

A Dislocated Quasi-Normed Space and Its Completeness for A Fixed Point Theorem

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Abstract

A concept of a dislocated quasi-normed space is introduced in this paper which is a spatial case of the concept of a quasi-normed space and related to the notion of a dislocated quasi-metric space. Its relationship with other concepts is given by illustrative examples. The completeness of it defines a dislocated quasi-Banach space that is important to our results with respect to a fixed-point theorem. Also, a quasi-contraction mapping is introduced, in which, Lipchitz constant depends on a constant of dislocated quasi-normed space, and then a Banach contraction principle theorem in a dislocated quasi-Banach space is studied with some important results.

Keywords: Dislocated quasi-Banach space, Dislocated quasi-metric space, Dislocated quasi-normed space, Fixed point theorem, Quasi-contraction mapping.

Introduction

Quasi-metric space (\mathcal{T}, d) is a nonempty set \mathcal{T} and a quasi-metric $d: \mathcal{T} \times \mathcal{T} \rightarrow [0, \infty)$ which differs from a metric function by the inequality: for all $u, v, w \in \mathcal{T}$ and $\beta \in [1, \infty)$, $d(u, v) \leq \beta(d(u, w) + d(w, v))$. A function d is a metric if $\beta = 1$. Also, Quasi-normed space $(\mathcal{T}, {}_q\|\cdot\|)$ is a real vector space with a quasi-norm ${}_q\|\cdot\|: \mathcal{T} \rightarrow [0, +\infty)$, which is a positive definite, absolutely homogeneous functional such that there is a constant $\beta \geq 1$, ${}_q\|v + w\| \leq \beta({}_q\|v\| + {}_q\|w\|)$, for all $v, w \in \mathcal{T}$. When $\beta = 1$; ${}_q\|\cdot\|$ be a norm function, and it is a semi-norm if it does not need to be positive or definite. A quasi-normed space is metrizable, thus the concept of completeness is correct, and it is called a quasi-Banach space¹⁻³.

Many authors have studied the concepts of dislocated metric space and dislocated quasi-metric

space with different definitions. Conditions of non-degeneracy and the symmetry of a metric function and a quasi-metric function are important to define these concepts⁴⁻⁷.

Fixed point theorems have been applied in many scientific fields such as physics, computer science, and others and have been studied in the above concepts. One of the most important theorems in mathematics which is Banach contraction principle theorem is found in many references⁸⁻¹¹.

This Paper introduces the notion of dislocated quasi-normed space and its completeness as new definitions with many illustrative examples. A Banach contraction principle theorem is studied and is proved with some results in dislocated quasi-Banach spaces.

Results and Discussion

Dislocated Quasi-Banach Spaces

Definition 1⁷: Let \mathcal{T} is a nonempty set and a function $d_q: \mathcal{T} \times \mathcal{T} \rightarrow [0, +\infty)$ does satisfy the conditions:

- 1) $d_q(u, v) = d_q(v, u) = 0$ implies $u = v$
- 2) $d_q(u, v) = d_q(v, u)$
 $d_q(u, v) \leq \beta (d_q(u, w) + d_q(w, v))$, for all $u, v, w \in \mathcal{T}$, where a constant $\beta \geq 1$

Then d_q it is called a dislocated metric and it is called a dislocated quasi-metric if satisfies only (1) and (3). (\mathcal{T}, d_q) is said to be a dislocated quasi-metric space. For simplicity, a dislocated quasi-metric space and other concepts are denoted by \mathcal{T} , and the examples show the relationship between these concepts.

Example 1⁶: Consider $\mathcal{T} = [0, +\infty)$, define a function $d_q: \mathcal{T} \times \mathcal{T} \rightarrow [0, +\infty)$ by $d_q(u, v) = \max\{u, v\}$ then \mathcal{T} is dislocated metric space, but is not metric since it is possible $d_q(u, v) = 0$ when $u \neq v$

Example 2⁶: Let $\mathcal{T} = \mathbb{R}^n$, and define the function $d_q: \mathcal{T} \times \mathcal{T} \rightarrow [0, +\infty)$ as: $d_q(u, v) = \sum_{i=1}^n |v_i - u_i| + |v_i|$, for all $u, v \in \mathcal{T}$, where $u = (u_1, u_2, u_3, \dots, u_n)$ and $v = (v_1, v_2, v_3, \dots, v_n)$. Clearly, \mathcal{T} is a dislocated quasi-metric space. Since condition (2) is failed then it is not dislocated metric
 Now, new definitions with new examples and results about them are given

Definition 2: A dislocated quasi-normed space is a vector space \mathcal{T} over \mathbb{R} with a function $d_q \|\cdot\|: \mathcal{T} \rightarrow [0, +\infty)$ which satisfies the properties: $\forall u, v \in \mathcal{T}$,

- (1) $d_q \|0\| = 0$
- (2) $d_q \|\alpha u\| = \alpha d_q \|u\|$, for any real number $\alpha \geq 0$.
- (3) There is a constant $\beta \geq 1$, $d_q \|u + v\| \leq \beta (d_q \|u\| + d_q \|v\|)$.

Example 3: Consider \mathcal{T} is a vector space consisting of the set of all bounded sequences of complex numbers over \mathbb{R} Let a function $d_q \|u\| = \sup_k |Re u_k|$, for all $u = \{u_k\} \in \mathcal{T}$, then \mathcal{T} is a dislocated quasi-normed space, indeed,

condition (1) is clear. For $\alpha \geq 0$,
 $d_q \|\alpha u\| = \sup_k |Re \alpha u_k| = \alpha \sup_k |Re u_k| = \alpha d_q \|u\|$. $\forall u, v \in \mathcal{T}$, and $d_q \|u + v\| = \sup_k |Re(u_k + v_k)| \leq \beta (\sup_k |Re u_k| + \sup_k |Re v_k|) = \beta (d_q \|u\| + d_q \|v\|)$, $\beta \geq 1$

It is clear that, a dislocated quasi-normed space is dislocated quasi-metric and a semi-normed space is dislocated metric.

Example 4: Let $\mathcal{T} = \mathbb{R}^n$, define a function $d_q: \mathcal{T} \times \mathcal{T} \rightarrow [0, +\infty)$ as $d_q(u, v) = \sum_{i=1}^n |u_i| + v_i^2$, for all $u, v \in \mathcal{T}$, then it is a quasi-dislocated metric space, since conditions (2) and (4) are satisfied in Definition 1, \mathcal{T} is not dislocated quasi-normed, because if there is a dislocated quasi-norm $d_q \|\cdot\|$ such that $d_q(u, v) = d_q \|u - v\|$ then $d_q(\alpha u, \alpha v) = d_q \|\alpha u - \alpha v\| = \alpha d_q \|u - v\| = \alpha d_q(u, v)$, $\alpha \geq 0$, must be satisfied, but this relation is not valid as: $d_q(\alpha u, \alpha v) = \alpha (\sum_{i=1}^n |u_i| + \alpha v_i^2)$ and $\alpha d_q(u, v) = \alpha (\sum_{i=1}^n |u_i| + v_i^2)$.

Example 5: Let \mathcal{T} be set of real numbers, and $d_q(u, v) = \max\{|u|, |v|^2\}$, then obviously, \mathcal{T} is a dislocated metric space. Similar to Example 4 with a function $d_q \|\cdot\|$ such that $d_q(u, v) = d_q \|u - v\|$ then $d_q(\alpha u, \alpha v) = d_q \|\alpha u - \alpha v\| = |\alpha| d_q(u, v)$, while: $d_q(\alpha u, \alpha v) = \max\{|\alpha u|, |\alpha v|^2\}$, and $|\alpha| d_q(u, v) = |\alpha| \max\{|u|, |v|^2\}$, so \mathcal{T} is not semi-normed.

Definition 3: A sequence $\{u_n\}$ in a dislocated quasi-normed space \mathcal{T} is called:

1. Convergent sequence if there exist $u \in \mathcal{T}$ and a positive integer ρ such that $\lim_{n \rightarrow \infty} d_q \|u_n - u\| = 0 = \lim_{n \rightarrow \infty} d_q \|u - u_n\|$, for all $n > \rho$.
2. Cauchy sequence if there exist a positive integer ρ such that $\lim_{n, m \rightarrow \infty} d_q \|u_n - u_m\| = 0$, $\forall m, n \geq \rho$.

Proposition 1: A limit point of a convergent sequence in a dislocated quasi-normed space is unique.

Proof: Let $\{u_n\}$ is a convergent sequence in \mathcal{T} such that u and v are limit points of it then $\lim_{n \rightarrow \infty} d_q \|u_n - u\| = 0$ and $\lim_{n \rightarrow \infty} d_q \|u_n - v\| = 0$. Since,

$dq\|u - v\| = \left\| \begin{matrix} u - u_n \\ +u_n - v \end{matrix} \right\| \leq \beta \left(\begin{matrix} dq\|u - u_n\| \\ + dq\|u_n - v\| \end{matrix} \right)$
 , where $\beta \geq 1$. By taking limit both sided as $n \rightarrow \infty$,
 $\lim_{n \rightarrow \infty} dq\|u - v\| = 0$, which implies that $u = v$.

Proposition 2: Every convergent sequence in a dislocated quasi-normed space is Cauchy, conversely is not true in general.

Proof: Let $\{u_n\}$ is a convergent sequence to u in \mathcal{T} then $\lim_{n \rightarrow \infty} dq\|u_n - u\| = 0 = \lim_{n \rightarrow \infty} dq\|u - u_n\|$. For $n, m \geq \rho$, $dq\|u_n - u_m\| =$

$dq\left\| \begin{matrix} u_n - u \\ -u_m + u \end{matrix} \right\| \leq \beta (dq\|u_n - u\| + dq\|u - u_m\|)$
 , where $\beta \geq 1$. So $dq\|u_n - u_m\| \rightarrow 0$ as $n, m \rightarrow \infty$.
 Hence $\{u_n\}$ is a Cauchy sequence .

Example 6 explains converse Proposition 2.

Definition 4: A dislocated quasi-normed space, in every Cauchy sequence is convergent, called a dislocated quasi-Banach space

Example 6: Let \mathcal{T} is a vector space consisting of the set of all continues complex-valued functions defined on $[0,1]$ with $dq\|f\| = \sup_{u \in [0,1]} |Re f(u)|$

, then, similarly to Example 3, \mathcal{T} is a dislocated quasi-normed space, but it is not complete, indeed,

let a sequence $\{f_n(u)\} \in \mathcal{T}$ such that $f_n(u) = \begin{cases} n, & \text{if } 0 \leq u \leq \frac{1}{n} \\ \frac{1}{u}, & \text{if } \frac{1}{n} \leq u \leq 1 \end{cases}$

To prove $\{f_n(u)\}$ a Cauchy sequence, let $n > m$, then

$$\begin{aligned} dq\|f_n(u) - f_m(u)\| &= \sup_{u \in [0,1]} |Re (f_n(u) - f_m(u))| \\ &= \max\{ \sup_{u \in [0, \frac{1}{n}]} |Re (n - m)|, \sup_{u \in [\frac{1}{n}, \frac{1}{m}]} |Re (\frac{1}{u} - m)|, 0\} \end{aligned}$$

Since in $[\frac{1}{n}, \frac{1}{m}]$, $|Re (n - m)| \leq |Re (\frac{1}{u} - m)|$ then

$$\begin{aligned} dq\|f_n(u) - f_m(u)\| &= \max\left\{ \frac{1}{n} |Re (n - m)|, \frac{1}{m} |Re (\frac{1}{u} - m)| \right\} \end{aligned}$$

As $n, m \rightarrow \infty$, $dq\|f_n(u) - f_m(u)\| \rightarrow 0$. then $\{f_n(u)\}$ is a Cauchy sequence.

But it is not convergent to $f(0)$, since, there is no continuous function at $u = 0$, that is $f \notin \mathcal{T}$.

Lemma 1: A subspace Ω of a dislocated quasi-Banach space itself a dislocated quasi-Banach space if and only if it is closed in \mathcal{T} .

Proof: The proof of lemma is very technical and can be found in ²with minor different.

Some Results of Fixed-Point Theorem in Dislocated Quasi-Banach Space

Definition 5: Let \mathcal{T} is a dislocated quasi-normed space, $J: \mathcal{T} \rightarrow \mathcal{T}$ is called a quasi-contraction mapping if for some $\beta \geq 1$ there exists a Lipchitz constant $\mu(\beta) \in [0,1)$, such that

$$dq\|Ju - Jv\| \leq \mu(\beta) dq\|u - v\|, \forall u, v \in \mathcal{T} \quad \text{----- } 1$$

Remark 1: In order to Eq. 1 satisfies, chose $\mu(\beta)$ a suitable belongs to interval $[0,1)$ such as $\mu(\beta) = \frac{\beta-1}{\beta^r}$ or $\mu(\beta) = \frac{\beta^r-1}{\beta^r}$, r is any positive integer, etc.

Lemma 2: Suppose \mathcal{T} is a dislocated quasi-normed space, a quasi-contraction mapping $J: \mathcal{T} \rightarrow \mathcal{T}$ is continuous.

Proof: Let $\varepsilon > 0$ be given. In Eq. 1, if $\mu(\beta) = 0$, the proof is trivial, since $dq\|Ju - Jv\| = 0 < \varepsilon$. If $\mu(\beta) \in (0,1)$, then for all $u, v \in \mathcal{T}$ with $dq\|u - v\| < \frac{\varepsilon}{\mu(\beta)} = \xi$, $dq\|Ju - Jv\| \leq \mu(\beta) dq\|u - v\| < \varepsilon$. Therefore, J is continuous.

Lemma 3: Suppose \mathcal{T} is a dislocated quasi-Banach space and $J: \mathcal{T} \rightarrow \mathcal{T}$ is a quasi-contraction mapping. If $\{\omega_n\}$ be a sequence in \mathcal{T} defined by $\omega_n = J\omega_{n-1}$, then $\{\omega_n\}$ is a Cauchy sequence and is convergent to some $\omega \in \mathcal{T}$ as $n \rightarrow \infty$.

Proof: For any integer $m > 0$, condition (3) implies that

$$\begin{aligned} dq\|\omega_n - \omega_{n+m}\| &\leq \beta (dq\|\omega_n - \omega_{n+1}\| + \\ &dq\|\omega_{n+1} - \omega_{n+2}\| + \dots + dq\|\omega_{n+m-1} - \omega_{n+m}\|) \\ &\leq \beta dq\|\omega_n - \omega_{n+1}\| + \beta^2 [dq\|\omega_{n+1} - \omega_{n+2}\| + \\ &dq\|\omega_{n+2} - \omega_{n+3}\| + \dots + dq\|\omega_{n+m-1} - \omega_{n+m}\|] \\ &\leq \beta dq\|\omega_n - \omega_{n+1}\| + \beta^2 dq\|\omega_{n+1} - \omega_{n+2}\| + \end{aligned}$$

$$\beta^3 [{}_{dq}\|\omega_{n+2} - \omega_{n+3}\| + \dots + {}_{dq}\|\omega_{n+m-1} - \omega_{n+m}\|]$$

Also, since J is a quasi-contraction mapping, then using Eq. 1,

$$\begin{aligned} & {}_{dq}\|\omega_n - \omega_{n+1}\| \leq \\ & \mu(\beta) {}_{dq}\|\omega_{n-1} - \omega_n\| \leq \mu(\beta)^2 {}_{dq}\|\omega_{n-2} - \omega_{n-1}\| \\ & \dots \leq \mu(\beta)^n {}_{dq}\|\omega_0 - \omega_1\|. \end{aligned}$$

Hence,

$$\begin{aligned} & {}_{dq}\|\omega_n - \omega_{n+m}\| \leq \beta \mu(\beta)^n {}_{dq}\|\omega_n - \omega_{n+1}\| + \\ & \beta^2 \mu(\beta)^{n+1} {}_{dq}\|\omega_{n+1} - \omega_{n+2}\| + \beta^3 \mu(\beta)^{n+2} \\ & {}_{dq}\|\omega_{n+2} - \omega_{n+3}\| + \dots \\ & + (\beta \mu(\beta))^{n+m-1} {}_{dq}\|\omega_{n+m-1} - \omega_{n+m}\| \leq (1 + \\ & \beta \mu(\beta) + (\beta \mu(\beta))^2 + \dots + \\ & (\beta \mu(\beta))^{n+m-1}) (\beta \mu(\beta))^n {}_{dq}\|\omega_0 - \omega_1\| = \\ & (\beta \mu(\beta))^n \left(\frac{1 - (\beta \mu(\beta))^{n+m}}{1 - \beta \mu(\beta)} \right) {}_{dq}\|\omega_0 - \omega_1\|. \end{aligned}$$

$$\text{Then, } {}_{dq}\|\omega_n - \omega_{n+m}\| \leq (\beta \mu(\beta))^n \left(\frac{1 - (\beta \mu(\beta))^{n+m}}{1 - \beta \mu(\beta)} \right) {}_{dq}\|\omega_0 - \omega_1\|.$$

As $n, n + m \rightarrow \infty$, ${}_{dq}\|\omega_n - \omega_{n+m}\| = 0$, (because, $\beta \mu(\beta) < 1$). Then $\{\omega_n\}$ is a Cauchy sequence in \mathbb{T} . Since \mathbb{T} is complete, $\{\omega_n\}$ converges to some $\omega \in \mathbb{T}$ as $n \rightarrow \infty$.

Definition 6: Let J be a continuous mapping from a dislocated quasi-Banach space \mathbb{T} into itself then an element u^* in J is said to be a fixed point of J if $Ju^* = u^*$.

Theorem 1 (Dislocated Quasi-Banach Fixed Point Theorem): Let $J : \mathbb{T} \rightarrow \mathbb{T}$ be a quasi-contraction mapping, where \mathbb{T} is a dislocated quasi-Banach space and $\{u_n\}$ is any sequence defined by $u_n = J(u_{n-1}), n \in \mathbb{N}$, then J has a unique fixed point which is a limit point of $\{u_n\}$.

Proof: Let u_0 be arbitrary point in \mathbb{T} . From the definition of a sequence $\{u_n\}$, $u_n = J^n(u_0), n \in \mathbb{N}$.

If $p = 1, 2, 3, \dots$, ${}_{dq}\|u_{n+p} - u_n\| = {}_{dq}\|J^{n+p}(u_0) - J^n(u_0)\| \leq \mu(\beta) {}_{dq}\|J^{n+p-1}(u_0) - J^{n-1}(u_0)\|$, because J is a quasi-contraction mapping continuing this process $n - 1$ times,

$$\begin{aligned} & {}_{dq}\|u_{n+p} - u_n\| \leq (\mu(\beta))^n {}_{dq}\|J^p(u_0) - u_0\| \text{ --2} \\ \text{But,} & \\ & {}_{dq}\|J^p(u_0) - u_0\| = \end{aligned}$$

$$\begin{aligned} & {}_{dq}\|J^p(u_0) - J^{p-1}(u_0) + J^{p-1}(u_0) - \\ & {}_{dq}\|J^{p-2}(u_0) + J^{p-2}(u_0) + \dots + Ju_0 - u_0\| \\ & \leq \\ & \beta ({}_{dq}\|J^p(u_0) - J^{p-1}(u_0)\| + \dots + {}_{dq}\|J(u_0) - u_0\|) \end{aligned}$$

Since $J^p(u_0) = J^{p-1}(u_1)$, $J^{p-1}(u_0) = J^{p-2}(u_1), \dots, Ju_0 = u_1$,

$$\begin{aligned} & {}_{dq}\|J^p(u_0) - u_0\| \leq \\ & \beta ({}_{dq}\|J^{p-1}(u_1) - J^{p-1}(u_0)\| \\ & + {}_{dq}\|J^{p-2}(u_1) - J^{p-2}(u_0)\| + \dots + {}_{dq}\|u_1 - u_0\|) \end{aligned}$$

By Eq. 2, ${}_{dq}\|u_{n+p} - u_n\| \leq (\beta \mu(\beta))^n [(\beta \mu(\beta))^{p-1} {}_{dq}\|u_1 - u_0\| + (\beta \mu(\beta))^{p-2} {}_{dq}\|u_1 - u_0\| + \dots + {}_{dq}\|u_1 - u_0\|] \leq (\beta \mu(\beta))^n {}_{dq}\|u_1 - u_0\| [1 + \beta \mu(\beta) + (\beta \mu(\beta))^2 + \dots + (\beta \mu(\beta))^{n+p-1}] = (\beta \mu(\beta))^n {}_{dq}\|u_1 - u_0\| \frac{1 - (\beta \mu(\beta))^{n+p}}{1 - \beta \mu(\beta)}$, where $0 \leq \beta \mu(\beta) < 1$. Then,

$$\begin{aligned} & {}_{dq}\|u_{n+p} - u_n\| \leq \\ & (\beta \mu(\beta))^n {}_{dq}\|u_1 - u_0\| \frac{1 - (\beta \mu(\beta))^{n+p}}{1 - \beta \mu(\beta)} \text{ ----- 3} \end{aligned}$$

As $n, n + p \rightarrow \infty$ in Eq. 3, ${}_{dq}\|u_{n+p} - u_n\| \rightarrow 0$, then $\{u_n\}$ is a Cauchy sequence in \mathbb{T} , hence, by Lemma 3, $\{u_n\}$ must be convergent, say $\lim_{n \rightarrow \infty} u_n = u^*$. Since J is a continuous mapping then $J u^* = J(\lim_{n \rightarrow \infty} u_n) = \lim_{n \rightarrow \infty} J u_{n+1}$, because $u_{n+1} = J u_n$, so $J u^* = u^*$, thus u^* is a fixed point of J . Now, let v^* is another fixed point, then $J v^* = v^*$, thus ${}_{dq}\|J u^* - J v^*\| \leq \mu(\beta) {}_{dq}\|u^* - v^*\|$, but ${}_{dq}\|J u^* - J v^*\| = {}_{dq}\|u^* - v^*\|$, so ${}_{dq}\|u^* - v^*\| \leq \mu(\beta) {}_{dq}\|u^* - v^*\|$, where $0 \leq \mu(\beta) < 1$, this implies that ${}_{dq}\|u^* - v^*\| = 0$, that is $u^* = v^*$. Therefore, u^* is unique.

Corollary 1: Let J be a quasi-contraction mapping defined on a dislocated quasi-Banach space \mathbb{T} into itself, and u^* is a unique fixed point of J then for any a sequence $\{u_n\}$, such that $u_n = J(u_{n-1}), n \in \mathbb{N}$, ${}_{dq}\|u_n - u^*\| \leq$

$$(\beta\mu(\beta))^n {}_{dq}\|u_1 - u_0\| \frac{1}{1-(\beta\mu(\beta))}, \quad 0 \leq \beta\mu(\beta) < 1.$$

Proof: From Theorem 1, u^* is a limit point of a sequence $\{u_n\}$, and

$${}_{dq}\|u_{n+p} - u_n\| \leq (\beta\mu(\beta))^n {}_{dq}\|u_1 - u_0\| \frac{1}{1-(\beta\mu(\beta))}, \quad 0 \leq \beta\mu(\beta) < 1,$$

where $p = 1, 2, 3, \dots$. As $p \rightarrow \infty$ in a above relation with the equality $\lim_{n \rightarrow \infty} u_{n+p} = \lim_{n \rightarrow \infty} u_n = u^*$, then the proof is completed.

Corollary 2: Let \mathcal{T} is a dislocated quasi-Banach space and Ω be a nonempty closed subset of \mathcal{T} and $J : \Omega \rightarrow \Omega$ be a quasi-contraction mapping, and $\{u_n\}$ is any sequence in Ω defined by $u_n = J(u_{n-1}), n \in \mathbb{N}$, then J has a unique fixed point.

Proof: Since Ω is closed subset of \mathcal{T} , then it is a dislocated quasi-Banach space by Lemma 1. Since J is a continues mapping then any a sequence $\{u_n\}$ be a Cauchy sequence and then converges to a limit point u^* according to Theorem 1, hence $Ju^* = J(\lim_{n \rightarrow \infty} u_n) = u^*$, so u^* is a unique fixed point.

Example 7: Let $\mathcal{T} = \mathbb{R}^n$ is a vector space over \mathbb{R} with a complete dislocated quasi-norm function ${}_{dq}\|u\| = |u_1|, \forall u = (u_1, u_2) \in \mathcal{T}$ and define a continuous self-mapping J by $(u) = \frac{\beta-1}{\beta^3}u \quad \forall u \in \mathcal{T}$

Conclusion

This paper has presented the notion of dislocated quasi-normed space, its completeness and the relationship with other concepts. It studies a Banach

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Authors' Declaration

- Conflicts of Interest: None.

, for some $\beta \geq 1$. It is clear that J be a quasi-contraction mapping with $\mu(\beta) = \frac{\beta-1}{\beta^2}$. Then from

Theorem 1, J has a unique fixed point which is $u^* = 0$ obviously.

Theorem 2: Let $J : \mathcal{T} \rightarrow \mathcal{T}$ be a mapping, where \mathcal{T} is a dislocated quasi-Banach space, such that J^n a quasi-contraction mapping for some integer $n > 0$, then J^n has a unique fixed point of J .

Proof : Since $J^n, n \in \mathbb{N}$, a quasi-contraction mapping, then by Theorem 1, $J^n, n > 0$, has a unique fixed point u^* , that is, $J^n u^* = u^*$. To prove u^* is a fixed point of J . Suppose $L = J^n$, so $L u^* = u^*$ this implies that $L^n u^* = L^{n-1}(L u^*) = L^{n-1}(u^*) = \dots = u^*$. So, $J u^* = J(L^n u^*) = L^n(J u^*)$.

$$\begin{aligned} & \|L^n(Ju^*) - u^*\| = \\ \text{Since, } & \left\| \begin{matrix} L^n(Ju^*) \\ -L^n(u^*) \end{matrix} \right\| \leq \mu(\beta) {}_{dq}\left\| \begin{matrix} L^{n-1}(Ju^*) \\ -L^{n-1}(u^*) \end{matrix} \right\| \\ & \leq (\mu(\beta))^2 {}_{dq}\|L^{n-2}(Ju^*) - L^{n-2}(u^*)\| \leq \dots \leq \\ & (\mu(\beta))^n {}_{dq}\|Ju^* - u^*\|, \text{ where } 0 \leq \mu(\beta) < 1. \\ \text{Hence, } & \lim_{n \rightarrow \infty} {}_{dq}\|L^n(Ju^*) - u^*\| = 0 \quad \text{as} \\ & (\mu(\beta))^n \rightarrow 0, \text{ thus } \lim_{n \rightarrow \infty} L^n(Ju^*) = u^* \text{ and} \\ & Ju^* = u^*. \end{aligned}$$

contraction principle theorem and proves some of its results in a dislocated quasi-Banach Space.

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الفضاء شبه المعياري المشتت وكماله لنظرية النقطة الصامدة

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الخلاصة

تم تقديم مفهوم الفضاء شبه المعياري المشتت في هذا البحث وهو حالة خاصة لمفهوم الفضاء شبه المعياري ومرتبطة بمفهوم الفضاء شبه المتري المشتت. العلاقة بين هذا المفهوم والمفاهيم الأخرى أعطيت من خلال أمثلة توضيحية. مفهوم شبه بناخ المشتت هو شبه المعياري المشتت الكامل والتي يعتبر مهم لنتائجنا فيما يتعلق بنظرية النقطة الصامدة. أيضاً، يتم تقديم التقابل شبه انكماشى، الذي فيه ثابت ليبنتشيز يعتمد على ثابت الفضاء شبه المعياري المشتت، ثم ندرس نظرية بناخ الانكماشية في شبه بناخ المشتت مع بعض النتائج المهمة.

الكلمات المفتاحية: الفضاء شبه بناخ المشتت، الفضاء شبه المتري المشتت، الفضاء شبه المعياري المشتت، نظرية النقطة الصامدة، التقابل شبه الانكماشى.