Sum of Squares of ‘m’ Consecutive Woodall Numbers

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Abstract:
This paper discusses the Sums of Squares of “m” consecutive Woodall Numbers. These discussions are made from the definition of Woodall numbers. Also learn the comparability of Woodall numbers and other special numbers. An attempt to communicate the formula for the sums of squares of ‘m’ Woodall numbers and its matrix form are discussed. Further, this study expresses some more correlations between Woodall numbers and other special numbers.

Keywords: Centered Polygonal Number, Gnomic Numbers, Mersenne Number, Sums of Squares, Sums of Squares of two Woodall Numbers, Woodall Numbers.

Introduction:
The general term of the Woodall number is given by the formula \(\omega(\tau) = \tau2^\tau - 1\). Duly, the first ten terms of this sequence are 1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239…⁴. Prodinger, H., and one other author derived a formula for the Sum of Squares of Tetranacci Number⁵. The Sums of Squares of Tribonacci Numbers and also Mersenne Numbers are studied by Soykan Y⁶–⁸. This article explains the generalization of sums of Woodall Numbers and its Matrix representation. These numbers were first discovered by Allan J.C. Cunningham and H.J. Woodall in 1917. Beginning with 63 and 159 each and every sixth Woodall number is a multiple of three. The indistinguishable identities for Pell, Lucas, Fibonacci, Jacobsthal, Jacobsthal − Lucas, and Tribonacci Numbers have recently been found by some authors in papers⁹,¹⁰. More related studies have been carried out¹¹–¹³. Motivated by some previous studies; this paper developed some new concepts and examined their properties.

Basic Definitions:
Definition. 1: For all \(\tau \geq 1\), the Woodall number is defined by the formula \(\omega(\tau) = \tau2^\tau - 1\).
Definition. 2: For all \(\tau \geq 1\), the Fermat number is defined as \((Fer)_\tau = 2^\tau + 1\).

Definition. 3: For all \(\tau \geq 1\), the general term of the Carol number is given by \((2\tau - 1)^2 - 2\).
Definition. 4: For all \(\tau \geq 1\), the Hex number is defined by the formula \((Hex)_\tau = 3\tau(\tau + 1) + 1\).
Definition. 5: For all \(\tau \geq 1\), the Hexagonal number is defined as \((H(6,\tau)) = (2\tau - 1)\).
Definition. 6: For all \(\tau \geq 1\), the Triangular number is defined by the formula \((Tri)_\tau = \tau(\tau + 1)\).
Definition. 7: For all \(\tau \geq 1\), the Cullan number is defined by the formula \(\omega(\tau) = \tau2^\tau + 1\).

Main theorem:
Sums of squares of ‘m’ consecutive Woodall numbers:
Theorem 1: For all \(\tau \geq 1\), the following equality hold,
\[
\sum_{l=1}^{m} \omega^2(\tau + l) = \frac{4}{3}[4^m - 1][\sum_{l=1}^{m}(A + 1)^2]X^2 - 4(2^m - 1)(A + 1)X + m \quad \text{where} \quad A = \tau + (l - 1) \quad \text{and} \quad X = 2^\tau.
\]

The following lemma is essential to execute the inductive pace in the Proof of the Main theorem.

Lemma. 1: For all \(\tau \geq 1\),
\[
\omega^2(\tau + l) = 4[(\tau + (l - 1)]^2 + 2[\tau + (l - 1) + 1])(2^2\tau) - 2^{l+1}(\tau + l)(2^\tau) + 1, \quad \text{where} \quad l = 1,2,3,\ldots m.
\]
Proof of the lemma: This lemma can be proved in two different methods, one is the direct method from the definition of Woodall number and the second one is the induction method on \( l = 1, 2, 3, \ldots m \).

**Method 1:** From the definition, \( \omega(\tau) = \tau 2^{\tau} - 1 \)

Therefore, \( \omega(\tau + 1) = (\tau + 1)2^{\tau+1} - 1 \)
\( \omega^2(\tau + 1) = [(\tau + 1)2^{\tau+1} - 1]^2 \)
\( \omega^2(\tau + 1) = 4(\tau^2 + 2\tau + 1)(2^{2\tau}) - 4(\tau + 1)(2^\tau) + 1 \)

Similarly, \( \omega^2(\tau + 2) = 16(\tau + 1)^2 + 2(\tau + 1) + 1)(2^{2\tau}) - 8(\tau + 2)(2^\tau) + 1 \)

And \( \omega^2(\tau + 3) = 64[(\tau + 2)^2 + 2(\tau + 2) + 1)(2^{2\tau}) - 16(\tau + 3)(2^\tau) + 1 \)

Finally, for \( l^{th} \) term, \( \omega^2(\tau + l) = 4^l[(\tau + (l - 1))^2 + 2[\tau + (l - 1) + 1)](2^{2\tau}) - 2^l(\tau + 1)(2^\tau) + 1 \)

**Method 2:**

Consider \( \omega^2(\tau + l) = 4^l[(\tau + (l - 1))^2 + 2[\tau + (l - 1) + 1)](2^{2\tau}) - 4l(\tau + l)(2^\tau) + 1 \)

For \( l = 1 \), from (1), \( \omega^2(\tau + 1) = 4(\tau^2 + 2\tau + 1)(2^\tau) + 1 \)

Therefore, by the induction hypothesis, the lemma is true for \( l = k \).

i.e., \( \omega^2[\tau + (k - 1)] = 4^{k-1}[(\tau + (k - 2))^2 + 2[\tau + (k - 1) + 1)](2^{2\tau}) - 4(k-1)(\tau + (k - 1))(2^\tau) + 1 \)

when \( l = k \), the lemma is proved by replacing \( k \) with \( k+1 \) on both sides of the above equation.

**Proof of main theorem:** From the lemma, \( \omega^2(\tau + l) = 4^l[(\tau + (l - 1))^2 + 2[\tau + (l - 1) + 1)](2^{2\tau}) - 4l(\tau + l)(2^\tau) + 1 \)

Secure summation \( l = 1, 2, 3, \ldots m \) on both sides, Eq.1 look right on.

\[
\sum_{l=1}^{m} \omega^2(\tau + l) = \sum_{l=1}^{m} 4^l[(\tau + (l - 1))^2 + 2[\tau + (l - 1) + 1)](2^{2\tau}) - \sum_{l=1}^{m} 4l(\tau + l)(2^\tau) + \sum_{l=1}^{m} 1
\]

\[
= \frac{4^m}{3}[4^m - 1] \text{ construct a geometric series with common ratio 4 and first term is 1. So, its summation leads to } \frac{4}{3}[4^m - 1]. 
\]

Also the summation \( \sum_{l=1}^{m} 4l \) construct the sum of first \( m \) natural numbers. Accordingly, that summation lead the way to \( 2m(m+1) \).

Finally, by presume \( A = [\tau + (l - 1)] \) and \( X = 2^\tau \), the above equation modified as

\[
\sum_{l=1}^{m} \omega^2(\tau + l) = \frac{4}{3}[4^m - 1] \sum_{l=1}^{m} (A + 1)^2 X^2 - 4(2^m - 1)(A + 1) X + m. 
\]

Hence the proof of main theorem.

**Corollary 1:** The sums of squares of Cullen number is

For all \( \tau \geq 1 \), \( \sum_{l=1}^{m} C \omega^2(\tau + l) = \frac{4}{3}[4^m - 1] \sum_{l=1}^{m} (A + 1)^2 X^2 - 4(2^m - 1)(A + 1) X + m \) where \( A = [\tau + (l - 1)] \), \( X = 2^\tau \)

**Sums of squares of \( m \) consecutive Woodall numbers in Matrix form.**

**Theorem 2:** For all \( \tau \geq 1 \) and \( l = 1, 2, 3, \ldots m \),
\[
W^2(\tau) = \begin{bmatrix}
E^{-2}(B-2)^2 & -(C-8)(B-2) & 1 & [X^2] \\
E^{-1}(B-1)^2 & -(C-4)(B-1) & 1 & X \\
EB^2 & -CB & 1 & 1 \\
\omega^2(B-2) & \omega^2(B) & \omega^2(B) & \omega^2(B)
\end{bmatrix}
\]
where \( E = 4^l \), \( B = (\tau + l) \), and \( C = (4l) \), and
\[
W^2(\tau) = \begin{bmatrix}
\omega^2(B-2) \\
\omega^2(B-1) \\
\omega^2(B) \\
\omega^2(B)
\end{bmatrix}
\]

**Proof:** By the definition, \( \omega(\tau) = \tau 2^\tau - 1 \)

Also, the square of Woodall number is \( \omega^2(\tau) = \tau^2(2^{2\tau}) - 2\tau(2^\tau) + 1 \)

In the similar way, \( \omega^2(\tau + 1) = 4(\tau + 1)^2(2^{2\tau}) - 4(\tau + 1)(2^\tau) + 1 \)

And \( \omega^2(\tau + 2) = 4^2(\tau + 2)^2(2^{2\tau}) - 8(\tau + 2)(2^\tau) + 1 \)

The above three equations can revise in matrix form as
\[
W^2_1(\tau) = D_1(\tau) Y(\tau), \text{ where }
W^2_1(\tau) = \begin{bmatrix}
\omega^2(\tau + 1) \\
\omega^2(\tau + 2) \\
\omega^2(\tau + 1) \\
\omega^2(\tau + 2)
\end{bmatrix}
D_1(\tau) = \begin{bmatrix}
\tau^2 & -2\tau & 1 \\
4(\tau + 1)^2 & -4(\tau + 1) & 1 \\
4^2(\tau + 2)^2 & -8(\tau + 2) & 1 \\
\end{bmatrix}
Y(\tau) = \begin{bmatrix}
(2^\tau) \\
(2^\tau) \\
(2^\tau) \\
(2^\tau)
\end{bmatrix}
\]

Indistinguishable \( W^2_2(\tau) \) and \( D_2(\tau) \) can be acquired by put back \( \tau \) by \( \tau + 1 \) in the respective matrix keeping \( Y(\tau) \) is fixed. In this fashion \( W^2_1(\tau) \) and \( D_1(\tau) \) gets the following matrix depiction as
\[
W^2_{l-1}(\tau) = D_{l-1}(\tau) Y(\tau) \text{ where } W^2_{l-1}(\tau) = \begin{bmatrix}
\omega^2(B + l - 2) \\
\omega^2(\tau + l - 1) \\
\omega^2(\tau + l) \\
\omega^2(\tau + l)
\end{bmatrix}
\]

\[
D_{l-1}(\tau) = \begin{bmatrix}
4^{l-2}(\tau + (l - 2))^2 & -4(l-2)(\tau + (l - 2)) & 1 \\
4^{l-1}(\tau + l - 1)^2 & -4(l-1)(\tau + l - 1) & 1 \\
4^l(\tau + l)^2 & -4l(\tau + l) & 1
\end{bmatrix}
\]

And \( Y(\tau) = \begin{bmatrix}
(2^\tau) \\
(2^\tau) \\
(2^\tau) \\
(2^\tau)
\end{bmatrix}
\]

By the appropriate substitution for \( E = 4^l \), \( B = (\tau + l) \) and \( C = (4l) \) the matrix representation reduced to \( W^2(\tau) = \begin{bmatrix}
E^{-2}(B-2)^2 & -(C-8)(B-2) & 1 & [X^2] \\
E^{-1}(B-1)^2 & -(C-4)(B-1) & 1 & X \\
EB^2 & -CB & 1 & 1 \\
\omega^2(B-2) & \omega^2(B) & \omega^2(B) & \omega^2(B)
\end{bmatrix}
\]

where \( W^2(\tau) = \begin{bmatrix}
\omega^2(B-2) \\
\omega^2(B-1) \\
\omega^2(B) \\
\omega^2(B)
\end{bmatrix}
\]
Hence the theorem.

**Corollary 2:** The matrix form of sums of squares of Cullan number is,

\[
Cu^2(τ) = 
\begin{bmatrix}
E^{-2}(B - 2)^2 & -(C - 8)(B - 2) & -1 \\
E^{-1}(B - 1)^2 & -(C - 4)(B - 1) & -1 \\
EB^2 & -CB & -1
\end{bmatrix}
[\begin{bmatrix}
X^2 \\
X \\
1
\end{bmatrix}]
\]

where \(Cu^2(τ) = \begin{bmatrix} cu^2(B - 2) \\ cu^2(B - 1) \end{bmatrix}\), \(E = 4^l\),
and \(C = (τ + l)\).

**Recurrent Matrix Form:**

**Theorem 3:** For all \(m \geq 1\) the recursive coefficient matrix get hold of the form

\[
W(m) = \begin{bmatrix}
F^2a_{11} & Fa_{12} & a_{13} \\
a_{21} + G & a_{22} - H & a_{23} \\
31 + 4[G + 2H] & a_{32} - 2H & a_{33}
\end{bmatrix}
\]

where \(F = m\), \(G = 4(m - 1)(3 + m)\) and \(H = 4(m - 1)\).

**Proof:** Consider the foremost coefficient matrix from \(D_1(τ)\) of theorem 2 as given below,

\[
W_1 = \begin{bmatrix}
1 & -2 & 1 \\
16 & -8 & 1 \\
144 & -24 & 1
\end{bmatrix}
\]

Let this matrix be denoted by

\[
W_1 = \begin{bmatrix}
1 & -2 & 1 \\
16 & -8 & 1 \\
144 & -24 & 1
\end{bmatrix}
\]

The components of succeeding order matrix \(W_2\) depends on the earlier order components in \(W_1\) excluding the components of last column.

\[
W_2 = \begin{bmatrix}
4 & -4 & 1 \\
36 & -12 & 1 \\
256 & -32 & 1
\end{bmatrix}
= \begin{bmatrix}
2^0a_{11} & 2a_{12} & a_{13} \\
a_{21} + 20 & a_{22} - 4 & a_{23} \\
a_{31} + 112 & a_{32} - 2(4) & a_{33}
\end{bmatrix}
\]

In this way the components of next order matrix \(W_3\) can be rewritten as follows,

\[
W_3 = \begin{bmatrix}
9 & -6 & 1 \\
64 & -16 & 1 \\
400 & -40 & 1
\end{bmatrix}
= \begin{bmatrix}
3^2a_{11} & 3a_{12} & a_{13} \\
a_{21} + 48 & a_{22} - 2(4) & a_{23} \\
a_{31} + 256 & a_{32} - 4(4) & a_{33}
\end{bmatrix}
\]

Finally, in general, the \(m^m\) order recursive coefficient matrix of sums of \(m\) consecutive Woodall numbers takes the form,

\[
W_m = \begin{bmatrix}
m^2a_{11} & ma_{12} & a_{13} \\
a_{21} + 4(m - 1)(3 + m) & a_{22} - 4(m - 1) & a_{23} \\
a_{31} + 16(m - 1)(5 + m) & a_{32} - 8(m - 1) & a_{33}
\end{bmatrix}
\]

By the suitable substitution like \(F = m\), \(G = 4(m - 1)(3 + m)\) and \(H = 4(m - 1)\) theorem concludes. Hence, \(W(m) = \begin{bmatrix} F^2a_{11} & Fa_{12} & a_{13} \\
a_{21} + G & a_{22} - H & a_{23} \\
a_{31} + 4[G + 2H] & a_{32} - 2H & a_{33}
\end{bmatrix}\)

**Corollary 3:** The recursive matrix form of sums of

\[
Cu(m) = \begin{bmatrix} F^2a_{11} & Fa_{12} & -a_{13} \\
a_{21} + G & a_{22} - H & -a_{23} \\
a_{31} + 4[G + 2H] & a_{32} - 2H & -a_{33}
\end{bmatrix}
\]

Where \(F = m\), \(G = 4(m - 1)(3 + m)\) and \(H = 4(m - 1)\).

**Applications:**

**Sums of squares of two Woodall numbers in terms of some other special numbers**

**Theorem 4:**

\[
\omega^2(τ + 1) + \omega^2(τ + 2) = \begin{bmatrix}
4(\text{carol})_τ + 8(\text{Fer})_τ \\
-4[\text{Hex})_τ + H(6, τ)] + (τ + 1)[64(\text{Fer})_τ]^2 - 140(\text{Fer})_τ + 76
\end{bmatrix}
\]

**Proof:** By the definition of Woodall number,

\[
\omega^2(τ + 1) + \omega^2(τ + 2) = 2^2[20(τ^2 + 2τ + 1) + 32(τ + 1) + 16] - 2^2[12τ + 12] + 2
\]

\[
\omega^2(τ + 1) + \omega^2(τ + 2) = 4[2^2(τ^2 + 18τ + 17)] - 12(2^2)[τ + 1] + 2
\]

\[
= 4(\text{X}^2)[\text{Hex})_τ + H(6, τ)] + (64X^2 - 12X)[τ + 1] + 2
\]

Where \(X = 2^τ\)

\[
= 4(\text{X}^2)[\text{Hex})_τ + H(6, τ)] + (64X^2 - 12X)[τ + 1] + 2
\]

Finally, in general, the \(m^m\) order recursive coefficient matrix of sums of \(m\) consecutive Woodall numbers takes the form,

\[
W_m = \begin{bmatrix} m^2a_{11} & ma_{12} & a_{13} \\
a_{21} + 4(m - 1)(3 + m) & a_{22} - 4(m - 1) & a_{23} \\
a_{31} + 16(m - 1)(5 + m) & a_{32} - 8(m - 1) & a_{33}
\end{bmatrix}
\]

Hence,

\[
\omega^2(τ + 1) + \omega^2(τ + 2) = \begin{bmatrix}
4(\text{carol})_τ + 8(\text{Fer})_τ \\
-4[\text{Hex})_τ + H(6, τ)] + (τ + 1)[64(\text{Fer})_τ]^2 - 140(\text{Fer})_τ + 76
\end{bmatrix}
\]
Sums of squares of three Woodall numbers in terms of some other special numbers

Theorem 5:
\[ \omega^2(\tau + 1) + \omega^2(\tau + 2) = \omega^2(\tau + 1) + \omega^2(\tau + 2) \]
\[ = [8C(5, \tau) + 25(Gno)_{\tau}] + 2(Tr)_{\tau} + (\tau + 85)][(Mer)_{\tau} + 1]^2 \]
\[ - 7(Gno)_{\tau} + 19][(Mer)_{\tau} + 1] + 3 \]

Proof: From Theorem 4,
\[ \omega^2(\tau + 1) + \omega^2(\tau + 2) = 2^{2\tau}[20(\tau^2 + 2\tau + 1) + 32(\tau + 1) + 16] - 2^r[12\tau + 12] + 2 \]
\[ \omega^2(\tau + 1) + \omega^2(\tau + 2) = 4(2^{2\tau})[5\tau^2 + 5\tau + 2] + (13\tau + 15) - 12(2^\tau)[\tau + 1] + 2 \]
\[ = 8(X^2)(C(5, \tau) + 4X^2[6(Gno)_{\tau} + (\tau + 1) + 20] - 12(X)(\tau + 1) + 2 \]
where \( X = 2^\tau \)

Adding the first term in the LHS of Eq.2,
\[ \omega^2(\tau) + \omega^2(\tau + 1) + \omega^2(\tau + 2) \]
\[ = 8[C(5, \tau) + 25(Gno)_{\tau}] + 2(Tr)_{\tau} + (\tau + 85)][X^2 \]
\[ - 7(Gno)_{\tau} + 19][X + 3 \]

It is easy to prove that, \( X = [(Mer)_{\tau} + 1] \).

Therefore, the above equation modified as follows,
\[ \omega^2(\tau) + \omega^2(\tau + 1) + \omega^2(\tau + 2) \]
\[ = [8C(5, \tau) + 25(Gno)_{\tau}] + 2(Tr)_{\tau} + (\tau + 85)][(Mer)_{\tau} + 1]^2 \]
\[ - 7(Gno)_{\tau} + 19][(Mer)_{\tau} + 1] + 3 \]

Hence the proof of the theorem.

Conclusions:
In this paper, authors have observed the sums of squares of Woodall Numbers and their matrix representation, which are exactly the Woodall Numbers and other specialization numbers. And it is interesting to see that the researcher can also proceed for further results in the problems. A similar study can be extended for other special numbers also.

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- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee at National College, India.

Authors’ contributions:
This work was carried out in collaboration between all authors. T.D. wrote and edited the manuscript with new ideas. P.S. reviewed the results with suggestions for corrections. All authors read and approved the final manuscript.

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مجموع مربعات "m" أرقام وودال المتتالية

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الخلاصة:
في هذه الورقة سوف نناقش مجموع مربعات "m" أرقام وودال المتتالية. يتم إجراء هذه المناقشات من تعريف أرقام وودال. نتعلم أيضا قابلية مقارنة أرقام وودال والأرقام الخاصة الأخرى. كما نحاول هنا توصيل صيغة مجموع مربعات أرقام وودال "m" وشكل المصفوفة الخاصة بها. علامة على ذلك، نعبر عن بعض الارتباطات بين أرقام وودال والأرقام الخاصة الأخرى.

الكلمات المفتاحية: رقم مضلع مركزي، أرقام جنومية، رقم ميرسين، مجموع المربعات، مجموع مربعات رقمين وودال، أرقام وودال.