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Sum of Squares of 'n' Consecutive Carol Numbers

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Abstract:
The discussion in this paper gives several theorems and lemmas on the Sums of Squares of "n" consecutive Carol Numbers. These theorems are proved by using the definition of carol numbers and mathematical induction method. Here the matrix form and the recursive form of sum of squares of "n" consecutive Carol numbers is also given. The properties of the Carol numbers are also derived.

Keywords: Carol Numbers, Cullen Numbers, Fibonacci, Sum of Squares, Woodall Numbers.

Introduction:

The formula $\zeta(\mu) = 4^\mu - 2^{\mu+1} - 1$ is the general form of Carol number. For the amenity of the reader, the sequence of carol numbers and some examples are exhibit obviously. The first five terms of this sequence are $-1, 7, 47, 223, 959, \ldots$[1-3]. In this sequence there is sub-sequence which has only prime numbers. For example, some of that elements are $7, 47, 223, 3967, 16127, \ldots$ A mathematician Soykan,Y derived Formulae for Sums of Squares of terms[1]. The sums of squares of generalized Fibonacci number and Tribonacci number[2-4]. Likewise, a study on Generalized Mersenne numbers by Soykan Y[5]. Now look into sums of squares of 'm' consecutive carol numbers and its matrices representation starting with 7, every third carol numbers is multiple of 7. The indistinguishable identities for sequence, Fibonacci, Jacobsthal, polygonal numbers have recently been found by some authors in paper[6,7]. More related studies have been carried out[8,9]. The following are some investigation and possessions of the above.

Basic definitions:

Definition. 1: Carol number

The Carol number is defined by the formula $\zeta(\mu) = 4^\mu - 2^{\mu+1} - 1$.

Definition. 2: Prime Number

A natural number greater than 1 and which is divisible by 1 and itself is called a Prime number.

Definition. 3: Modulo

Let 'a' and 'b' be two positive numbers. They are congruent modulo a given number 'n', if they give the same remainder when divided by 'n'.

Example. 1: 12 and 22 are congruent modulo '5'.

Main Results:

Theorem. 1: For $\mu \geq 0$, $l = 1,2,3,\ldots n$ the following equality is true,

$$\sum_{l=1}^{n} \zeta^2[\mu + (l-1)] = \sum_{l=1}^{n} \frac{X-M}{105}\{7[(X^3 + M^3) + XM(X+M)] - 60(X^2 + XM + M^2) + 70(X+M) + 420\} + (N - \mu)$$

where $X = 2^N, M = 2^\mu$ and $N = \mu + l$.

Lemma. 1: For $\mu \geq 0$, $l = 1,2,3,\ldots n$, the upcoming is true

$$\sum_{l=1}^{n} \zeta^2[\mu + (l-1)] = \frac{1}{15}\sum_{l=1}^{n} 2^{4(\mu+l)} - \frac{1}{7}\sum_{l=1}^{n} 2^{2+3(\mu+l)} + \frac{1}{3}\sum_{l=1}^{n} 2^{1+2(\mu+l)} + \sum_{l=1}^{n} 2^{2+(\mu+l)} + \left(\frac{2^{3\mu+2}}{7} - \frac{2^{4\mu}}{15} - \frac{2^{2\mu+1}}{3} - 2^{(\mu+2)}\right) + l$$

Proof: By Definition 1, $\zeta(\mu) = 4^\mu - 2^{\mu+1} - 1$.

Therefore,

$$\zeta^2(\mu) = 2^{4\mu} - 4(2^{3\mu}) + 2(2^{2\mu}) + 4(2^\mu) + 1$$

By replacing μ *by* $\mu + 1$, yields $\zeta^2(\mu+1) = 2^4(2^{4\mu}) - 4(2^3)(2^{3\mu}) + 2(2^2)(2^{2\mu}) + 4(2)(2^\mu) + 1$

In the similar manner, by replacing μ by $\mu + 1$ successively the values of $\zeta^2(\mu+2), \zeta^2(\mu+3), \zeta^2(\mu+4)\ldots$ are obtained.

Finally,

$$\zeta^2[\mu+(n-1)] = 2^{4(n-1)}(2^{4\mu}) - 4\left(2^{3(n-1)}\right)(2^{3\mu}) + 2\left(2^{2(n-1)}\right)(2^{2\mu}) + 4\left(2^{(n-1)}\right)(2^{\mu}) + 1 \qquad 1$$

adding all these 'n' terms, arrives in the stage,

$$\sum_{l=1}^{n}\zeta^2\,[\mu+(l-1)] = \frac{1}{15}\sum_{l=1}^{n}2^{4(\mu+l)} - \frac{1}{7}\sum_{l=1}^{n}2^{2+3(\mu+l)} + \frac{1}{3}\sum_{l=1}^{n}2^{1+2(\mu+l)}\sum_{l=1}^{n}2^{2+(\mu+l)} + \left(\frac{2^{3\mu+2}}{7} - \frac{2^{4\mu}}{15} - \frac{2^{2\mu+1}}{3} - 2^{(\mu+2)}\right) \qquad 2$$

In Eq.1, the first, second, third and fourth terms of RHS of 1, consists of Geometric Series with common ratio $2^4, 2^3, 2^2, 2$ respectively. Using the corresponding formulae, and after some algebraic simplification the proof of lemma is obtained.

Proof of Theorem. 1:
By choosing $\mu + l = N$ *and* $2^{\mu} = M$ in Eq.2,

$$\sum_{l=1}^{n}\zeta^2\,[\mu+(l-1)] = \frac{1}{105}\sum_{l=1}^{n}[7(2^{4N}) - 60(2^{3N}) + 70(2^{2N}) + 420(2^{N})]$$

$$-\frac{1}{105}\sum_{l=1}^{n}[7(M^4) - 60(M^3) + 70(M^2) + 420(M)] + \sum_{l=1}^{n} l$$

Collecting corresponding terms in Eq. 3 yields

$$\sum_{l=1}^{n}\zeta^2[\mu+(l-1)] = \sum_{l=1}^{n}\frac{X-M}{105}\{7[(X^3+M^3) + XM(X+M)] - 60(X^2+XM + M^2) + 70(X+M) + 420\} + (N - \mu)$$

where $X = 2^N, M = 2^{\mu}$ *and* $N = \mu + l$

Matrix form of Sum of Squares of 'n' consecutive Carol Number

Theorem 2: For $\mu \geq 0,\ l = 0,1,2\ldots(n-1)$ the matrix form of 'n' consecutive carol number is

$$\begin{bmatrix}\zeta^2[\mu+(n-5)]\\ \zeta^2[\mu+(n-4)]\\ \zeta^2[\mu+(n-3)]\\ \zeta^2[\mu+(n-2)]\\ \zeta^2[\mu+(n-1)]\end{bmatrix} = \begin{bmatrix}2^{4(n-5)} & -4(2^{3(n-5)}) & 2(2^{2(n-5)}) & 4(2^{(n-5)}) & 1\\ 2^{4(n-4)} & -4(2^{3(n-4)}) & 2(2^{2(n-4)}) & 4(2^{(n-4)}) & 1\\ 2^{4(n-3)} & -4(2^{3(n-3)}) & 2(2^{2(n-3)}) & 4(2^{(n-3)}) & 1\\ 2^{4(n-2)} & -4(2^{3(n-2)}) & 2(2^{2(n-2)}) & 4(2^{(n-2)}) & 1\\ 2^{4(n-1)} & -4(2^{3(n-1)}) & 2(2^{2(n-1)}) & 4(2^{(n-1)}) & 1\end{bmatrix}\begin{bmatrix}Y^4\\ Y^3\\ Y^2\\ Y^1\\ 1\end{bmatrix}$$

where $Y = (2^{\mu})$

Proof: By the definition of Carol number,

$$\zeta(\mu) = 4^{\mu} - 2^{\mu+1} - 1$$

Therefore,

$$\zeta^2(\mu) = (2^{4\mu}) - 4(2^{3\mu}) + 2(2^{2\mu}) + 4(2^{\mu}) + 1$$

Consider the system of following equations,

$$\zeta^2(\mu) = (2^{4\mu}) - 4(2^{3\mu}) + 2(2^{2\mu}) + 4(2^{\mu}) + 1$$
$$\zeta^2(\mu) = (2^{4\mu}) - 4(2^{3\mu}) + 2(2^{2\mu}) + 4(2^{\mu}) + 1$$
$$\zeta^2(\mu) = (2^{4\mu}) - 4(2^{3\mu}) + 2(2^{2\mu}) + 4(2^{\mu}) + 1$$
$$\zeta^2(\mu) = (2^{4\mu}) - 4(2^{3\mu}) + 2(2^{2\mu}) + 4(2^{\mu}) + 1$$
$$\zeta^2(\mu) = (2^{4\mu}) - 4(2^{3\mu}) + 2(2^{2\mu}) + 4(2^{\mu}) + 1$$

This system of equations can be written in the matrix form $AX = B$ where

$$A = \begin{bmatrix}\zeta^2[\mu]\\ \zeta^2[\mu+1]\\ \zeta^2[\mu+2]\\ \zeta^2[\mu+3]\\ \zeta^2[\mu+4]\end{bmatrix} \qquad X = \begin{bmatrix}1 & -4 & 2 & 4 & 1\\ 2^4 & -4(2^3) & 2(2^2) & 4(2) & 1\\ 2^8 & -4(2^6) & 2(2^4) & 4(2^2) & 1\\ 2^{12} & -4(2^9) & 2(2^6) & 4(2^3) & 1\\ 2^{16} & -4(2^{12}) & 2(2^8) & 4(2^4) & 1\end{bmatrix} \text{ and } B = \begin{bmatrix}Y^4\\ Y^3\\ Y^2\\ Y^1\\ 1\end{bmatrix}$$

For the next set of system of equations from $\zeta^2(\mu + 1)$ to $\zeta^2(\mu + 5)$, the matrix representation is,

$$\begin{bmatrix}\zeta^2[\mu+1]\\ \zeta^2[\mu+2]\\ \zeta^2[\mu+3]\\ \zeta^2[\mu+4]\\ \zeta^2[\mu+5]\end{bmatrix} = \begin{bmatrix}2^4 & -4(2^3) & 2(2^2) & 4(2) & 1\\ 2^8 & -4(2^6) & 2(2^4) & 4(2^2) & 1\\ 2^{12} & -4(2^9) & 2(2^6) & 4(2^3) & 1\\ 2^{16} & -4(2^{12}) & 2(2^8) & 4(2^4) & 1\\ 2^{20} & -4(2^{15}) & 2(2^{10}) & 4(2^5) & 1\end{bmatrix}\begin{bmatrix}Y^4\\ Y^3\\ Y^2\\ Y^1\\ 1\end{bmatrix}$$

Continuing in this manner, the general matrix form of the sum of 'n' consecutive Carol number is,

$$\begin{bmatrix}\zeta^2[\mu+(n-5)]\\ \zeta^2[\mu+(n-4)]\\ \zeta^2[\mu+(n-3)]\\ \zeta^2[\mu+(n-2)]\\ \zeta^2[\mu+(n-1)]\end{bmatrix} = \begin{bmatrix}2^{4(n-5)} & -4(2^{3(n-5)}) & 2(2^{2(n-5)}) & 4(2^{(n-5)}) & 1\\ 2^{4(n-4)} & -4(2^{3(n-4)}) & 2(2^{2(n-4)}) & 4(2^{(n-4)}) & 1\\ 2^{4(n-3)} & -4(2^{3(n-3)}) & 2(2^{2(n-3)}) & 4(2^{(n-3)}) & 1\\ 2^{4(n-2)} & -4(2^{3(n-2)}) & 2(2^{2(n-2)}) & 4(2^{(n-2)}) & 1\\ 2^{4(n-1)} & -4(2^{3(n-1)}) & 2(2^{2(n-1)}) & 4(2^{(n-1)}) & 1\end{bmatrix}$$

$$\begin{bmatrix} Y^4 \\ Y^3 \\ Y^2 \\ Y^1 \\ 1 \end{bmatrix} \text{ where } Y = (2^{\mu})$$

Hence the proof.

Recursive Matrix form of Sum of Squares of 'n' consecutive Carol Number

Theorem 3: The recursive matrix form of 'n' consecutive carol number is

$$A(\zeta_n) = \begin{bmatrix} Q^4a_{11} & Q^3a_{12} & Q^2a_{13} & Qa_{14} & a_{15} \\ Q^4a_{21} & Q^3a_{22} & Q^2a_{23} & Qa_{24} & a_{25} \\ Q^4a_{31} & Q^3a_{32} & Q^2a_{33} & Qa_{34} & a_{35} \\ Q^4a_{41} & Q^3a_{42} & Q^2a_{43} & Qa_{44} & a_{45} \\ Q^4a_{51} & Q^3a_{52} & Q^2a_{53} & Qa_{54} & a_{55} \end{bmatrix} \textit{where } Q = 2^{n-1}$$

$$\zeta^2(\mu) = (2^{4\mu}) - 4(2^{3\mu}) + 2(2^{2\mu}) + 4(2^{\mu}) + 1$$

$$\zeta^2(\mu+1) = 2^4(2^{4\mu}) - 4(2^3)(2^{3\mu}) + 2(2^2)(2^{2\mu}) + 4(2)(2^{\mu}) + 1$$

$$\zeta^2(\mu+2) = 2^8(2^{4\mu}) - 4(2^6)(2^{3\mu}) + 2(2^4)(2^{2\mu}) + 4(2^2)(2^{\mu}) + 1$$

$$\zeta^2(\mu+3) = 2^{12}(2^{4\mu}) - 4(2^9)(2^{3\mu}) + 2(2^6)(2^{2\mu}) + 4(2^3)(2^{\mu}) + 1$$

$$\zeta^2(\mu+4) = 2^{16}(2^{4\mu}) - 4(2^{12})(2^{3\mu}) + 2(2^8)(2^{2\mu}) + 4(2^4)(2^{\mu}) + 1$$

The matrix representation of this system is

$$\begin{bmatrix} \zeta^2[\mu] \\ \zeta^2[\mu+1] \\ \zeta^2[\mu+2] \\ \zeta^2[\mu+3] \\ \zeta^2[\mu+4] \end{bmatrix} = \begin{bmatrix} 1 & -4 & 2 & 4 & 1 \\ 2^4 & -4(2^3) & 2(2^2) & 4(2) & 1 \\ 2^8 & -4(2^6) & 2(2^4) & 4(2^2) & 1 \\ 2^{12} & -4(2^9) & 2(2^6) & 4(2^3) & 1 \\ 2^{16} & -4(2^{12}) & 2(2^8) & 4(2^4) & 1 \end{bmatrix} \begin{bmatrix} Y^4 \\ Y^3 \\ Y^2 \\ Y^1 \\ 1 \end{bmatrix}$$

Consider the initial matrix

$$A(\zeta_1) = \begin{bmatrix} 1 & -4 & 2 & 4 & 1 \\ 2^4 & -4(2^3) & 2(2^2) & 4(2) & 1 \\ 2^8 & -4(2^6) & 2(2^4) & 4(2^2) & 1 \\ 2^{12} & -4(2^9) & 2(2^6) & 4(2^3) & 1 \\ 2^{16} & -4(2^{12}) & 2(2^8) & 4(2^4) & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

The elements of second set of matrices $A(\zeta_2)$ can be written by using the elements of first set of matrices $A(\zeta_1)$ as follows,

$$A(\zeta_2) = \begin{bmatrix} 2^4 & -4(2^3) & 2(2^2) & 4(2) & 1 \\ 2^8 & -4(2^6) & 2(2^4) & 4(2^2) & 1 \\ 2^{12} & -4(2^9) & 2(2^6) & 4(2^3) & 1 \\ 2^{16} & -4(2^{12}) & 2(2^8) & 4(2^4) & 1 \\ 2^{20} & -4(2^{15}) & 2(2^{10}) & 4(2^5) & 1 \end{bmatrix} = \begin{bmatrix} 2^4a_{11} & 2^3a_{12} & 2^2a_{13} & 2a_{14} & a_{15} \\ 2^4a_{21} & 2^3a_{22} & 2^2a_{23} & 2a_{24} & a_{25} \\ 2^4a_{31} & 2^3a_{32} & 2^2a_{33} & 2a_{34} & a_{35} \\ 2^4a_{41} & 2^3a_{42} & 2^2a_{43} & 2a_{44} & a_{45} \\ 2^4a_{51} & 2^3a_{52} & 2^2a_{53} & 2a_{54} & a_{55} \end{bmatrix}$$

Finally, the elements of nth set of matrices $A(\zeta_n)$ has the following form

$$A(\zeta_n) = \begin{bmatrix} 2^{4(n-1)}a_{11} & 2^{3(n-1)}a_{12} & 2^{2(n-1)}a_{13} & 2^{(n-1)}a_{14} & a_{15} \\ 2^{4(n-1)}a_{21} & 2^{3(n-1)}a_{22} & 2^{2(n-1)}a_{23} & 2^{4(n-1)}a_{24} & a_{25} \\ 2^{4(n-1)}a_{31} & 2^{3(n-1)}a_{32} & 2^{2(n-1)}a_{33} & 2^{4(n-1)}a_{34} & a_{35} \\ 2^{4(n-1)}a_{41} & 2^{3(n-1)}a_{42} & 2^{2(n-1)}a_{43} & 2^{4(n-1)}a_{44} & a_{45} \\ 2^{4(n-1)}a_{51} & 2^{3(n-1)}a_{52} & 2^{2(n-1)}a_{53} & 2^{4(n-1)}a_{54} & a_{55} \end{bmatrix} \quad 4$$

By choosing $Q = 2^{n-1}$ Eq. 3 takes the form

$$A(\zeta_n) = \begin{bmatrix} Q^4a_{11} & Q^3a_{12} & Q^2a_{13} & Qa_{14} & a_{15} \\ Q^4a_{21} & Q^3a_{22} & Q^2a_{23} & Qa_{24} & a_{25} \\ Q^4a_{31} & Q^3a_{32} & Q^2a_{33} & Qa_{34} & a_{35} \\ Q^4a_{41} & Q^3a_{42} & Q^2a_{43} & Qa_{44} & a_{45} \\ Q^4a_{51} & Q^3a_{52} & Q^2a_{53} & Qa_{54} & a_{55} \end{bmatrix} \textit{where } Q = 2^{n-1}$$

which gives the recursive matrix form of sum of 'n' consecutive Carol number.

Generalized sum of Carol numbers

Theorem 4: For all $\mu_1 \geq 0$ *and* $\mu_2 \geq 0$, $\zeta(\mu_1 + \mu_2) = [1 + 2(2^{\mu_2})]\zeta(\mu_1) + [1 + 2(2^{\mu_1})]\zeta(\mu_2) + \zeta(\mu_1)\zeta(\mu_2) + 2[(2^{\mu_1} + 1)(2^{\mu_2} + 1)] - 2$.

Proof: By the definition of Carol number, $\zeta(\mu) = (2^{\mu} - 1)^2 - 2$

therefore, $\zeta(\mu_1 + \mu_2) = (2^{\mu_1+\mu_2} - 1)^2 - 2$

$$= [2^{\mu_2}(2^{\mu_1} - 1) + (2^{\mu_2} - 1)]^2 - 2$$

$$= \zeta(\mu_1) + \zeta(\mu_2) + \{[(2^{\mu_1} - 1)^2 - 2] + 2\}\{\zeta(\mu_2) + 2(2^{\mu_2})\} + 2(2^{\mu_1} - 1)[\zeta(\mu_2) + (2^{\mu_2} + 1)] + 2$$

$$= \zeta(\mu_1) + \zeta(\mu_2) + \zeta(\mu_1)\zeta(\mu_2) + 2\zeta(\mu_1)(2^{\mu_2})\} + 2\zeta(\mu_2) + 4(2^{\mu_2}) + 2(2^{\mu_1} - 1)\zeta(\mu_2) + 2(2^{\mu_1} - 1)\,(2^{\mu_2} + 1) + 2$$

$$= [1 + 2(2^{\mu_2})]\zeta(\mu_1) + [1 + 2(2^{\mu_1}) + \zeta(\mu_1)]\zeta(\mu_2) + 4(2^{\mu_2}) + 2[(2^{\mu_1} - 1)(2^{\mu_2} + 1) + 1]$$

Hence,

$$\zeta(\mu_1 + \mu_2) = [1 + 2(2^{\mu_2})]\zeta(\mu_1) + [1 + 2(2^{\mu_1})]\zeta(\mu_2) + \zeta(\mu_1)\zeta(\mu_2) + 2[(2^{\mu_1} + 1)(2^{\mu_2} + 1)] - 2$$

Properties of Carol Numbers:

Theorem 5: $\zeta(\mu_3) - 2\zeta(\mu_1) = \zeta^2(\mu_2)$ where $\zeta(\mu_1), \zeta(\mu_2), \zeta(\mu_3)$ represents first three Carol numbers.

Proof: By the definition of Carol number, proceeding like $\zeta(\mu_1) = 4^{\mu_1} - 2^{(\mu_1+1)} - 1$

when $\mu_1 = 1$, $\mu_2 = 2$ *and* $\mu_3 = 3$ the first, second and third Carol numbers are -1, 7 and 47 respectively, which satisfies the required equation. Hence the theorem.

Theorem 6: $\zeta(3n - 1) \equiv 0(mod\ 7)$.

Proof: By definition, $\zeta(\mu) = 4^{\mu} - 2^{(\mu+1)} - 1$.

The prove of this theorem is given by induction method.

When $n = 1$, $\zeta(2) = \zeta(\mu_2) = 4^2 - 2^3 - 1 = 7 \equiv 0(mod\ 7)$

Therefore, the theorem is true for $n = 1$.

By induction, one can assume that the theorem is true for $\zeta(3n - 4) = \zeta(\mu_{3n-4})$ is divisible by 7.

Now, $\zeta(3n - 1) = \zeta(\mu_{3n-1}) = 4^{3n-1} - 2^{3n-1} - 1$

$$\zeta(3n - 1) = \zeta(\mu_{3n-1}) = 4^{[(3n-4)+3]} - 2^{[(3n-4)+3]+1} - 1$$

$$\zeta(3n - 1) = \zeta(\mu_{3n-1}) = (4^3)4^{(3n-4)} - (2^3)2^{(3n-4)+1} - 1$$

$$\zeta(3n - 1) = \zeta(\mu_{3n-1}) = (7m + 1)4^{(3n-4)} - (7m + 1)2^{(3n-4)+1} - 1$$
$$= 4^{(3n-4)} - 2^{(3n-4)+1} - 1$$
$$\equiv 0(mod\ 7)$$

Hence, the theorem is true for all values of n.

Theorem 7: $\zeta(\mu_n)$ where $n \neq (3m + 2)$ $for\ all\ m > 0$ is a prime Carol number.

Proof: By induction method.

$\zeta(\mu) = 4^{\mu} - 2^{(\mu+1)} - 1$

When $m = 1$ *and* $n = 5$, $\zeta(5) = \zeta(\mu_5) = 4^5 - 2^6 - 1 = 959$, which is not a prime.

Therefore, by induction the theorem is true for $m = k - 1$ *and* $n = 3k - 1$,

Hence, $\zeta(\mu_n)$ *where* $n \neq 3k - 1$, is a prime Carol number.

i.e., $\zeta(\mu_{n-1}) = \zeta(n - 1) = 4^{3(k-1)+2} - 2^{[3(k-1)+2]+1} - 1$ is not a prime, when $m = k - 1$ *and* $n = 3k - 1$

when $m = k$, $\zeta(\mu_n) = \zeta(n) = \zeta(3k + 1) = 4^{3k+2} - 2^{[3k+2]+1} - 1$

$$= 4^{[3(k-1)+2]+3} - 2^{[3(k-1)+2]+4} - 1$$
$$= (4^3)4^{[3(k-1)+2]} - (2^3)2^{[3(k-1)+2]+1} - 1$$
$$= (7m + 1)4^{[3(k-1)+2]} - (7m + 1)2^{[3(k-1)+2]+1} - 1$$
$$= 4^{[3(k-1)+2]} - 2^{[3(k-1)+2]+1} - 1$$

which is not a prime

Therefore, $\zeta(\mu_n)$ where $n \neq 3m + 2$ for all $m > 0$ is a prime Carol number.

Hence the theorem.

Conclusion:

In this paper the authors have studied the sum of squares of carol numbers and its matrix representation. Also, the sum of squares is expressed in terms of other special numbers. Similar study can be extended for other special numbers.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures in the manuscript are ours. Besides, the Figures and images, which are not ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Sanatana Dharma College Alappuzha.

Authors' contributions:

This work was carried out in collaboration between all authors. C.D wrote and edited the manuscript with new ideas. P.S reviewed the results with suggestions for corrections. All authors read and approved the final manuscript.

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[2]باحث، الكلية الوطنية، تيروتشيرابالي، الهند.

الخلاصة:
تعطينا مناقشة مجموع مربعات أعداد كارول المتتالية عدة نظريات في هذه الورقة. لإثبات هذه النظريات نستخدم تعريف أرقام كارول وطريقة الاستقراء. هنا يتم أيضا إعطاء شكل المصفوفة والشكل العودي لمجموع مربعات أرقام كارول المتتالية. يتم اشتقاق خصائص أرقام كارول."n""n".

الكلمات المفتاحية: أرقام كارول، أرقام كولين، فيبوناتشي، مجموع المربعات، أرقام وودال.