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On $\breve{\alpha}-\breve{\varphi}$ −Fuzzy Contractive Mapping in Fuzzy Normed Space

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Abstract

The idea of fixed points represents one of the most potent mathematical tools. This paper's main purpose is to introduce a new kind of fuzzy contractive mapping in a fuzzy normed space (briefly $\breve{\mathcal{F}}_{\eta}$ space) namely "$\breve{\alpha}-\breve{\varphi}$ −fuzzy contractive mapping". We proved some fixed point results for this mapping in the setting of $\breve{\mathcal{F}}_{\eta}$ space using the triangular property of fuzzy norm. Moreover, under specific conditions, some other results for such type of mapping are established. Finally, an example is offered to show the results' usefulness.

Keywords: Fixed point, $\breve{\mathcal{F}}_{\eta}$ space, $\breve{\alpha}$ − admissible mapping, $\breve{\alpha}-\breve{\varphi}$ −fuzzy contractive mapping, Fuzzy Banach space.

Introduction

Functional analysis is one of the most important areas of contemporary mathematics. It plays an essential role in the theory of differential equations, representation theory, and probability, as well as in the study of many different properties of different spaces, such as metric space, Hilbert space, Banach space, and others see references [1-4]. Fuzzy sets were first proposed by Zadeh in 1965. This theory is currently being developed, studied, and used in a wide variety of contexts. Fuzzy metric spaces were pioneered by Kramosil, I. and Michalek, J. in 1975. There is a significant amount of research on fuzzy metric spaces; see [5-8]. The fuzzy norm was first established in a linear space by Katsaras in 1984. Numerous articles on $\breve{\mathcal{F}}_{\eta}$ space have been written, for instance, see references [9-12]. On other hand, the presence of a solution in theory or practice is comparable to the presence of a fixed point for a suitable map or operator in a broad variety of problems in mathematics, computers, economics, modeling, and engineering. As a result, many branches of mathematics, science, and technology greatly depend on the presence of fixed points. Considerable studies about fixed point theory were published, for example, see [13-16]. In this paper a new kind of fuzzy contractive mapping in a $\breve{\mathcal{F}}_{\eta}$ space namely "$\breve{\alpha}-\breve{\varphi}$ −fuzzy contractive mapping" is presented. We introduce the idea of triangular property of fuzzy norm and by using this idea, fixed point theorem for self-mapping is proved. Also under specific conditions, other results in the framework of $\breve{\mathcal{F}}_{\eta}$ space are established.

Preliminaries:

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This section includes some terminology and outcomes which are used throughout the study. First, we'll go through basic terminology in the fuzzy setting.

Definition 1: [17] A binary operation $\circledcirc: [0,1] \times [0,1] \to [0,1]$ is referred to as a t-norm if it fulfills the requirements below for all $\mathfrak{s}, \mathfrak{z}, \xi, \varsigma \in [0,1]$,

(i) $1 \circledcirc \mathfrak{z} = \mathfrak{z}$,

(ii) $\mathfrak{z} \circledcirc \varsigma = \varsigma \circledast \mathfrak{z}$,

(iii) $\mathfrak{z} \circledcirc (\xi \circledcirc \varsigma) = (\mathfrak{z} \circledcirc \xi) \circledcirc \varsigma$

(iv) If $\mathfrak{z} \le \varsigma$ and $\xi \le \mathfrak{s}$ then $\mathfrak{z} \circledcirc \xi \le \varsigma \circledcirc \mathfrak{s}$.

Definition 2: [18] A $\breve{\mathcal{F}}_{\eta}$ space is a triplet $(\mathcal{V}, \breve{\mathcal{F}}_{\eta}, \circledcirc)$, where $\mathcal{V}$ is vector space, $\circledcirc$ is a t-norm and $\breve{\mathcal{F}}_{\eta}$ represents a fuzzy set on $\mathcal{V} \times R$ (R is a field) that fulfills the requirements for each $\omega, \mathfrak{p} \in \mathcal{V}$:

(1) $\breve{\mathcal{F}}_{\eta}(\omega, 0) = 0$,

(2) $\breve{\mathcal{F}}_{\eta}(\omega, \mathfrak{z}) = 1$, for each $\mathfrak{z} > 0$ if and only if $\omega = 0$,

(3) $\breve{\mathcal{F}}_{\eta}(\gamma\omega, \mathfrak{z}) = \breve{\mathcal{F}}_{\eta}(\omega, \mathfrak{z}/|\gamma|)$, for all $0 \neq \gamma \in R$, $\mathfrak{z} \ge 0$

(4) $\breve{\mathcal{F}}_{\eta}(\omega, \mathfrak{z}) \circledcirc \breve{\mathcal{F}}_{\eta}(\mathfrak{p}, \varsigma) \le \breve{\mathcal{F}}_{\eta}(\omega + \mathfrak{p}, \mathfrak{z} + \varsigma)$, for all $\mathfrak{z}, \varsigma \ge 0$

(5) $\breve{\mathcal{F}}_{\eta}(\omega, .)$is left continuous for all $\omega \in \mathcal{V}$, and $\lim_{\mathfrak{z} \to \infty} \breve{\mathcal{F}}_{\eta}(\omega, \mathfrak{z}) = 1$.

Samet et al [19] established the idea of α -admissible mapping in the manner described below.

Definition 3: [19] Suppose $\mathcal{V}$ is a nonempty set, $\acute{\Omega}: \mathcal{V} \to \mathcal{V}$, and $\breve{\alpha}: \mathcal{V} \times \mathcal{V} \to [0, \infty)$. $\acute{\Omega}$ is termed as $\breve{\alpha}$ -admissible mapping if for every $\omega, \mathfrak{p} \in \mathcal{V}$,

$\breve{\alpha}(\omega, \mathfrak{p}) \ge 1$ then $\breve{\alpha}(\acute{\Omega}\omega, \acute{\Omega}\mathfrak{p}) \ge 1$.

The idea of α -admissible mappings was then generalized by Salimi et al. [20] in the manner described below.

Definition 4: [20] Suppose $\mathcal{V}$ is a nonempty set, $\acute{\Omega}: \mathcal{V} \to \mathcal{V}$, and $\alpha, \eta: \mathcal{V} \times \mathcal{V} \to [0, \infty)$. Then $\acute{\Omega}$ is termed as α -admissible mapping with respect to η if for every $\omega, \mathfrak{p} \in \mathcal{V}$, $\alpha(\omega, \mathfrak{p}) \ge \eta(\omega, \mathfrak{p})$ then $\alpha(\acute{\Omega}\omega, \acute{\Omega}\mathfrak{p}) \ge \eta(\acute{\Omega}\omega, \acute{\Omega}\mathfrak{p})$.

Results and Discussion

Main Results

Definition 5: Suppose $(\mathcal{V}, \breve{\mathcal{F}}_{\eta}, \circledast)$ is a $\breve{\mathcal{F}}_{\eta}$ space. A mapping $\acute{\Omega}: \mathcal{V} \to \mathcal{V}$ is termed as $\breve{\alpha}$ −admissible mapping with respect to $\breve{\eta}$ where $\breve{\alpha}, \breve{\eta}: \mathcal{V} \times \mathcal{V} \times (0, \infty) \to (0, \infty)$ if for each $\omega, \mathfrak{p} \in \mathcal{V}$, and $\tau > 0$,

$$\breve{\alpha}(\omega, \mathfrak{p}, \tau) \ge \breve{\eta}(\omega, \mathfrak{p}, \tau) \quad \Rightarrow \breve{\alpha}(\acute{\Omega}\omega, \acute{\Omega}\mathfrak{p}, \tau) \ge \breve{\eta}(\acute{\Omega}\omega, \acute{\Omega}\mathfrak{p}, \tau).$$

The collection of all continuous functions $\breve{\varphi}: [0, \infty) \to [0, \infty)$ is denoted Φ with $\breve{\varphi}(\beta) < \beta$ for all $\beta > 0$.

Definition 6: Consider $(V, \breve{\mathcal{F}}_{\eta}, \circledcirc)$ be a $\breve{\mathcal{F}}_{\eta}$ space. The fuzzy norm $\breve{\mathcal{F}}_{\eta}$ is called triangular if the following condition holds:

$$\frac{1}{\breve{\mathcal{F}}_{\eta}(\omega-\mathfrak{p},\tau)} - 1 \le \left(\frac{1}{\breve{\mathcal{F}}_{\eta}(\omega-z,\tau)} - 1\right) + \left(\frac{1}{\breve{\mathcal{F}}_{\eta}(\mathfrak{p}-z,\tau)} - 1\right) \quad 1$$

for all $\omega, \mathfrak{p}, z \in \mathcal{V}$, and $\tau > 0$.

Definition 7: Consider $(\mathcal{V}, \breve{\mathcal{F}}_{\eta}, \circledcirc)$ be a $\breve{\mathcal{F}}_{\eta}$ space. $\acute{\Omega}: \mathcal{V} \to \mathcal{V}$ is termed as $\breve{\alpha}$−$\breve{\varphi}$ −fuzzy contractive mapping (briefly $\breve{\alpha}$−$\breve{\varphi}$ −FCM) if there exist functions $\breve{\alpha}, \breve{\eta}: \mathcal{V} \times \mathcal{V} \times (0, \infty) \to [0, \infty)$ and $\breve{\varphi} \in \Phi$ such that, for all $\omega, \mathfrak{p} \in \mathcal{V}$, and $\tau > 0$

$$\breve{\alpha}(\omega, \mathfrak{p}, \tau) \ge \breve{\eta}(\omega, \mathfrak{p}, \tau) \quad \Rightarrow \frac{1}{\breve{\mathcal{F}}_{\eta}(\acute{\Omega}\omega-\acute{\Omega}\mathfrak{p},\tau)} - 1 \le \breve{\varphi}\left(\frac{1}{\mathfrak{B}(\omega,\mathfrak{p},\tau)} - 1\right) \quad 2$$

where

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$$\mathfrak{B}(\omega, \mathfrak{p}, \tau) = \min\{\breve{\mathcal{F}}_\eta(\omega - \mathfrak{p}, \tau), \breve{\mathcal{F}}_\eta(\mathfrak{p} - \Omega\mathfrak{p}, \tau), \breve{\mathcal{F}}_\eta(\omega - \Omega\mathfrak{p}, \tau) \circledcirc \breve{\mathcal{F}}_\eta(\mathfrak{p} - \Omega\mathfrak{p}, \tau)\}$$

The following theorem is our first interesting result

Theorem 1: Suppose that $(\mathcal{V}, \breve{\mathcal{F}}_\eta, \circledcirc)$ is a fuzzy Banach space. Let $\Omega: \mathcal{V} \to \mathcal{V}$ be $\breve{\alpha}-\breve{\varphi}$ −FCM satisfies:

(i) Ω is $\breve{\alpha}$ − admissible mapping with respect to $\breve{\eta}$.

(ii) there exist $\omega_\circ \in \mathcal{V}$ such that $\breve{\alpha}(\omega_\circ, \Omega\omega_\circ, \tau) \geq \breve{\eta}(\omega_\circ, \Omega\omega_\circ, \tau)$

(iii) Ω is continuous.

Then Ω possesses a fixed point.

Proof: Consider $\omega_\circ \in \mathcal{V}$ such that $\breve{\alpha}(\omega_\circ, \Omega\omega_\circ, \tau) \geq \breve{\eta}(\omega_\circ, \Omega\omega_\circ, \tau)$ for each $\tau > 0$. Define a sequence $\{\omega_n\}$ in $\mathcal{V}$ by $\omega_n = \Omega\omega_{n-1} = \Omega^n\omega_\circ$ for all $n \in N$. If $\omega_n = \omega_{n+1}$ then $\omega = \omega_n$ is a fixed point of Ω. Hence suppose that $\omega_n \neq \omega_{n+1}$ for all $n \in N$. Because Ω is an $\breve{\alpha}$ − admissible mapping and $\breve{\alpha}(\omega_\circ, \Omega\omega_\circ, \tau) \geq \breve{\eta}(\omega_\circ, \Omega\omega_\circ, \tau)$, thus conclude

$$\breve{\alpha}(\omega_1, \omega_2, \tau) = \breve{\alpha}(\Omega\omega_\circ, \Omega^2\omega_\circ, \tau) \geq \breve{\eta}(\Omega\omega_\circ, \Omega^2\omega_\circ, \tau) = \breve{\eta}(\omega_1, \omega_2, \tau)$$

Keeping up with this procedure, acquire

$\breve{\alpha}(\omega_n, \omega_{n+1}, \tau) \geq \breve{\eta}(\omega_n, \omega_{n+1}, \tau)$ for all $n \in N$. Now by (2) with $\omega = \omega_{n-1}$, $\mathfrak{p} = \omega_n$, obtain,

$$\frac{1}{\breve{\mathcal{F}}_\eta(\Omega\omega_{n-1} - \Omega\omega_n, \tau)} - 1 \leq \breve{\varphi}\left(\frac{1}{\mathfrak{B}(\omega_{n-1}, \omega_n, \tau)} - 1\right)$$

where

$$\mathfrak{B}(\omega_{n-1}, \omega_n, \tau) = \min\{\breve{\mathcal{F}}_\eta(\omega_{n-1} - \omega_n, \tau), \breve{\mathcal{F}}_\eta(\omega_n - \Omega\omega_n, \tau), \breve{\mathcal{F}}_\eta(\omega_{n-1} - \Omega\omega_n, \tau) \circledcirc \breve{\mathcal{F}}_\eta(\omega_n - \Omega\omega_n, \tau)\}$$

$$= \min\{\breve{\mathcal{F}}_\eta(\omega_{n-1} - \omega_n, \tau), \breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau), \breve{\mathcal{F}}_\eta(\omega_{n-1} - \omega_{n+1}, \tau) \circledcirc \breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau)\}$$

$$= \min\{\breve{\mathcal{F}}_\eta(\omega_{n-1} - \omega_n, \tau), \breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau)\}$$

It follows

$$\frac{1}{\breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau)} - 1 \leq \breve{\varphi}\left(\frac{1}{\min\{\breve{\mathcal{F}}_\eta(\omega_{n-1} - \omega_n, \tau), \breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau)\}} - 1\right)$$ for all $n \in N$.

Now, if $\min\{\breve{\mathcal{F}}_\eta(\omega_{n-1} - \omega_n, \tau), \breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau)\} = \breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau)$ for some $n \in N$, then

$$\frac{1}{\breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau)} - 1 \leq \breve{\varphi}\left(\frac{1}{\breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau)} - 1\right) < \frac{1}{\breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau)} - 1$$

which is a contradiction. Thus ,

$$\frac{1}{\breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau)} - 1 < \frac{1}{\breve{\mathcal{F}}_\eta(\omega_{n-1} - \omega_n, \tau)} - 1$$

Consequently, $\breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau) > \breve{\mathcal{F}}_\eta(\omega_{n-1} - \omega_n, \tau)$ for all $n \in N$, thus $\{\breve{\mathcal{F}}_\eta(\omega_{n-1} - \omega_n, \tau)\}$ is an increasing sequence of positive reals in $[0, 1]$.

Consider $\mu(\tau) = \lim_{n\to\infty} \breve{\mathcal{F}}_\eta(\omega_{n-1} - \omega_n, \tau)$; To show that $\mu(\tau) = 1$. Presume that there $\tau_\circ > 0$ such that $\mu(\tau_\circ) < 1$.

From

$$\frac{1}{\breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau_\circ)} - 1 \leq \breve{\varphi}\left(\frac{1}{\breve{\mathcal{F}}_\eta(\omega_{n-1} - \omega_n, \tau_\circ)} - 1\right)$$

Utilizing the continuity of $\breve{\varphi}$ and letting $\to \infty$, obtain a contradiction

$$\frac{1}{\mu(\tau_\circ)} - 1 \leq \breve{\varphi}\left(\frac{1}{\mu(\tau_\circ)} - 1\right) < \frac{1}{\mu(\tau_\circ)} - 1$$

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This indicates that $\lim_{n\to\infty} \breve{\mathcal{F}}_\eta(\omega_{n-1} - \omega_n, \tau) = 1$. Then, for a fixed $i \in N$,

$$\breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+i}, \tau) \geq \breve{\mathcal{F}}_\eta(\omega_n - \omega_{n+1}, \tau) \circledast \breve{\mathcal{F}}_\eta(\omega_{n+1} - \omega_{n+2}, \tau) \circledcirc \ldots$$

$$\circledcirc \breve{\mathcal{F}}_\eta(\omega_{n+i-1} - \omega_{n+i}, \tau) \to 1 \circledcirc 1 \circledcirc \ldots \circledcirc 1 = 1$$

as $n \to \infty$

Hence $\{\omega_n\}$ is Cauchy sequence**.** Since $(\mathcal{V}, \breve{\mathcal{F}}_\eta, \circledcirc)$ is complete, then

$\{\omega_n\}$ converges to some $\omega^* \in \mathcal{V}$ and the continuity of $\acute{\Omega}$ lead to $\acute{\Omega}\omega_n \to \acute{\Omega}\omega^*$.

Hence

$$\lim_{n\to\infty} \breve{\mathcal{F}}_\eta(\omega_{n+1} - \acute{\Omega}\omega^*, \tau) = \lim_{n\to\infty} \breve{\mathcal{F}}_\eta(\acute{\Omega}\omega_n - \acute{\Omega}\omega^*, \tau) = 1$$

for all $\tau > 0$, that is, the sequence $\{\omega_n\}$ converges to $\acute{\Omega}\omega^*$. By the limit's uniqueness, conclude that $\acute{\Omega}\omega^* = \omega^*$.

In the next theorem, the continuity hypothesis is swapped out for a different regularity hypothesis.

Theorem 2: Suppose $(\mathcal{V}, \breve{\mathcal{F}}_\eta, \circledcirc)$ is a fuzzy Banach space where $\breve{\mathcal{F}}_\eta$ is triangular. Let $\acute{\Omega}: \mathcal{V} \to \mathcal{V}$ be $\breve{\alpha}-\breve{\varphi}$ −FCM satisfies:

(i) $\acute{\Omega}$ is $\breve{\alpha}$ − admissible mapping.

(ii) there exist $\omega_\circ \in \mathcal{V}$ such that $\breve{\alpha}(\omega_\circ, \acute{\Omega}\omega_\circ, \tau) \geq \breve{\eta}(\omega_\circ, \acute{\Omega}\omega_\circ, \tau)$

(iii) for any sequence $\{\omega_n\}$ in $\mathcal{V}$ with $\breve{\alpha}(\omega_n, \omega_{n+1}, \tau) \geq \breve{\eta}(\omega_n, \omega_{n+1}, \tau)$ for all $n \in N$,$\tau > 0$ and $\omega_n \to \omega$ as $n \to \infty$, then $\breve{\alpha}(\omega_n, \omega, \tau) \geq \breve{\eta}(\omega_n, \omega, \tau)$. Then $\acute{\Omega}$ possesses a fixed point.

Proof: From the previous proof steps of Theorem1, we see that $\{\omega_n\}$ is a Cauchy sequence in $(\mathcal{V}, \breve{\mathcal{F}}_\eta, \circledcirc)$ such that $\breve{\alpha}(\omega_n, \omega_{n+1}, \tau) \geq \breve{\eta}(\omega_n, \omega_{n+1}, \tau)$ for each $n \in N$. Then there is a point $\omega^\star$ in $\mathcal{V}$ such that $\omega_n \to \omega^\star$ as $n \to \infty$. The theorem's hypothesis (iii) implies that

$$\breve{\alpha}(\omega_n, \omega^\star, \tau) \geq \breve{\eta}(\omega_n, \omega^\star, \tau) \qquad 3$$

If $\omega^\star \neq \acute{\Omega}\omega^\star$, that is, $\breve{\mathcal{F}}_\eta(\omega^\star - \acute{\Omega}\omega^\star, \tau) < 1$ for some $\tau > 0$, then, from (1), (2), and (3), since $\breve{\mathcal{F}}_\eta$ is triangular,

$$\frac{1}{\breve{\mathcal{F}}_\eta(\omega^\star - \acute{\Omega}\omega^\star, \tau)} - 1 \leq \left(\frac{1}{\breve{\mathcal{F}}_\eta(\omega^\star - \omega_{n+1}, \tau)} - 1\right) + \left(\frac{1}{\breve{\mathcal{F}}_\eta(\acute{\Omega}\omega_n - \acute{\Omega}\omega^\star, \tau)} - 1\right)$$

$$\leq \left(\frac{1}{\breve{\mathcal{F}}_\eta(\omega^\star - \omega_{n+1}, \tau)} - 1\right) + \breve{\varphi}\left(\frac{1}{\mathfrak{B}(\omega_n, \omega^\star, \tau)} - 1\right)$$

Letting $n \to \infty$, obtain

$$\mathfrak{B}(\omega_n, \omega^\star, \tau) = \min\{\breve{\mathcal{F}}_\eta(\omega_n - \omega^\star, \tau), \breve{\mathcal{F}}_\eta(\omega^\star - \acute{\Omega}\omega^\star, \tau), \breve{\mathcal{F}}_\eta(\omega_n - \acute{\Omega}\omega^\star, \tau) \circledcirc \breve{\mathcal{F}}_\eta(\omega^\star - \acute{\Omega}\omega^\star, \tau)\}$$

$$= \min\{\breve{\mathcal{F}}_\eta(\omega_n - \omega^\star, \tau), \breve{\mathcal{F}}_\eta(\omega^\star - \acute{\Omega}\omega^\star, \tau)\}$$

$$= \min\{1, \breve{\mathcal{F}}_\eta(\omega^\star - \acute{\Omega}\omega^\star, \tau)\}$$

$$= \breve{\mathcal{F}}_\eta(\omega^\star - \acute{\Omega}\omega^\star, \tau).$$

in order to prevent contradiction with $\breve{\varphi}(\beta) < \beta$ for $\beta > 0$, it must be concluded that

$$\frac{1}{\breve{\mathcal{F}}_\eta(\omega^\star - \acute{\Omega}\omega^\star, \tau)} - 1 = 0$$

Thus, it follows that $\acute{\Omega}\omega^\star = \omega^\star$.

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From the aforementioned results, some of the following corollaries can be drawn. For instance, the following corollary is obtained from Theorems 4 and 5 (with Ñ triangular) by assuming $\breve{\eta}(\omega, \mathfrak{p}, \tau) = 1$.

Corollary 1: Suppose $(\mathcal{V}, \breve{\mathcal{F}}_\eta, \circledcirc)$ is a fuzzy Banach space (with $\breve{\mathcal{F}}_\eta$ triangular) and $\Omega: \mathcal{V} \to \mathcal{V}$ be an $\breve{\alpha}-$ admissible mapping. Suppose that there is $\breve{\varphi} \in \Phi$ such that, for each $\omega, \mathfrak{p} \in \mathcal{V}$, and $\tau > 0$,

$$\breve{\alpha}(\omega, \mathfrak{p}, \tau) \geq 1 \quad \Rightarrow \frac{1}{\breve{\mathcal{F}}_\eta(\Omega\omega - \Omega\mathfrak{p}, \tau)} - 1 \leq \breve{\varphi}\left(\frac{1}{\mathfrak{B}(\omega, \mathfrak{p}, \tau)} - 1\right)$$

where

$$\mathfrak{B}(\omega, \mathfrak{p}, \tau) = \min\{\breve{\mathcal{F}}_\eta(\omega - \mathfrak{p}, \tau), \breve{\mathcal{F}}_\eta(\mathfrak{p} - \Omega\mathfrak{p}, \tau), \breve{\mathcal{F}}_\eta(\omega - \Omega\mathfrak{p}, \tau) \circledcirc \breve{\mathcal{F}}_\eta(\mathfrak{p} - \Omega\mathfrak{p}, \tau)\}$$

and assuming the following is hold::

i) there exist $\omega_\circ \in \mathcal{V}$ such that $\breve{\alpha}(\omega_\circ, \Omega\omega_\circ, \tau) \geq 1$
ii) either Ω is continuous or for any sequence $\{\omega_n\}$ in $\mathcal{V}$ with $\breve{\alpha}(\omega_n, \omega_{n+1}, \tau) \geq 1$ for each $n \in N$, $\tau > 0$ and $\omega_n \to \omega$ as $n \to \infty$, then $\breve{\alpha}(\omega_n, \omega, \tau) \geq 1$.

Then Ω possesses a fixed point.

Example 1: Consider $\mathcal{V} = \mathbb{R}$ with the fuzzy norm, $\breve{\mathcal{F}}_\eta: \mathcal{V} \times \mathbb{R} \to [0,1]$ given by:

$\breve{\mathcal{F}}_\eta(\omega, \tau) = \frac{\tau}{\tau + \|\omega\|}$ for all $\omega \in \mathcal{V}$ and $\tau > 0$ where $\|\omega\| = |\omega|$.

Define $\Omega : \mathcal{V} \to \mathcal{V}$ by

$$\Omega\omega = \begin{cases} \frac{\omega^2}{4} & , \omega \in [0,1] \\ 3 & \text{otherwise} \end{cases}$$

and $\breve{\alpha}: \mathcal{V} \times \mathcal{V} \times [0, \infty) \to [0, \infty)$ by :

$$\breve{\alpha}(\omega, \mathfrak{p}, \tau) = \begin{cases} 1 & if \quad \omega, \mathfrak{p} \in [0, 1] \\ 0 & \quad otherwise \end{cases}$$

and $\breve{\eta}(\omega, \mathfrak{p}, \tau) = \frac{1}{2}$ for all $\omega, \mathfrak{p} \in \mathcal{V}$ and $\tau > 0$. Then applying Theorem5 with $\breve{\varphi}(\beta) = \frac{\beta}{2}$ for all $\beta \geq 0$ it is inferred that Ω possesses a fixed point.

Proof: At first to prove that Ω is $\breve{\alpha}$–admissible mapping. Let $\omega, \mathfrak{p} \in \mathcal{V}$, if $\breve{\alpha}(\omega, \mathfrak{p}, \tau) > \breve{\eta}(\omega, \mathfrak{p}, \tau)$ for each $\tau > 0$ then $\omega, \mathfrak{p} \in [0, 1]$. On other hand, for each $\omega, \mathfrak{p} \in \mathcal{V}$,then $\Omega\omega, \Omega\mathfrak{p} \in [0, 1]$ which indicates $\breve{\alpha}(\Omega\omega, \Omega\mathfrak{p}, \tau) > \breve{\eta}(\Omega\omega, \Omega\mathfrak{p}, \tau)$. Moreover, if $\{\omega_n\}$ is a sequence in $\mathcal{V}$ such that $\breve{\alpha}(\omega_n, \omega_{n+1}, \tau) \geq \breve{\eta}(\omega_n, \omega_{n+1}, \tau)$, for all $n \in N$ and $\tau > 0$ such that $\omega_n \to \omega \; as \; n \to \infty$, then $\{\omega_n\} \subset [0, 1]$ and hence $\omega \in [0, 1]$. Consequently, $\breve{\alpha}(\omega_n, \omega, \tau) \geq \breve{\eta}(\omega_n, \omega, \tau)$ for each $n \in N$ and each $\tau > 0$. Now let $\breve{\alpha}(\omega, \mathfrak{p}, \tau) > \breve{\eta}(\omega, \mathfrak{p}, \tau)$ for each $\tau > 0$, that is $\omega, \mathfrak{p} \in [0, 1]$. Consequently, Theorem 5's contractive requirement is fulfilled, that is

$$\frac{1}{\breve{\mathcal{F}}_\eta(\Omega\omega - \Omega\mathfrak{p}, \tau)} - 1 = \frac{1}{4\tau}|\omega^2 - \mathfrak{p}^2| \leq \frac{1}{2\tau}|\omega - \mathfrak{p}|$$

$$\leq \frac{1}{2\tau}\max\{|\omega - \mathfrak{p}|, |\omega - \Omega\omega|, |\mathfrak{p} - \Omega\mathfrak{p}|\}$$

$$= \frac{1}{2}\left(\frac{\tau + \max\{|\omega - \mathfrak{p}|, |\omega - \Omega\omega|, |\mathfrak{p} - \Omega\mathfrak{p}|\}}{\tau} - 1\right)$$

$$= \frac{1}{2}\left(\frac{1}{\frac{\tau}{\tau + \max\{|\omega - \mathfrak{p}|, |\omega - \Omega\omega|, |\mathfrak{p} - \Omega\mathfrak{p}|\}}} - 1\right)$$

$$= \breve{\varphi}\left(\frac{1}{\mathfrak{B}(\omega, \mathfrak{p}, \tau)} - 1\right)$$

Hence each condition of Theorem 5 holds and Ω possesses a fixed point. The fixed points of Ω in this example are 0 and 3.

In the subsequent stage, which involves determining whether or not the $\breve{\alpha}-\breve{\varphi}$ −FCM possesses a unique fixed point, the following assumptions will be taken into account:

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(v) for all $\omega, \mathfrak{p} \in \mathcal{V}$ and $\tau > 0$, $\exists\, z \in \mathcal{V}$ such that $\breve{\alpha}(\omega, z, \tau) \geq \breve{\eta}(\omega, z, \tau)$, $\breve{\alpha}(\mathfrak{p}, z, \tau) \geq \breve{\eta}(\mathfrak{p}, z, \tau)$ and $\lim_{n\to\infty} \breve{\mathcal{F}}_{\eta}(\Omega^{n-1}z - \Omega^{n}z, \tau) = 1$.

Theorem 3: By including hypothesis (v) in Theorem 4 (Theorem 5), one arrives to the conclusion that ω^* is the only fixed point of Ω under the condition that $\widetilde{\phi} \in \Phi$ is non-decreasing.

Proof: Suppose that $\omega^*$and $\mathfrak{p}^*$represent two fixed points of Ω. If $\breve{\alpha}(\omega^*, \mathfrak{p}^*, \tau) \geq \breve{\eta}(\omega^*, \mathfrak{p}^*, \tau)$ then by condition (2), conclude $\omega^* = \mathfrak{p}^*$. Assume if $\breve{\alpha}(\omega^*, \mathfrak{p}^*, \tau) < \breve{\eta}(\omega^*, \mathfrak{p}^*, \tau)$, then from assumption (v), $\exists\, z \in \mathcal{V}$ such that

$$\breve{\alpha}(\omega^*, z, \tau) \geq \breve{\eta}(\omega, z, \tau) \quad \text{and} \quad \breve{\alpha}(\mathfrak{p}^*, z, \tau) \geq \breve{\eta}(\mathfrak{p}^*, z, \tau) \qquad 4$$

Because Ω is $\breve{\alpha}-$ admissible, obtain

$\breve{\alpha}(\omega^*, \Omega^{n}z, \tau) \geq \breve{\eta}(\omega^*, \Omega^{n}z, \tau)$. Now to prove that $\breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n}z, \tau) \to 1$ as $\to \infty$.

Suppose that there is $\tau > 0$ with $\lim_{n\to\infty} \breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n}z, \tau) < 1$. Consequently, from (2) and (4) it is obtained

$$\frac{1}{\breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n}z, \tau)} - 1 = \frac{1}{\breve{\mathcal{F}}_{\eta}(\Omega\omega^* - \Omega(\Omega^{n-1}z), \tau)} - 1 \leq \breve{\varphi}\left(\frac{1}{\mathfrak{B}(\omega^*, \Omega^{n-1}z, \tau)} - 1\right)$$

Where

$$\mathfrak{B}(\omega^*, \Omega^{n-1}z, \tau) = \min\{\breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n-1}z, \tau), \breve{\mathcal{F}}_{\eta}(\Omega^{n-1}z - \Omega(\Omega^{n-1}z), \tau), \breve{\mathcal{F}}_{\eta}(\omega^* - \Omega(\Omega^{n-1}z), \tau) \circledcirc \breve{\mathcal{F}}_{\eta}(\Omega^{n-1}z - \Omega(\Omega^{n-1}z), \tau)\}$$

$$= \min\{\breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n-1}z, \tau), \breve{\mathcal{F}}_{\eta}(\Omega^{n-1}z - \Omega^{n}z, \tau), \breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n}z, \tau) \circledcirc \breve{\mathcal{F}}_{\eta}(\Omega^{n-1}z - \Omega^{n}z, \tau)\}$$

$$= \min\{\breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n-1}z, \tau), \breve{\mathcal{F}}_{\eta}(\Omega^{n-1}z - \Omega^{n}z, \tau)\}$$

Let $n_\circ \in N$ such that $\breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n}z, \tau) \leq \breve{\mathcal{F}}_{\eta}(\Omega^{n}z - \Omega^{n+1}z, \tau)$ for all $n \geq n_\circ$.Hence, for each $n \geq n_\circ$

$$\frac{1}{\breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n}z, \tau)} - 1 = \frac{1}{\breve{\mathcal{F}}_{\eta}(\Omega\omega^* - \Omega(\Omega^{n-1}z), \tau)} - 1 \leq \breve{\varphi}\left(\frac{1}{\min\{\breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n-1}z, \tau), \breve{\mathcal{F}}_{\eta}(\Omega^{n-1}z - \Omega^{n}z, \tau)\}} - 1\right) = \breve{\varphi}\left(\frac{1}{\breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n-1}z, \tau)} - 1\right)$$

This indicates that

$$\frac{1}{\breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n}z, \tau)} - 1 = \frac{1}{\breve{\mathcal{F}}_{\eta}(\Omega\omega^* - \Omega(\Omega^{n-1}z), \tau)} - 1 \leq \breve{\varphi}^{n-n_\circ}\left(\frac{1}{\breve{\mathcal{F}}_{\eta}(\omega^* - \Omega^{n_\circ}z, \tau)} - 1\right)$$

Letting $n \to \infty$, then $\Omega^{n}z \to \omega^*$. Likewise, one may get $\Omega^{n}z \to \mathfrak{p}^*$ as $n \to \infty$. Therefore, it is concluded that $\omega^* = \mathfrak{p}^*$.

Conclusion

This work presents the concept of $\breve{\alpha}-\breve{\varphi}$ –FCM in fuzzy normed space and proves the fixed point theorem for this kind of mapping. To illustrate the significance of the results, a specific example is provided. Further work is required to generalize this contraction mapping and investigate its potential applications in the fuzzy normed space.

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Authors' Declaration

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Technology.

Authors' Contribution Statement

R. I.S contributed to the research design and implementation, as well as the analysis of the results. B. A. A. participated in the revision of the results and proofreading. Both authors worked together to enhance the final research by discussing the results.

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حول دالة الانكماش الضبابية $\breve{\alpha}-\breve{\varphi}$ في الفضاء المعياري الضبابي

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الخلاصة

تمثل فكرة النقاط الصامدة إحدى أقوى الأدوات الرياضية. الغرض الرئيسي من هذه الورقة هو تقديم نوع جديد من دالة الانكماش الضبابي في فضاء معياري ضبابي وهي "الدالة الانكماشية الضبابية $\breve{\alpha}-\breve{\varphi}$". نثبت بعض نتائج النقطه الصامدة لهذه الداله في سياق الفضاء المعياري الضبابي باستخدام الخاصية المثلثية للمعيار الضبابي. بالإضافة إلى ذلك ، في ظل ظروف محددة ، تم إثبات بعض النتائج الأخرى لهذا النوع من الدالة. أخيرًا ، يتم تقديم مثال لإظهار فائدة النتائج.

الكلمات المفتاحية: النقطة الصامدة، فضاء معياري ضبابي، الدالة المقبولة القريبه -$\tilde{\alpha}$، دالة الانكماش الضبابية $\tilde{\alpha}-\tilde{\phi}$، فضاء بناخ الضبابي.