On Semigroup Ideals and Right n-Derivation in 3-Prime Near-Rings

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Abstract:
The current paper studied the concept of right n-derivation satisfying certified conditions on semigroup ideals of near-rings and some related properties. Interesting results have been reached, the most prominent of which are the following: Let \( \mathcal{M} \) be a 3-prime left near-ring and \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \) are nonzero semigroup ideals of \( \mathcal{M} \), if \( d \) is a right n-derivation of \( \mathcal{M} \) satisfies on of the following conditions,

(i) \( d \left( (u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n) \right) = 0 \) for each \( u_i \in \mathcal{A}_1, v_i \in \mathcal{A}_2, \ldots, u_n \in \mathcal{A}_n \);

(ii) \( d \left( (u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n) \right) = 0 \) for each \( u_i \cdot v_i \in \mathcal{A}_i, v_i \in \mathcal{A}_2, \ldots, u_n \cdot v_n \in \mathcal{A}_n \);

(iii) \( d \left( (u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n) \right) = 0 \) for each \( u_i = v_i \in \mathcal{A}_1, v_i \in \mathcal{A}_2, \ldots, u_n = v_n \in \mathcal{A}_n \);

(iv) \( d + d \) is an n-additive mapping from \( \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_n \) to \( \mathcal{M} \);

(v) \( d \left( (u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n) \right) \in \mathcal{Z}(\mathcal{M}) \) for each \( u_i = v_i \in \mathcal{A}_1, v_i \in \mathcal{A}_2, \ldots, u_n = v_n \in \mathcal{A}_n \);

Then \( \mathcal{M} \) is a commutative ring.

Keywords: Generalized right derivations, Prime near-ring, Right derivations, Right \( n \)-derivations, Semigroup ideals.

Introduction:
A left near-ring is a nonempty set \( \mathcal{M} \) with two binary operations \((+), (\cdot)\) which satisfies (i) \((\mathcal{M}, +)\) is a group that is not necessarily abelian, (ii) \((\mathcal{M}, \cdot)\) is a semi group, (iii) \(a \cdot (b + c) = a \cdot b + a \cdot c\) for each \(a, b, c \in \mathcal{M}\), (recall that when \( \mathcal{M} \) satisfies the right distributive law, \((a + b) \cdot c = a \cdot c + b \cdot c\) for each \(a, b, c \in \mathcal{M}\), then \( \mathcal{M} \) will be called right near-ring ), usually \( \mathcal{M} \) will be 3-prime, if for \(x, y \in \mathcal{M}\) with \(x \cdot y = 0\) implies \(x = 0\) or \(y = 0\). A left near-ring \( \mathcal{M} \) is called zero-symmetric if \(0x = 0\), for all \(x \in \mathcal{M}\) (left distributivity yields \(x0 = 0\)). \(\mathcal{Z}(\mathcal{M})\) will refer to the multiplicative center of \( \mathcal{M} \). Let \(0 \neq \mathcal{A} \subseteq \mathcal{M} \), then \( \mathcal{A} \) is said to be a semigroup ideal of \( \mathcal{M} \) if \( \mathcal{A} \cdot \mathcal{M} \subseteq \mathcal{M} \) and \( \mathcal{M} \cdot \mathcal{A} \subseteq \mathcal{M} \). For each \( m, n \in \mathcal{M}\), then \(m \cdot n\) = \(m + n - m \cdot n\), \([m, n] = mn - nm\) and \(m \cdot n = mn + nm\) will be denoted to the additive commutator, Lie product, and Jordan product, respectively. For more about near-ring, make reference to Pilz.1

Certain mappings, involving some algebraic identities, defined on rings,3-3 or near-rings,4 and sometimes on an appropriate subset of them, and the effect of these mappings on the algebraic structure of the near-rings, how the near-rings can be converted into rings or commutative rings, was a study project that has attracted the interest of many researchers over the past three decades.

Different types of mappings, such as derivations, generalized derivations, left derivations, homoderivations and multipliers on near-rings or rings have been studied and some related properties have been discussed, see7-10. Also, the derivation concepts generalization has been studied by various means according to different authors such as \( n \)-derivations, \((\sigma, \tau)\)-n-derivation, right \( n \)-derivation and generalized right \( n \)-derivation, on near-ring and obtained new interest results for
researchers in this field\textsuperscript{11-14}. Majed and Farhan\textsuperscript{15} are the first to define the concepts of right derivation and right $n$-derivation on the near-ring.

Let $d$ be an additive mapping from $\mathcal{M}$ into itself, $d$ is said to be a right derivation of $\mathcal{M}$ if $d(mn) = d(m)n + d(n)m$ for each $m, n \in \mathcal{M}$.

Let $d: \mathcal{M} \times \mathcal{M} \times \ldots \times \mathcal{M} \rightarrow \mathcal{M}$ be an $n$-times mapping (i.e. additive in each argument), $d$ is said to be right $n$-derivation of $\mathcal{M}$ if the following equations hold for each $m_1, m_1', m_2, m_2', \ldots, m_n, m_n' \in \mathcal{M}$:

\[
d(m_1, m_1', m_2, m_2', \ldots, m_n, m_n') = d(m_1, m_2, \ldots, m_n) m_1' + d(m_1, m_2', \ldots, m_n) m_1 + d(m_1, m_2, m_2', \ldots, m_n) m_n + d(m_1, m_2, m_2', \ldots, m_n') m_n'
\]

In this line of inspection, this work will give new essential results in this field and generalize some known results presented.

Note that from now, $\mathcal{M}$ will be 3-prime left near-ring, the abbreviation $\mathcal{C}, \mathcal{R}$ will refer to the commutative ring while $\mathcal{R}, \mathcal{D}$ and $\mathcal{R}, \mathcal{n}, \mathcal{D}$ are a brief to the derivation right and derivation $n$-derivation respectively.

Preliminaries

Lemma 1: \textsuperscript{4} [Lemmas 1.2(ii) and 1.3(ii)]

(i) If $3 \in Z(\mathcal{M})/[0]$ for which $3 + 3 \in Z(\mathcal{M})$, then $(\mathcal{M}, +)$ is abelian.

(ii) If $3 \in Z(\mathcal{M})/[0]$ and $3s \in Z(\mathcal{M})$ or $3s \in Z(\mathcal{M})$, where $s \in \mathcal{M}$, then $s \in Z(\mathcal{M})$.

Lemma 2: \textsuperscript{4} [Lemma 1.3(i), 1.4(i) and 1.5] Let $\mathcal{A} \neq 0$ be a semigroup ideal of $\mathcal{M}$, $s \in \mathcal{M}$.

(i) If $As = \{0\}$ or $sA = \{0\}$, then $s = 0$.

(ii) If $tAs = \{0\}, t, s \in \mathcal{M}$, then either $t = 0$ or $s = 0$.

(iii) If $\mathcal{A} \subseteq Z(\mathcal{M})$, then $\mathcal{M}$ is a $\mathcal{C}, \mathcal{R}$.

(iv) If $[s, u] = 0$, for any $v \in \mathcal{A}$, then $s \in Z(\mathcal{M})$.

Lemma 3: \textsuperscript{14} [Lemma 2.6] $\mathcal{M}$ is zero symmetric if and only if $\mathcal{M}$ admitting $\mathcal{R}, \mathcal{n}, \mathcal{D}$.

Lemma 4: Let $\mathcal{A} \neq 0$ be a semigroup ideal of $\mathcal{M}$. If $(\mathcal{A}, +)$ is abelian then $(\mathcal{M}, +)$ is abelian.

Proof: Since $a + b = b + a$ for any $a, b \in \mathcal{A}$, substitute as for $a$ and $b$ for $a$ to get as $+ a = a = b + as$ for any $a, s \in \mathcal{M}$ it follows $a(s + t - s\ t) = 0$, $a, s \in \mathcal{M}$, so $(\mathcal{M}, +)$ is abelian by Lemma 2(i).

Lemma 5: Let $\mathcal{A}$ and $\mathcal{B}$ be nonzero semigroup ideals of $\mathcal{M}$, if the additive commutator $(a, b) \in Z(\mathcal{M})$ for any $a, b \in \mathcal{A}$ and $b \in \mathcal{B}$ then $\mathcal{M}$ is abelian.

Proof: From assumption $(a, b) = s(a, b) \in Z(\mathcal{M})$ for any $a, b \in \mathcal{A}$, $b \in \mathcal{B}$, $s \in \mathcal{M}$, using Lemma 1(ii) leads to either $(a, b) = 0$ for any $a, b \in \mathcal{A}, b \in \mathcal{B}$ or $s \in Z(\mathcal{M})$ for any $s \in \mathcal{M}$, by Lemma 2(iii) the last result can be reduced to either $(a, b) = 0$ for any $a, b \in \mathcal{A}$, $b \in \mathcal{B}$ or $s \in Z(\mathcal{M})$ is a $\mathcal{C}, \mathcal{R}$.

If $(a, b) = 0$ for any $a, b \in \mathcal{A}$, $b \in \mathcal{B}$, then substitute as for $a$ and $ab$ for $b$ to get $a + ab = ab + as$ for any $a, b \in \mathcal{A}, b \in \mathcal{B}$, $s \in \mathcal{M}$ it follows $a(s + b - s - b) = 0$ for any $a, b \in \mathcal{A}, b \in \mathcal{B}, s \in \mathcal{M}$ so $a(s + b - s - b) = \{0\}$ for any $b \in \mathcal{B}, s \in \mathcal{M}$, thus $(\mathcal{M}, +)$ abelian by Lemma 2(i) and Lemma 4 and this complete the proof.

Corollary 1: Let $\mathcal{A}$ be a nonzero semigroup ideals of $\mathcal{M}$, if $(a, b) \in Z(\mathcal{M})$ for any $a, b \in \mathcal{A}$, then $\mathcal{M}$ is abelian.

Corollary 2: Let $\mathcal{A}$ be a nonzero semigroup ideals of $\mathcal{M}$, if $(a, s) \in Z(\mathcal{M})$ or $(s, a) \in Z(\mathcal{M})$ for any $a, b \in \mathcal{A}$ and $s \in \mathcal{M}$ then $\mathcal{M}$ is abelian.

Lemma 6: Let $d$ be a $\mathcal{R}, \mathcal{n}, \mathcal{D}$ of $\mathcal{M}$ and $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are nonzero semigroup ideals of $\mathcal{M}$, if $d(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \{0\}$, then $d = 0$.

Proof: For any $u_1 \in \mathcal{A}_1, u_2 \in \mathcal{A}_2, \ldots, u_n \in \mathcal{A}_n$, $0 = d(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = d(u_1, u_2, \ldots, u_n) = d(u_1, u_2, \ldots, u_n) u_1$, that is

\[
d(m_1, u_2, \ldots, u_n) u_1 = 0\] for any $m_1 \in \mathcal{M}, u_2 \in \mathcal{A}_2, \ldots, u_n \in \mathcal{A}_n$, so $d(m_1, u_2, \ldots, u_n) = 0$ for any $m_1 \in \mathcal{M}, u_2 \in \mathcal{A}_2, \ldots, u_n \in \mathcal{A}_n$, $u_1 \in \mathcal{A}_1, \ldots, u_n \in \mathcal{A}_n$, according to Lemma 2(i), replace $u_2$ by $m_2 u_2$ in the last result, to obtain $d(m_1, m_2, \ldots, u_n) = 0$ for any $m_1, m_2 \in \mathcal{M}, u_2 \in \mathcal{A}_2, \ldots, u_n \in \mathcal{A}_n$, and $u_1 \in \mathcal{A}_1, \ldots, u_n \in \mathcal{A}_n$, and proceeding inductively to conclude $d = 0$.

Lemma 7: Let $d$ be a $\mathcal{R}, \mathcal{n}, \mathcal{D}$ of $\mathcal{M}$ and $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are nonzero semigroup ideals of $\mathcal{M}$, and $s \in \mathcal{M}, d(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) s = \{0\}$, then $s = 0$.

Proof: For any $u_1 \in \mathcal{A}_1, u_2 \in \mathcal{A}_2, \ldots, u_n \in \mathcal{A}_n$, $d(u_1, u_2, \ldots, u_n) s = 0$, that is $0 = d(u_1, u_2, \ldots, u_n) s = d(u_1, u_2, \ldots, u_n) u_1 s$, it follows $d(u_1, u_2, \ldots, u_n) A_1 = \{0\}$, using Lemma 2(ii) implies $\mathcal{A}_1(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) s = \{0\}$, so $d(u_1, u_2, \ldots, u_n) s = 0$.

As well,
proof: If $(\mathcal{M} , +)$ is abelian, then
d\left( [x, y], x_2, \ldots, x_n \right) = 0

for each \(x, y, x_2, \ldots, x_n \in \mathcal{M}\) and thus \(\mathcal{M}\) is a C.R. according to Lemma 8.

Lemma 10: Let \(A_1, A_2, \ldots, A_n\) be nonzero simigroup ideals of \(\mathcal{M}\) and \(d\) is a \(\mathcal{R}, n, D\) of \(\mathcal{M}\). If there is \(z \in A_1\) such that
\(d(a_1, a_2, \ldots, z, a_n) = 0\)

for any \(a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\), then either \(d = 0\) or \(z \in Z(\mathcal{M})\).

Proof: If \(a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\), then
\(d(a_1, a_2, \ldots, a_n) = 0\)

and this result leads to \(d(z) = 0\), then either \(d = 0\) or \(z \in Z(\mathcal{M})\).

Main Results:

Theorem 1: Let \(d\) be a nonzero \(\mathcal{R}, n, D\) of \(\mathcal{M}\) and
\(A_1, A_2, \ldots, A_n\) be nonzero simigroup ideals of \(\mathcal{M}\), if
\(d(u_1, u_2, \ldots, u_n) = 0\)

for any \(u_1, u_2, \ldots, u_n \in A_1, A_2, \ldots, A_n\), then \(\mathcal{M}\) is a C.R. according to Lemma 8.

Proof: If \(\mathcal{M}\) is a \(\mathcal{R}, n, D\) of \(\mathcal{M}\), then
\(d(u_1, u_2, \ldots, u_n) = 0\)

for any \(u_1, u_2, \ldots, u_n \in A_1, A_2, \ldots, A_n\), then \(\mathcal{M}\) is a C.R. according to Lemma 8.

Theorem 2: Let \(d\) be a nonzero \(\mathcal{R}, n, D\) of \(\mathcal{M}\) and
\(A_1, A_2, \ldots, A_n\) be nonzero simigroup ideals of \(\mathcal{M}\), if
\(d(\mathcal{u}_1, \mathcal{v}_1, \ldots, \mathcal{v}_n) = 0\)

for any \(\mathcal{u}_1, \mathcal{v}_1, \ldots, \mathcal{v}_n \in A_1, A_2, \ldots, A_n\), then \(\mathcal{M}\) is a C.R. according to Lemma 8.

Proof: If \(\mathcal{M}\) is a \(\mathcal{R}, n, D\) of \(\mathcal{M}\), then
\(d(\mathcal{u}_1, \mathcal{v}_1, \ldots, \mathcal{v}_n) = 0\)

for any \(\mathcal{u}_1, \mathcal{v}_1, \ldots, \mathcal{v}_n \in A_1, A_2, \ldots, A_n\), then \(\mathcal{M}\) is a C.R. according to Lemma 8.
\( u_1, v_1 \in A_1 \), \( u_2, v_2 \in A_2 \), ..., \( u_n, v_n \in A_n \),
then \( M \) is a \( C.R. \)

**Proof:** By assumption
\[
d\left( (u_1, v_1), (u_2, v_2), ..., (u_i, v_i), ..., (u_n, v_n) \right) = 0
\]
for any \( u_1, v_1 \in A_1 \), \( u_2, v_2 \in A_2 \), ..., \( u_i, v_i \in A_i \), ..., \( u_n, v_n \in A_n \).

Thus, \[
0 = d\left( (u_1, v_1), (u_2, v_2), ..., s(u_i, v_i), ..., (u_n, v_n) \right)
\]
\[
= d\left( (u_1, v_1), (u_2, v_2), ..., s(u_i, v_i), ..., (u_n, v_n) \right)
\]
\[
= d\left( (u_1, v_1), (u_2, v_2), ..., s, ..., (u_n, v_n) \right) (u_i, v_i)
\]
for any \( u_1, v_1 \in A_1 \), \( u_2, v_2 \in A_2 \), ..., \( u_i, v_i \in A_i \), ..., \( u_n, v_n \in A_n \), \( s \in M \).

If \( m \in M \), then
\[
0 = d\left( (u_1, v_1), (u_2, v_2), ..., s, ..., (u_n, v_n) \right) m(u_i, v_i)
\]
for any \( u_1, v_1 \in A_1 \), \( u_2, v_2 \in A_2 \), ..., \( u_i, v_i \in A_i \), ..., \( u_n, v_n \in A_n \), \( s \in M \).

Three primeness of \( M \) implies either
\[
d\left( (u_1, v_1), (u_2, v_2), ..., s, ..., (u_n, v_n) \right) = 0
\]
or \( (u, v_i) \neq 0 \) for any \( u_1, v_1 \in A_1 \), \( u_2, v_2 \in A_2 \), ..., \( u_i, v_i \in A_i \), ..., \( u_n, v_n \in A_n \), \( s \in M \),
proceeding as above to arrive at \( d = 0 \) (a contradiction) or there is \( i \in \{1, 2, ..., n\} \) s.t \( u_i, v_i = 0 \) which implies \( M \) is \( C.R \).

**Corollary 14:** Let \( d \) be a nonzero \( R. \) \( n. \) \( D. \) of \( M \) if \( d(s_1, s_2, ..., (s_n, t_n)) = 0 \) for any \( s_1, t_1, s_2, t_2, ..., s_n, t_n \in M \), then \( M \) is a \( C.R. \).

**Theorem 3:** Let \( d \) be a \( R. \) \( n. \) \( D. \) of \( M \) and \( A_1, A_2, ..., A_n \) are a nonzero semigroup ideals of \( M \), if \( d(u_1, u_2, ..., (u_i, v_i), ..., u_n) = (u, v_i) \) for any \( u_1 \in A_1 \), \( u_2 \in A_2 \), ..., \( u_i \in A_i \), ..., \( u_n \in A_n \), then \( M \) is a \( C.R. \).

**Proof:** For any \( u_1 \in A_1 \), \( u_2 \in A_2 \), ..., \( u_i \in A_i \), ..., \( u_n \in A_n \),
\[
d(u_1, u_2, ..., (u_i, v_i), ..., u_n) = (u, v_i)
\]
Therefore
\[
d(u_1, u_2, ..., (u_i, v_i), ..., u_n) = (u_i, v_i)
\]
for any \( u_1 \in A_1 \), \( u_2 \in A_2 \), ..., \( u_i \in A_i \), ..., \( u_n \in A_n \), \( s \in M \).

It follows \( s(u_i, v_i) = d(u_1, u_2, ..., s(u_i, v_i), ..., u_n) = d(u_1, u_2, ..., s, ..., u_n) (u, v_i) + (u_i, v_i) \) for any \( u_1 \in A_1 \), \( u_2 \in A_2 \), ..., \( u_i \in A_i \), ..., \( u_n \in A_n \), \( s \in M \).
Corollary 19: Let \( d \) be a nonzero \( R \), \( n \), \( D \) of \( M \), if \( d \left( (s_1, t_1), (s_2, t_2), \ldots, (s_\ell, t_\ell) \right) = (s_\ell, t_\ell) \) for any \( s_1, t_1, s_2, t_2, \ldots, s_\ell, t_\ell \in M \), then \( M \) is a \( C.R. \).

Theorem 5: Let \( d \) be a \( R \), \( n \), \( D \) of \( M \) and \( A_1, A_2, \ldots, A_n \) are nonzero semigroup ideals of \( M \), if \( d + d \) is an \( n \) - additive mapping from \( A_1 \times A_2 \times \ldots \times A_n \) to \( M \), then \( M \) is \( C.R. \).

Proof: From hypothesis: For any \( u_1 \in A_1, u_2 \in A_2, \ldots, u_n \in A_n \)
\[
(d + d)\left( u_1, u_2, \ldots, u_n \right) = (d + d)\left( u_1, u_2, \ldots, u_n \right) + \ldots + (d + d)\left( u_1, u_2, \ldots, u_n \right)
\]
As well,
\[
(d + d)\left( u_1, u_2, \ldots, u_n \right) = d\left( u_1, u_2, \ldots, u_n \right) + \ldots + d\left( u_1, u_2, \ldots, u_n \right)
\]
Comparing the last two expressions to conclude \( d\left( u_1, u_2, \ldots, u_n \right) = 0 \), the required result obtained by Theorem 1.

Corollary 20: Let \( d \) be a nonzero \( R \), \( n \), \( D \) of \( M \) and \( A \) is a nonzero semigroup ideals of \( M \), if \( d + d \) is an \( n \) - additive mapping from \( A \times A \times \ldots \times A \) to \( M \), then \( M \) is \( C.R. \).

Corollary 21: Let \( d \) be a nonzero \( R \), \( n \), \( D \) of \( M \), if \( d + d \) is an \( n \) - additive mapping on \( M \), then \( M \) is \( C.R. \).

Corollary 22: Let \( d \) be a nonzero \( R \), \( D \) of \( M \) and \( A \) is a nonzero semigroup ideals of \( M \), if \( d + d \) is an additive on \( A \) then \( M \) is \( C.R. \).

Corollary 23: Let \( d \) be a nonzero \( R \), \( D \) of \( M \), if \( d + d \) is an additive on \( M \) then \( M \) is \( C.R. \).

Theorem 6: Let \( d_1 \) and \( d_2 \) are two nonzero \( R \), \( n \), \( D \)'s of \( M \) ( \( M \) is two torsion free) and \( A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n \) are nonzero semigroup ideals of \( M \), if \( d_1 (A_1, A_2, \ldots, A_n) d_2 (B_1, B_2, \ldots, B_n) \subseteq Z(M) \), then \( M \) is a \( C.R. \).

Proof: By assumption: for any \( a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n, b_1 \in B_1, b_2 \in B_2, \ldots, b_n \in B_n \)
\[
d_1 (a_1, a_2, \ldots, a_n) d_2 (b_1, b_2, \ldots, b_n) \subseteq Z(M) .
\]
Therefore
\[
d_1 (a_1, a_2, \ldots, a_n) d_2 (b_1, b_2, b_1, b_2, \ldots, b_n) = d_1 (a_1, a_2, \ldots, a_n) d_2 (b_1, b_2, b_1, b_2, \ldots, b_n) b_1 = d_1 (a_1, a_2, \ldots, a_n) d_2 (b_1, b_2, b_1, b_2, \ldots, b_n) (b_1 + b_n) \in Z(M).
\]
Use Lemma 1(ii) to get, for any \( a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n, b_1 \in B_1, b_2 \in B_2, \ldots, b_n \in B_n \)
\[
d_1 (a_1, a_2, \ldots, a_n) d_2 (b_1, b_2, b_1, b_2, \ldots, b_n) (b_1 + b_n) = 0 \text{ or } b_1 + b_n \in Z(M).
\]
From Lemma 7, Eq.4 can be written as
\[
d_2 (b_1, b_2, b_1, b_2, \ldots, b_n) = 0 \text{ or } b_1 + b_n \in Z(M).
\]
If there is \( b \in B \) and
\[
d_2 (b_1, b_2, b_1, b_2, \ldots, b_n) = 0
\]
Hence \( b \in B \), because of Lemma 10.

Again use Lemma 1(ii) to get:
\[
\forall a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n, b_1 \in B_1, b_2 \in B_2, \ldots, b_n \in B_n, \exists M.
\]
If
\[
d_1 (a_1, a_2, \ldots, a_n) d_2 (b_1, b_2, \ldots, b_n) \subseteq Z(M) \text{ or } b = 0.
\]
Then
\[
d_1 (a_1, a_2, \ldots, a_n) d_2 (b_1, b_2, \ldots, b_n) = d_1 (a_1, a_2, \ldots, a_n) d_2 (b_1, b_2, \ldots, b_n) (b + b) \in Z(M).
\]
For the same reason as above, the following result can be satisfied
\[
d_1 (a_1, a_2, \ldots, a_n) d_2 (b_1, b_2, \ldots, b_n) = 0 \text{ or } (b + b) \in Z(M).
\]
Because of Lemma 7, the last result can be reduce to \( b + b \in Z(M) \) and this result, Eq.5 becomes \( b_1 + b_n \subseteq Z(M) \) for every \( b_1 \in B_1 \), it follows \( s b_1 + s b_2 \subseteq s (b_1 + b_2) \subseteq Z(M) \) for every \( b_1 \in B_1, s \in M \). So \( b_1 + b_n = 0 \text{ or } s \subseteq Z(M) \) for any \( s \in M \), so this torsion freeness and Lemma 2(iii), leads to the required result.

Corollary 24: Let \( d_1 \) and \( d_2 \) are two nonzero \( R \), \( n \), \( D \)'s of \( M \) ( \( M \) is two torsion free) and \( A \), \( B \) are nonzero semigroup ideals of \( M \), if \( d_1 (A, A, \ldots, A) d_2 (B, B, \ldots, B) \subseteq Z(M) \), then \( M \) is \( C.R. \).
Corollary 25: Let $d_1$ and $d_2$ are two nonzero $R, n, D$'s of $M$ ($M$ is two torsion free) and $A$ is a nonzero semigroup ideals of $M$, if $d_1(A, A, \ldots, A) d_2(A, A, \ldots, A) \subseteq Z(M)$, then $M$ is $C.R.$

Corollary 26: Let $d_1$ and $d_2$ are two nonzero $R, n, D$'s of $M$ ($M$ is two torsion free), $A$ and $B$ are nonzero semigroup ideals of $M$, if $d_1(A) d_2(B) \subseteq Z(M)$, then $M$ is $C.R.$

Corollary 27: Let $d_1$ and $d_2$ are two nonzero $R, n, D$'s of $M$ ($M$ is two torsion free), $A$ is a nonzero semigroup ideals of $M$, if $d_1(A) d_2(A) \subseteq Z(M)$, then $M$ is $C.R.$

Corollary 28: Let $d_1$ and $d_2$ are two nonzero $R, n, D$'s of $M$ ($M$ is two torsion free), if $d_1(M, M, \ldots, M) d_2(M, M, \ldots, M) \subseteq Z(M)$, then $M$ is $C.R.$

Corollary 29: Let $d_1$ and $d_2$ are two $R, D$'s of $M$ ($M$ is two torsion free), if $d_1(M) d_2(M) \subseteq Z(M)$, then $M$ is $C.R.$

Theorem 7: Let $d$ be a nonzero $R, n, D$ of $M$, where $M$ is two torsion free, $A_1, A_2, \ldots, A_n$ are nonzero semigroup ideals of $M$, if $d((u_1, u_2), \ldots, (u_j, v_j), \ldots, u_n) \in Z(M)$ for any $u_i, v_i \in A_i$, $i \neq j$, then $M$ is $C.R.$

Proof: By assumption

$$d(u_1, u_2, \ldots, (u_j, v_j), \ldots, u_n) \in Z(M)$$

for any $u_i, v_i \in A_i$, $i \neq j$, $u_i, v_i \in A_j$, $i \neq j$, $u_i, v_i \in A_n$, then $M$ is $C.R.$

Therefore,

$$d(u_1, u_2, \ldots, (u_j, v_j), \ldots, u_n) = d(u_1, u_2, \ldots, (u_j, v_j), \ldots, u_n) + d(u_1, u_2, \ldots, (u_j, v_j), \ldots, u_n)$$

for any $u_i, v_i \in A_i$, $i \neq j$, $u_i, v_i \in A_j$, $i \neq j$, $u_i, v_i \in A_n$, $i \neq j$. Replace $s$ by $(u_j, v_j)$ in last equation to get

$$d(u_1, u_2, \ldots, (u_j, v_j), \ldots, u_n) \in Z(M)$$

for any $u_i, v_i \in A_i$, $i \neq j$, $u_i, v_i \in A_j$, $i \neq j$, $u_i, v_i \in A_n$, $i \neq j$. Using Lemma 1(ii) implies

$$d(u_1, u_2, \ldots, (u_j, v_j), \ldots, u_n) = 0 \text{ or } \left(2(u_j, v_j)\right) \in Z(M)$$

for any $u_i, v_i \in A_i$, $i \neq j$, $u_i, v_i \in A_j$, $i \neq j$, $u_i, v_i \in A_n$. If there is $u_0, v_0 \in A_j$ such that

$$d(u_1, u_2, \ldots, (u_0, v_0), \ldots, u_n) = 0 \text{ for any } u_i, v_i \in A_i, \text{ } i \neq j, u_i, v_i \in A_n$$

according to Lemma 6.

Return to the hypothesis: for any $u_i \in A_1, u_2 \in A_2, \ldots, u_n \in A_n, s \in M$.

$$d(u_1, u_2, \ldots, s(u_0, s), \ldots, u_n) = d(u_1, u_2, \ldots, s(u_0, v_0), \ldots, u_n) = d(u_1, u_2, \ldots, s(u_1, u_2, \ldots, s(u_0, v_0), \ldots, u_n) \in Z(M)$$

Using Lemma 1(ii) in last result forces

$$d(u_1, u_2, \ldots, s, \ldots, u_n) \in Z(M) \text{ or } (u_0, v_0) = 0, \text{ if } d(u_1, u_2, \ldots, s, \ldots, u_n) = \in Z(M), \text{ replace } s \text{ by } 2s(u_0, v_0), \text{ last expression can be written as:}$$

$$d(u_1, u_2, \ldots, s, \ldots, u_n)(2(u_0, v_0)) \in Z(M) \text{ or } (u_0, v_0) = 0, \text{ which conclude that}$$

$$d(u_1, u_2, \ldots, s, \ldots, u_n)(2(u_0, v_0)) \in Z(M), \text{ thus}$$

$$2(u_0, v_0) \in Z(M) \text{ according to Lemma 1(ii) and Lemma 6.}$$

Therefore, Eq.8 becomes $2(u_j, v_j) \in Z(M)$ for any $u_j, v_j \in A_j$, it follows $2(s(u_j, v_j)) = s(2(u_j, v_j)) \in Z(M)$ for any $u_j, v_j \in A_j$, s.e. M, Lemma 1(i) and two torsion freeness ensures that $(u_j, v_j) = 0$ for any $u_j, v_j \in A_j$, or $s \in Z(M)$ for any $s \in M$. Therefore $M$ is $C.R$ by Lemma 4, Lemma 9 and Lemma 2(ii).

Corollary 30: Let $d$ be a nonzero $R, n, D$ of $M$, where $M$ is two torsion free, $A$ is a nonzero semigroup ideals of $M$, if $d((u_1, u_2, \ldots, (u_j, v_j), \ldots, u_n) \in Z(M)$ for any $u_i, v_i \in A_i$, $i \neq j$, $u_i, v_i \in A_n$, then $M$ is $C.R.$

Corollary 31: Let $d$ be a nonzero $R, D$ of $M$, where $M$ is two torsion free, $A$ is a nonzero semigroup ideals of $M$, if $d((u, v)) \in Z(M)$ for any $u, v \in A$, then $M$ is $C.R.$

Corollary 32: Let $d$ be a nonzero $R, n, D$ of $M$, where $M$ is two torsion free, if $d((s, t)) \in Z(M)$ for any $s, t \in M$, then $M$ is $C.R.$

Theorem 8: Let $d$ be a nonzero $R, n, D$ of $M$, where $M$ is two torsion free, $A_1, A_2, \ldots, A_n$ are nonzero semigroup ideals of $M$, if $d((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)) \in Z(M)$ for any $u_i, v_i \in A_i, \text{ } i \neq j, u_i, v_i \in A_n, \text{ then } M$ is $C.R.$

Proof: By assumption: for any $u_1, v_1 \in A_1$, $u_2, v_2 \in A_2, \ldots, u_n, v_n \in A_n$,

$$d((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)) = d(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)) \in Z(M) \text{ or } (u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)) \in Z(M) \text{ or } (u_0, v_0) = 0, \text{ which conclude that}$$

$$d((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)) (u_0, v_0) \in Z(M), \text{ thus}$$

$$2(u_0, v_0) \in Z(M) \text{ according to Lemma 1(ii) and Lemma 6.}$$

Therefore, Eq.8 becomes $2(u_j, v_j) \in Z(M)$ for any $u_j, v_j \in A_j$, it follows $2(s(u_j, v_j)) = s(2(u_j, v_j)) \in Z(M)$ for any $u_j, v_j \in A_j$, s.e. M, Lemma 1(i) and two torsion freeness ensures that $(u_j, v_j) = 0$ for any $u_j, v_j \in A_j$, or $s \in Z(M)$ for any $s \in M$. Therefore $M$ is $C.R$ by Lemma 4, Lemma 9 and Lemma 2(ii).
... \in A_u, s \in M.
Replace s by (u_i, v_j) in last equation to get:
\[
d( (u_1, v_1), (u_2, v_2), ..., (u_i, v_j), ..., (u_n, v_n) )
\]
\[
(2(u_i, v_j)) \in Z(M)
\]
For any \( u_1, v_1 \in A_1, u_2, v_2 \in A_2, ..., u_i, v_j \in A_j, ..., u_n, v_n \in A_u \)
Using Lemma 1(ii) implies
\[
d( (u_1, v_1), (u_2, v_2), ..., (u_i, v_j), ..., (u_n, v_n) ) =
\]
0 or \( (2(u_i, v_j)) \in Z(M) \)
for any
\[
( (u_1, v_1), (u_2, v_2), ..., u_i, v_j \in A_j, ..., u_n, v_n \in A_u )
\]
If there is \( u_0, v_0 \in A_j \) such that
\[
d( (u_1, v_1), (u_2, v_2), ..., (u_i, v_j), ..., (u_n, v_n) ) =
\]
0 for any \( u_1, v_1 \in A_1, u_2, v_2 \in A_2, ..., u_n, v_n \in A_u \).
Return to the hypothesis: for any
\[
( (u_1, v_1), (u_2, v_2), ..., u_i, v_j \in A_j, ..., u_n, v_n \in A_u, s \in M ).
\]
\[
d( (u_1, v_1), (u_2, v_2), ..., (u_i, v_j), ..., (u_n, v_n) ) =
\]
\[
d( (u_1, v_1), (u_2, v_2), ..., s( (u_0, v_0) ) ) = \cdots (u_n, v_n) ) \in Z(M)
\]
11
Replace s by \( (u_i, v_j) \) where \( u_i, v_j \in A_j \) and use Lemma 1(ii) in Eq.11 to get:
Either
\[
d( (u_1, v_1), (u_2, v_2), ..., (u_i, v_j), ..., (u_n, v_n) ) = 0 \text{ or } (u_0, v_0) \in Z(M)
\]
If \( d( (u_1, v_1), (u_2, v_2), ..., (u_i, v_j), ..., (u_n, v_n) ) = 0 \) for
\[
( (u_1, v_1), (u_2, v_2), ..., u_i, v_j \in A_j, ..., u_n, v_n \in A_u, then M is a C.R because of Theorem 2.
\]
If \( (u_0, v_0) \in Z(M) \), then using Eq.11 leads to
\[
d( (u_1, v_1), (u_2, v_2), ..., (u_i, v_j), ..., (u_n, v_n) ) \in Z(M)
\]
for any \( u_1, v_1 \in A_1, u_2, v_2 \in A_2, ..., u_n, v_n \in A_u, s \in M \)
or \( (u_0, v_0) = 0 \).
If \( d( (u_1, v_1), (u_2, v_2), ..., s, ..., (u_n, v_n) ) \in Z(M) \)
Replace s by 2s \( (u_0, v_0) \), the last expression can be written as:
\[
d( (u_1, v_1), (u_2, v_2), ..., s, ..., (u_n, v_n)) (2 (u_0, v_0) ) \in Z(M)
\]
Use Lemma 1(ii) to get
\[
d( (u_1, v_1), (u_2, v_2), ..., s, ..., (u_n, v_n)) = 0
\]
for any \( u_1, v_1 \in A_1, u_2, v_2 \in A_2, ..., u_n, v_n \in A_u, s \in M \)
or \( 2 (u_0, v_0) \in Z(M). \)
Therefore, Eq.9 becomes
\[
d( (u_1, v_1), (u_2, v_2), ..., s, ..., (u_n, v_n) = 0 \text{ for any } u_1, v_1 \in A_1, u_2, v_2 \in A_2, ..., u_n, v_n \in A_u, s \in M \)
or \( 2 (u_0, v_0) \in Z(M) \).
Lemma 1(i) and two torsion freeness ensures that \( (u_0, v_0) = 0 \) for any \( u_i, v_j \in A_j \), hence
\[
d( (u_1, v_1), (u_2, v_2), ..., s( (u_0, v_0) ) \in Z(M) \)
for any \( u_1, v_1 \in A_j, s \in M \).
Therefore \( M \) is C.R by Lemma 4, Lemma 9 and Lemma 2(iii).

Corollary 34: Let \( d \) be a nonzero \( R_n \in D \) of \( M \), where \( M \) is two torsion free, if
\[
d((s_1 t_1), (s_2 t_2), ..., (s_n t_n) ) \in Z(M)
\]
for any \( s_1, t_1, s_2, t_2, ..., s_n, t_n \in M \), then \( M \) is a C.R.

Conclusion:
By using the semigroup ideals with right \( n \)-derivations involving some algebraic identities, this work gives very attractive results about the commutativity of the near ring.

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Authors’ Declaration:
- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Al-Qadisyah.

References:
على مثاليات شبه الزمرة والاشتقاقات \( n \)-اليمنية على الحلقات المقترحة

نظام فرحان

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الخلاصات:

في الورقة حالياً تم دراسة مفهوم الاشتقاقات اليمنية - والتي تحمل شروط معينة على مثاليات شبه الزمرة على الحلقات المقترحة وتم مناقشة بعض الخصائص المرتبطة بها. وقد تم التوصل إلى نتائج مثيرة للاهتمام، من أبرزها ما يلي: لكن مثالية يمينية بيمارضية ثانوية أولية وكل مثالية يمينية على شبه الزمرة مثالية من \( M \) إذا كان \( d \) اشتقاق \( n \)-يمني على \( A_1, A_2, \ldots, A_n \) من شبه الزمرة مثالية من \( M \) بحيث: 

\[
\begin{align*}
(\text{i}) & \quad d(u_1, u_2, \ldots, u_n, v_1, \ldots, v_n, u_1) = 0 \quad \forall\ u_1 \in A_1, u_2 \in A_2, \ldots, u_n \in A_n; \\
(\text{ii}) & \quad d((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)) = 0 \\
& \quad \forall\ u_1, v_1 \in A_1, u_2 \in A_2, \ldots, u_n, v_n \in A_n; \\
(\text{iii}) & \quad d((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)) = 0 \quad \forall\ u_1, v_1 \in A_1, \\
& \quad u_2 \in A_2, \ldots, u_n, v_n \in A_n; \\
(\text{iv}) & \quad d + d = n \text{-additive mapping from } A_1 \times A_2 \times \ldots \times A_n \\
& \quad \text{to } M; \\
(\text{v}) & \quad d(u_1, u_2, \ldots, (u_1, v_1), \ldots, u_n) \in Z(M) \quad \forall\ u_1 \in A_1, u_2 \in A_2, \ldots, u_n \in A_n; \\
(\text{vi}) & \quad d((u_1, u_2, \ldots, (u_1, v_1), \ldots, u_n, v_1, \ldots, u_n) \in Z(M) \quad \forall u_1, v_1 \in A_1, \\
& \quad u_2, v_2 \in A_2, \ldots, u_n, v_n \in A_n.
\end{align*}
\]

للمثلية \( M \) فإن حلقة ابتدائية.

الكلمات المفتاحية: تعميم الاشتقاقات اليمنية، الحلقات المقترحة الأولية، الاشتقاقات اليمنية، الاشتقاقات، شبه الزمرة، مثاليات شبه الزمرة.