.0		-0.2142	-0.2152	-0.2163	-0.2169
0.2	1.0	-0.0058	-0.0064	-0.0069	-0.0073
0.5		-0.0546	-0.0561	-0.0574	-0.0583
0.7		-0.0905	-0.0925	-0.0944	-0.0956
1.0		-0.1093	-0.1122	-0.1149	-0.1166

Example 3: Consider the following NT-FDPDE[44].

$${}^{c}D_t^{\mu}\varphi(z,t) = \frac{z^2}{2}\varphi_{zz}(z,t), \ 0 < \mu \le 1. \quad 18$$

Depending on the I.C: $\varphi(z,0) = z^2$.

The following result is obtained by employing the differentiation property and applying DSET, including both sides of Eq.18:

$$S_z E_t\left[\frac{\partial^{\mu}\varphi(z,t)}{\partial t^{\mu}}\right] = S_z E_t\left[\frac{z^2}{2}\varphi_{zz}(z,t)\right], \quad 19$$

Depending on the derivative properties of DSET, Eq.19 becomes:

$$\delta^{-\mu}\overline{\Phi}(\gamma,\delta) - \sum_{j=0}^{n-1}\delta^{-\mu+j+2} S_z\left(\frac{\partial^j\varphi(z,0)}{\partial t^j}\right) = S_z E_t\left[\frac{z^2}{2}\varphi_{zz}(z,t)\right],$$

where S_z is a single ST.

$$\overline{\Phi}(\gamma,\delta) - \delta^2 S_z(\varphi(z,0)) = \delta^{\mu} S_z E_t\left[\left[\frac{z^2}{2}\varphi_{zz}(z,t)\right]\right],$$

$$\overline{\Phi}(\gamma,\delta) = 2\delta^2\gamma^2 + \delta^{\mu} S_z E_t\left[\left[\frac{z^2}{2}\varphi_{zz}(z,t)\right]\right], \quad 20$$

Now, taking IDSET on each side of the Eq.20:

$$S_z E_t^{-1}\left(\overline{\Phi}(\gamma,\delta)\right) = 2\delta^2\gamma^2 + S_z E_t^{-1}\left(\delta^{\mu} S_z E_t\left[\left[\frac{z^2}{2}\varphi_{zz}(z,t)\right]\right]\right),$$

The formula shown below may be created using Eq.7:

$$\varphi_{m+1}(z,y) = z^2 + S_z E_y^{-1}\left(\delta^{\mu} S_z E_t\left[\left[\frac{z^2}{2}\varphi_{zz}(z,t)\right]\right]\right), \quad 21$$

Using the iteration formula Eq.21 becomes:

$\varphi_0(z,t) = z^2,$
$\varphi_1(z,t) = z^2 + S_z E_t{}^{-1}(2\delta^{\mu+2}\gamma^2) = z^2 + z^2 \frac{t^\mu}{\Gamma(\mu+1)},$
$\varphi_2(z,t) = z^2 + z^2 \frac{t^\mu}{\Gamma(\mu+1)} + z^2 \frac{t^{2\mu}}{\Gamma(2\mu+1)},$
$\varphi_3(z,t) = z^2 + z^2 \frac{t^\mu}{\Gamma(\mu+1)} + z^2 \frac{t^{2\mu}}{\Gamma(2\mu+1)} + z^2 \frac{t^{3\mu}}{\Gamma(3\mu+1)},$
$\vdots$
And so on...
Proceeding in this manner, gaining

$$\varphi_m(z,t) = \sum_{k=0}^{m} \frac{z^2 t^{k\mu}}{\Gamma(k\mu+1)}, \qquad 22$$

This can be assumed to be the m^th AS of Eq.22. The ES when $\mu = 1$ of Eq.18 is given by:

$$\varphi(z,t) = \lim_{m\to\infty} \varphi_m(z,t) = z^2 e^t.$$

The comparison between the suggested method with the method that combines Yang transform with the variational iteration method described in reference[44] to some of the 4-order approximate solutions for Eq.18 for various values of μ and various values of $z, t,$ as well as the absolute error between the ES and AS when $\mu = 1$ are included in Table 2, also Fig. 3 illustrates the Numerical solution: (a) the ES and (b) the AS in the case of $\mu = 1$, while Fig. 4 illustrated the absolute error between the ES and AS.

Table 2. AS and ES for the different values of μ in Example 3

Z	t	$\mu = 0.5$		$\mu = 0.7$		$\mu = 0.9$		Exact	Absolute Error
		DSET VIM	Yang transform[31]	DSET VIM	Yang transform[31]	DSET VIM	Yang transform[31]		
0.25	**0.2**	0.1107	0.1119	0.0910	0.0911	0.0800	0.0800	0.0763	5.7183e-09
0.5		0.4430	0.4479	0.3642	0.3647	0.3201	0.3201	0.3053	2.2873e-08
0.75		0.9967	1.0079	0.8194	0.8207	0.7202	0.7202	0.6870	5.1465e-08
1.0		1.7719	1.7919	1.4568	1.4591	1.2802	1.2805	1.2214	9.1493e-08
0.25	**0.4**	0.1440	0.1489	0.1168	0.1178	0.0994	0.0995	0.0932	3.7693e-07
0.5		0.5760	0.5959	0.4672	0.4713	0.3977	0.3983	0.3729	1.5077e-06
0.75		1.2960	1.3409	1.0514	1.0605	0.8947	0.8962	0.8391	3.3924e-06
1.0		2.3040	2.3839	1.8691	1.8855	1.5906	1.5934	1.4918	6.0309e-06
0.25	**0.6**	0.1765	0.1877	0.1449	0.1481	0.1222	0.1229	0.1138	4.4250e-06
0.5		0.7059	0.7509	0.5798	0.5925	0.4887	0.4916	0.4555	1.7700e-05
0.75		1.5883	1.6895	1.3045	1.3331	1.0995	1.1062	1.0249	3.9825e-05
1.0		2.8237	3.0036	2.3191	2.3700	1.9547	1.9666	1.8221	7.0800e-05

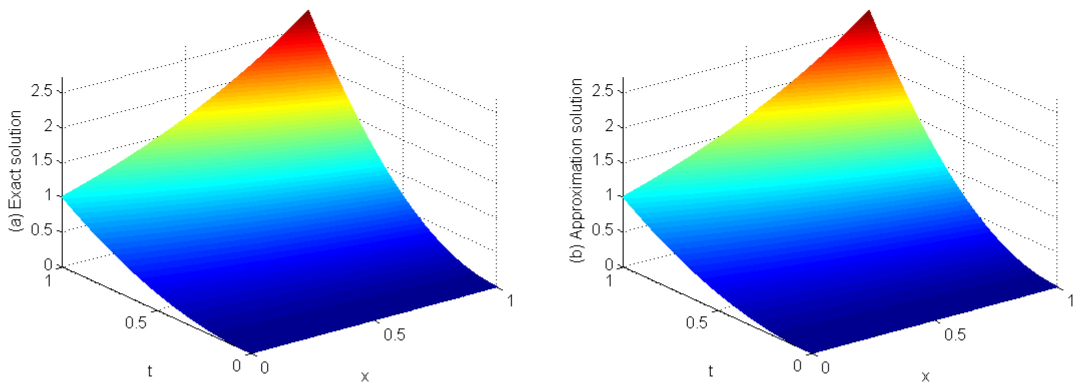

Figure 3. In the case $\mu = 1$, (a) the ES and (b) the AS

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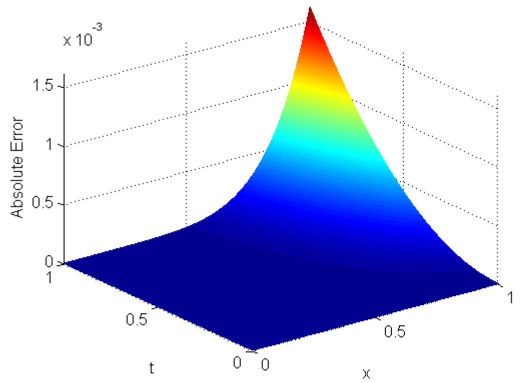

Figure 4. The Absolute Error when $\mu = 1$

Example 4: Let's examine the NT-FDPDE[45].

$${}^{c}D_t^{\mu}\varphi(z,t) - 2\frac{z^2}{t}\varphi(z,t)\varphi_z(z,t) = 0, z, t \geq 0, 1 < \mu \leq 2. \quad 23$$

Depending on the I.C: $\varphi(z,0) = 0, \varphi_z(z,0) = z$.

The following result is obtained by employing the differentiation property and applying DSET, including both sides of Eq.23:

$$\mathrm{S_zE}_t\left[\frac{\partial^{\mu}\varphi(z,t)}{\partial t^{\mu}}\right] - \mathrm{S_zE}_t\left[2\frac{z^2}{t}\varphi(z,t)\varphi_z(z,t)\right] = 0. \quad 24$$

Depending on the derivative properties of DSET, the Eq.24 becomes

$$\delta^{-\mu}\overline{\Phi}(\gamma,\delta) - \sum_{j=0}^{n-1}\delta^{-\mu+j+2}\mathrm{S_z}\left(\frac{\partial^j\varphi(z,0)}{\partial t^j}\right) - \mathrm{S_zE}_t\left[2\frac{z^2}{t}\varphi(z,t)\varphi_z(z,t)\right] = 0,$$

where $\mathrm{S_z}$ is a single ST.

$$\overline{\Phi}(\gamma,\delta) - \delta^2\mathrm{S_z}(\varphi(z,0)) - \delta^3\mathrm{S_z}\left(\varphi_y(z,0)\right) - \delta^{\mu}\mathrm{S_zE}_t\left[2\frac{z^2}{t}\varphi(z,t)\varphi_z(z,t)\right] = 0,$$

$$\overline{\Phi}(\gamma,\delta) = \delta^3\gamma + \delta^{\mu}\mathrm{S_zE}_t\left[2\frac{z^2}{t}\varphi(z,t)\varphi_z(z,t)\right]. \quad 25$$

Now, taking IDSET on each side of the Eq.25

$$S_zE_t^{-1}\left(\overline{\Phi}(\gamma,\delta)\right) = S_zE_t^{-1}(\delta^3\gamma) + S_zE_t^{-1}\left(\delta^{\mu}\mathrm{S_zE}_t\left[2\frac{z^2}{t}\varphi(z,t)\varphi_z(z,t)\right]\right),$$

The formula shown below may be created using Eq.7:

$$\varphi_{m+1}(z,t) = zt + S_zE_t^{-1}\left(\delta^{\mu}\mathrm{S_zE}_t\left[2\frac{z^2}{t}\varphi(z,t)\varphi_z(z,t)\right]\right). \quad 26$$

Using the iteration formula Eq.26 becomes:

$$\varphi_0(z,t) = zt,$$

$$\varphi_1(z,t) = zt + S_zE_t^{-1}(12\gamma^3\delta^{\mu+3}) = zt + \frac{2z^3t^{\mu+1}}{\Gamma(\mu+2)},$$

$$\varphi_2(z,t) = zt + \frac{2z^3t^{\mu+1}}{\Gamma(\mu+2)} + \frac{16z^5t^{2\mu+1}}{\Gamma(2\mu+2)}, \quad 27$$

$\vdots$

So, the AS of Eq.23 is calculated by:

$$\varphi(z,t) = zt + \frac{2z^3t^{\mu+1}}{\Gamma(\mu+2)} + \frac{16z^5t^{2\mu+1}}{\Gamma(2\mu+2)} + \frac{24\Gamma(2\mu+2)z^7t^{3\mu+1}}{\Gamma(3\mu+2)\Gamma^2(\mu+2)} + \cdots.$$

And in the special case $\mu \to 2$, is

$$\varphi(z,t) = zt + \frac{z^3t^3}{3} + \frac{2z^5t^5}{15} + \frac{17z^7t^7}{315} + \cdots.$$

Recall that the ES of Eq.23 is calculated by:

$$\varphi(z,t) = \lim_{m\to\infty}\varphi_m(z,t),$$

$$\varphi(z,t) = \tan(zt).$$

Which is an ES to the NT-FDPDE when $\mu = 2$.

The comparison between the suggested method with the method that combines Elzaki transform with Adomian decomposition method described in reference[45] to some of the 4-order approximate solutions for Eq.23 for various values of z and $t = 1.1$, along with the absolute error between the ES and AS when $\mu = 2$ are included in Table 3, while Fig. 5, illustrates the Numerical solution: (a) the ES and (b) the AS in the case of $\mu = 2$.

Table 3. The AS of Example 4 uses four terms DSETVIM.

Z	t	$\mu = 1.94$		$\mu = 1.96$		$\mu = 1.98$		$\mu = 2$			
		DSET VIM	EADM[32]	DSET VIM	EADM[32]	DSET VIM	EADM[32]	DSET VIM	EADM[32]	Exact	Absolute Error
0	1.1	0	0	0	0	0	0	0	0	0	0
0.1		0.1104	0.1105	0.1104	0.1105	0.1104	0.1105	0.1104	0.1104	0.1104	5.1821e-11
0.2		0.2238	0.2239	0.2237	0.2238	0.2237	0.2237	0.2236	0.2236	0.2236	2.6930e-08
1.3		0.3434	0.3435	0.3431	0.3432	0.3428	0.3429	0.3425	0.3425	0.3425	1.0618e-06
0.4		0.4732	0.4735	0.4724	0.4726	0.4715	0.4718	0.4707	0.4709	0.4707	1.4668e-05
0.5		0.6183	0.6197	0.6165	0.6176	0.6147	0.6156	0.6129	0.6137	0.6131	1.1478e-04
0.6		0.7858	0.7906	0.7822	0.7862	0.7788	0.7820	0.7754	0.7781	0.7761	6.3109e-04
0.7		0.9853	0.9993	0.9789	0.9906	0.9728	0.9824	0.9669	0.9745	0.9696	2.7390e-03
0.8		1.2303	1.2659	1.2196	1.2493	1.2094	1.2337	1.1995	1.2190	1.2096	1.0087e-02
0.9		1.5396	1.6208	1.5226	1.5902	1.5062	1.5616	1.4905	1.5348	1.5236	3.3143e-02
1.0		1.9389	2.1086	1.9127	2.0540	1.8876	2.0033	1.8636	1.9560	1.9647	1.0118e-01

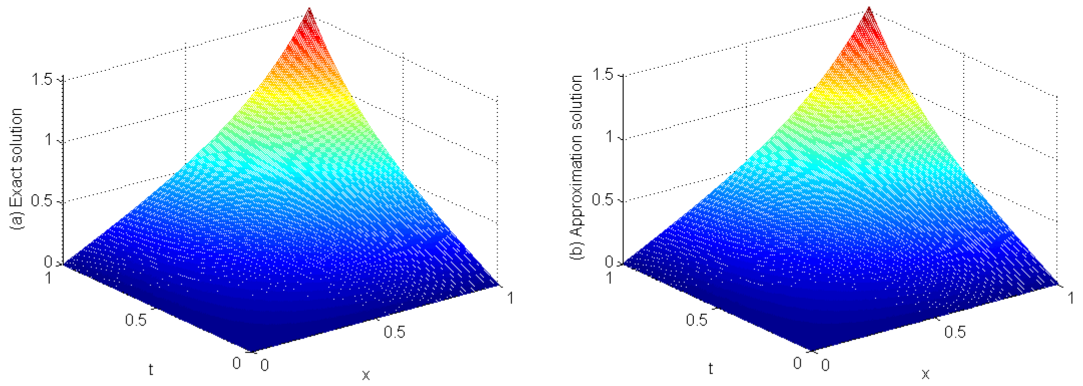

Figure 5. In the case $\mu = 2$, (a) the ES and (b) the AS

Conclusion

Combining Sumudu-Elzaki Transforms and the Variational Iteration Method is an effective strategy for solving NT-FDPDEs. The suggested method is very effective and appropriate for these types of problems. The results demonstrate that the DSETVIM produces very accurate approximations with just a few iterations. The numerical results demonstrate how effective, simple, and speedy this new analytical approach is, producing a series solution that rapidly converges to the right solution.

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Author's Declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images that are not ours have been included with the necessary permission for re-publication, which is attached to the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Al-Esraa University College.

Author's Contribution Statement

M. G. S. A.: Designed and conceptualized the idea and drafted the concept note. He participated also in the validation of the work and manuscript preparation. R. M. F. and W. R. A. A. participated in the review of literature, running simulations, and drafting the results and discussions.

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الحلول العددية للمعادلات التفاضلية الجزئية غير الخطية ذات الرتبة الكسرية بأستخدام تحويل تكاملي مزدوج جديد مع طريقة التكرار المتغاير

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الخلاصة

يتناول هذه البحث تحويلاً تكامليا مزدوجًا جديدًا يسمى تحويل سومودو إلزاكي المزدوج DSET. نقوم بدمج DSET مع طريقة شبه تحليلية، وهي طريقة التكرار المتغاير DSETVIM ، للوصول إلى حل عددي للمعادلات التفاضلية الجزئية الغير خطية ذات الرتب الكسرية. تقلل خاصية الطريقة المزدوجة المقترحة من عدد العمليات الحسابية المطلوبة، لذا فإن الدمج بين هاتين الطريقتين يؤدي إلى التسريع في الوصول للحل. تم اختبار التقنية المقترحة على أربعة امثلة. أظهرت النتائج أن حل هذه الأنواع من المعادلات باستخدام DSETVIM كان أكثر فائدة وكفاءة.

الكلمات المفتاحية: تحويل سومودو إلزاكي المزدوج، التفاضل الكسري، المعادلات التفاضلية الجزئية اللاخطية ذات الرتب الكسرية، الحل العددي، طريقة التكرار المتغاير.