Open Access
Baghdad Science Journal
P-ISSN: 2078-8665
2023, 20(1 Special Issue) ICAAM: 345-349
E-ISSN: 2411-7986

DOI: https://dx.doi.org/10.21123/bsj.2023.8409

Sum of Squares of 'm' Consecutive Woodall Numbers

P. Shanmuganandham [1] ***T. Deepika*** [1,2*]

[1]National College, Tiruchirappalli, India.
[2]Research Scholar, Affiliated to Bharadhidasan University, Tiruchirappalli, India.
*Corresponding author: deepikathangam92@gmail.com
E-mail address: thamaraishanmugam@yahoo.co.in
ICAAM= International Conference on Analysis and Applied Mathematics 2022.

Received 20/1/2023, Revised 11/2/2023, Accepted 12/2/2023, Published 1/3/2023

Abstract:

This paper discusses the Sums of Squares of "m" consecutive Woodall Numbers. These discussions are made from the definition of Woodall numbers. Also learn the comparability of Woodall numbers and other special numbers. An attempt to communicate the formula for the sums of squares of 'm' Woodall numbers and its matrix form are discussed. Further, this study expresses some more correlations between Woodall numbers and other special numbers.

Keywords: Centered Polygonal Number, Gnomic Numbers, Mersenne Number, Sums of Squares, Sums of Squares of two Woodall Numbers, Woodall Numbers.

Introduction:

The general term of the Woodall number is given by the formula $\omega(\tau) = \tau 2^{\tau} - 1$. Duly, the first ten terms of this sequence are 1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239…[1-4]. Prodinger, H., and one other author derived a formula for the Sum of Squares of Tetranacci Number[5]. The Sums of Squares of Tribonacci Numbers and also Mersenne Numbers are studied by Soykan Y[6-8]. This article explains the generalization of sums of squares of 'm' consecutive Woodall numbers and its Matrix representation. These numbers were first discovered by Allan J.C. Cunningham and H.J. Woodall in 1917. Beginning with 63 and 159 each and every sixth Woodall number is a multiple of three. The indistinguishable identities for Pell, Lucas, Fibonacci, Jacobsthal, Jacobsthal – Lucas, and Tribonacci Numbers have recently been found by some authors in papers[9,10]. More related studies have been carried out[11-14]Motivated by some previous studies; this paper developed some new concepts and examined their properties.

Basic Definitions:

Definition. 1: For all $\tau \geq 1$, the Woodall number is defined by the formula $\omega(\tau) = \tau 2^{\tau} - 1$.

Definition. 2: For all $\tau \geq 1$, the Fermat number is defined as $(Fer)_{\tau} = 2^{\tau} + 1$.

Definition. 3: For all $\tau \geq 1$, the general term of the Carol number is given by $(2^{\tau} - 1)^2 - 2$.

Definition. 4: For all $\tau \geq 1$, the Hex number is defined by the formula $(Hex)_{\tau} = 3\tau(\tau + 1) + 1$.

Definition. 5: For all $\tau \geq 1$, the Hexagonal number is defined as $H(6, \tau) = \tau(2\tau - 1)$.

Definition. 6: For all $\tau \geq 1$, the Triangular number is given by the formula $(Tri)_{\tau} = \frac{\tau(\tau+1)}{2}$

Definition. 7: For all $\tau \geq 1$, the Cullan number is defined by the formula $\omega(\tau) = \tau 2^{\tau} + 1$

Main theorem:

Sums of squares of 'm' consecutive Woodall numbers:

Theorem 1: For all $\tau \geq 1$, the following equality hold,

$\sum_{l=1}^{m} \omega^2(\tau + l) = \frac{4}{3}[4^m - 1]\{\sum_{l=1}^{m}(A + 1)^2\}X^2 - 4(2^m - 1)(A + 1)X + m$ where $A = \tau + (l - 1)$ and $X = 2^{\tau}$.

The following lemma is essential to execute the inductive pace in the Proof of the Main theorem.

Lemma. 1: For all $\tau \geq 1$,

$\omega^2(\tau + l) = 4^l\{[\tau + (l - 1)]^2 + 2[\tau + (l - 1) + 1]\}(2^{2\tau}) - 2^{l+1}(\tau + l)(2^{\tau}) + 1,$ where $l = 1,2,3, \dots m.$ (1)

Proof of the lemma: This lemma can be proved in two different methods, one is the direct method from the definition of Woodall number and the second one is the induction method on $l = 1,2,3,\dots m$.

Method. 1: From the definition, $\omega(\tau) = \tau 2^{\tau} - 1$

Therefore, $\omega(\tau+1) = (\tau+1)2^{\tau+1} - 1$

$\omega^2(\tau+1) = [(\tau+1)2^{\tau+1} - 1]^2$

$\omega^2(\tau+1) = 4(\tau^2 + 2\tau + 1)(2^{2\tau}) - 4(\tau+1)(2^{\tau}) + 1$

Similarly, $\omega^2(\tau+2) = 16[(\tau+1)^2 + 2(\tau+1) + 1](2^{2\tau}) - 8(\tau+2)(2^{\tau}) + 1$

And $\omega^2(\tau+3) = 64[(\tau+2)^2 + 2(\tau+2) + 1](2^{2\tau}) - 16(\tau+3)(2^{\tau}) + 1$

Finally, for l^{th}term, $\omega^2(\tau+l) = 4^l\{[\tau+(l-1)]^2 + 2[\tau+(l-1)+1]\}(2^{2\tau}) - 2^\wedge(l+1)(2^{\tau}) + 1$.

Method. 2:

Consider $\omega^2(\tau+l) = 4^l\{[\tau+(l-1)]^2 + 2[\tau+(l-1)+1]\}(2^{2\tau}) - 4l(\tau+l)(2^{\tau}) + 1$

For $l = 1$, from (1), $\omega^2(\tau+1) = 4(\tau^2+2\tau+1)(2^{2\tau}) - 4(\tau+1)(2^{\tau}) + 1$

Therefore, by the induction hypothesis, the lemma is true for $l = k-1$.

i.e., $\omega^2[\tau+(k-1)] = 4^{k-1}\{[\tau+(k-2)]^2 + 2[\tau+(k-1)]\}(2^{2\tau}) - 4(k-1)[\tau+(k-1)](2^{\tau}) + 1$

when $l = k$, the lemma is proved by replacing k with k+1 on both sides of the above equation.

Proof of main theorem: From the lemma,

$$\omega^2(\tau+l) = 4^l\{[\tau+(l-1)]^2 + 2[\tau+(l-1)+1]\}(2^{2\tau}) - 4l(\tau+l)(2^{\tau}) + 1$$

Secure summation $l = 1,2,3,\dots m$ on the both sides, Eq.1 look right on,

$\sum_{l=1}^{m}\omega^2(\tau+l) = \sum_{l=1}^{m}4^l\{[\tau+(l-1)]^2 + 2[\tau+(l-1)+1]\}(2^{2\tau}) - \sum_{l=1}^{m}4l(\tau+l)(2^{\tau}) + \sum_{l=1}^{m}1$

$\sum_{l=1}^{m}4^l$ constitute a geometric series with common ratio 4 and first term is 1. So, its summation leads to $\frac{4}{3}[4^m - 1]$. Also the summation $\sum_{l=1}^{m}4l$ construct the sum of first m natural numbers. Accordingly, that summation lead the way to $2m(m+1)$.

Finally, by presume $A = [\tau+(l-1)]$and $X = 2^{\tau}$, the above equation modified as

For all $\tau \geq 1$, $\sum_{l=1}^{m}\omega^2(\tau+l) = \frac{4}{3}[4^m - 1]\{\sum_{l=1}^{m}(A+1)^2\}X^2 - 4(2^m-1)(A+1)X + m.$

Hence the proof of main theorem.

Corollary 1: The sums of squares of Cullan number is,

For all $\tau \geq 1$, $\sum_{l=1}^{m}Cu^2(\tau+l) = \frac{4}{3}[4^m - 1]\{\sum_{l=1}^{m}(A+1)^2\}X^2 - 4(2^m-1)(A+1)X - m$

where$A = [\tau+(l-1)]$, $X = 2^{\tau}$

Sums of squares of 'm' consecutive Woodall numbers in Matrix form.

Theorem 2: For all $\tau \geq 1$, and $l = 1,2,3,\dots m$,

$$W^2(\tau) = \begin{bmatrix} E^{-2}(B-2)^2 & -(C-8)(B-2) & 1 \\ E^{-1}(B-1)^2 & -(C-4)(B-1) & 1 \\ EB^2 & -CB & 1 \end{bmatrix}\begin{bmatrix} X^2 \\ X \\ 1 \end{bmatrix}$$

where$E = 4^l$, $B = (\tau+l)$, and $C = (4l)$, and $W^2(\tau) = \begin{bmatrix} \omega^2(B-2) \\ \omega^2(B-1) \\ \omega^2(B) \end{bmatrix}$.

Proof: By the definition, $\omega(\tau) = \tau 2^{\tau} - 1$

Also, the square of Woodall number is $\omega^2(\tau) = \tau^2(2^{2\tau}) - 2\tau(2^{\tau}) + 1$

In the similar way, $\omega^2(\tau+1) = 4(\tau+1)^2(2^{2\tau}) - 4(\tau+1)(2^{\tau}) + 1$

And $\omega^2(\tau+2) = 4^2(\tau+2)^2(2^{2\tau}) - 8(\tau+2)(2^{\tau}) + 1$.

The above three equations can revise in matrix form as ${W_1}^2(\tau) = D_1(\tau)Y(\tau)$, where

${W_1}^2(\tau) = \begin{bmatrix} \omega^2(\tau) \\ \omega^2(\tau+1) \\ \omega^2(\tau+2) \end{bmatrix}$ $D_1(\tau) = \begin{bmatrix} \tau^2 & -2\tau & 1 \\ 4(\tau+1)^2 & -4(\tau+1) & 1 \\ 4^2(\tau+2)^2 & -8(\tau+2) & 1 \end{bmatrix}$ and $Y(\tau) = \begin{bmatrix} (2^{2\tau}) \\ (2^{\tau}) \\ 1 \end{bmatrix}$

Indistinguishable $W_2^2(\tau)\ and\ D_2(\tau)$can be acquired by put back τ by $\tau+1$ in the respective matrix keeping $Y(\tau)$ is fixed. In this fashion $W_{l-1}^2(\tau)\ and\ D_{l-1}(\tau)$ gets the following matrix depiction as

${W_{l-1}}^2(\tau) = D_{l-1}(\tau)Y(\tau)$ where ${W_{l-1}}^2(\tau) = \begin{bmatrix} \omega^2(\tau+l-2) \\ \omega^2(\tau+l-1) \\ \omega^2(\tau+l) \end{bmatrix}$

,$D_{l-1}(\tau) = \begin{bmatrix} 4^{l-2}[\tau+(l-2)]^2 & -4(l-2)[\tau+(l-2)] & 1 \\ 4^{l-1}(\tau+l-1)^2 & -4(l-1)(\tau+l-1) & 1 \\ 4^l(\tau+l)^2 & -4l(\tau+l) & 1 \end{bmatrix}$

and $Y(\tau) = \begin{bmatrix} (2^{2\tau}) \\ (2^{\tau}) \\ 1 \end{bmatrix}$

By the appropriate substitution for $E = 4^l$, $B = (\tau+l)$ and $C = (4l)$ the matrix representation reduced to $W^2(\tau) = \begin{bmatrix} E^{-2}(B-2)^2 & -(C-8)(B-2) & 1 \\ E^{-1}(B-1)^2 & -(C-4)(B-1) & 1 \\ EB^2 & -CB & 1 \end{bmatrix}\begin{bmatrix} X^2 \\ X \\ 1 \end{bmatrix}$

where $W^2(\tau) = \begin{bmatrix} \omega^2(B-2) \\ \omega^2(B-1) \\ \omega^2(B) \end{bmatrix}$

Hence the theorem.

Corollary 2: The matrix form of sums of squares of Cullan number is,

$$Cu^2(\tau) = \begin{bmatrix} E^{-2}(B-2)^2 & -(C-8)(B-2) & -1 \\ E^{-1}(B-1)^2 & -(C-4)(B-1) & -1 \\ EB^2 & -CB & -1 \end{bmatrix} \begin{bmatrix} X^2 \\ X \\ 1 \end{bmatrix}$$

where $Cu^2(\tau) = \begin{bmatrix} cu^2(B-2) \\ cu^2(B-1) \\ cu^2(B) \end{bmatrix}$, $E = 4^l$,

and $= (\tau + l)$, $C = (4l)$

Recursive Matrix form:

Theorem 3: For all $m \geq 1$ the recursive coefficient matrix get hold of the form

$$W(m) = \begin{bmatrix} F^2 a_{11} & F a_{12} & a_{13} \\ a_{21} + G & a_{22} - H & a_{23} \\ a_{31} + 4[G + 2H] & a_{32} - 2H & a_{33} \end{bmatrix}$$

where $F = m, G = 4(m-1)(3+m)$ and $H = 4(m-1)$.

Proof: Consider the foremost coefficient matrix from $D_1(\tau)$ of theorem 2 as given below,

$$W_1 = \begin{bmatrix} 1 & -2 & 1 \\ 16 & -8 & 1 \\ 144 & -24 & 1 \end{bmatrix}.$$

Let this matrix be denoted by

$$W_1 = \begin{bmatrix} 1 & -2 & 1 \\ 16 & -8 & 1 \\ 144 & -24 & 1 \end{bmatrix}$$

The components of succeeding order matrix W_2 depends on the earlier order components in W_1 excluding the components of last column.

$$W_2 = \begin{bmatrix} 4 & -4 & 1 \\ 36 & -12 & 1 \\ 256 & -32 & 1 \end{bmatrix} = \begin{bmatrix} 2^2 a_{11} & 2a_{12} & a_{13} \\ a_{21} + 20 & a_{22} - 4 & a_{23} \\ a_{31} + 112 & a_{32} - 2(4) & a_{33} \end{bmatrix}$$

In this way the components of next order matrix W_3 can be rewritten as follows,

$$W_3 = \begin{bmatrix} 9 & -6 & 1 \\ 64 & -16 & 1 \\ 400 & -40 & 1 \end{bmatrix} = \begin{bmatrix} 3^2 a_{11} & 3a_{12} & a_{13} \\ a_{21} + 48 & a_{22} - 2(4) & a_{23} \\ a_{31} + 256 & a_{32} - 4(4) & a_{33} \end{bmatrix}$$

Finally, in general, the m^{th} order recursive coefficient matrix of sums of m consecutive Woodall numbers takes the form,

$$W_m = \begin{bmatrix} m^2 a_{11} & m a_{12} & a_{13} \\ a_{21} + 4(m-1)(3+m) & a_{22} - 4(m-1) & a_{23} \\ a_{31} + 16(m-1)(5+m) & a_{32} - 8(m-1) & a_{33} \end{bmatrix}$$

By the suitable substitution like $F = m, G = 4(m-1)(3+m)$ and $H = 4(m-1)$ theorem concludes.

Hence, $W(m) =$

$$\begin{bmatrix} F^2 a_{11} & F a_{12} & a_{13} \\ a_{21} + G & a_{22} - H & a_{23} \\ a_{31} + 4[G + 2H] & a_{32} - 2H & a_{33} \end{bmatrix}$$

Corollary 3: The recursive matrix form of sums of squares of Cullan numbers is,

$$Cu(m) = \begin{bmatrix} F^2 a_{11} & F a_{12} & -a_{13} \\ a_{21} + G & a_{22} - H & -a_{23} \\ a_{31} + 4[G + 2H] & a_{32} - 2H & -a_{33} \end{bmatrix}$$

Where $F = m, G = 4(m-1)(3+m)$ and $H = 4(m-1)$

Applications:

Sums of squares of two Woodall numbers in terms of some other special numbers

Theorem 4:

$$\omega^2(\tau+1) + \omega^2(\tau+2) = [4(carol)_\tau + 8(Fer)_\tau - 4][(Hex)_\tau + H(6,\tau)] + (\tau + 1)[64(Fer)_\tau^2 - 140(Fer)_\tau + 76]$$

Proof: By the definition of Woodall number,

$$\omega^2(\tau+1) + \omega^2(\tau+2) = 2^{2\tau}[20(\tau^2 + 2\tau + 1) + 32(\tau + 1) + 16] - 2^\tau[12\tau + 12] + 2$$

$$\omega^2(\tau+1) + \omega^2(\tau+2) = 4(2^{2\tau})[5\tau^2 + 18\tau + 17] - 12(2^\tau)[\tau + 1] + 2$$

$$= 4(X^2)[(Hex)_\tau + H(6,\tau)] + (64X^2 - 12X)[\tau + 1] + 2$$

Where $X = 2^\tau$

$$= 4(X^2)[(Hex)_\tau + H(6,\tau)] + (64X^2 - 12X)[\tau + 1] + 2$$

$$= 4(X^2)[(Hex)_\tau + H(6,\tau)] + [64(Fer)_\tau^2 - 140(Fer)_\tau + 76](\tau + 1) + 2$$

Hence,

$$\omega^2(\tau+1) + \omega^2(\tau+2) = [4(carol)_\tau + 8(Fer)_\tau - 4][(Hex)_\tau + H(6,\tau)] + (\tau + 1)[64(Fer)_\tau^2 - 140(Fer)_\tau + 76]$$

Sums of squares of three Woodall numbers in terms of some other special numbers

Theorem 5:

$$\omega^2(\tau)+\omega^2(\tau+1)+\omega^2(\tau+2) = [8C(5,\tau)+25(Gno)_\tau + 2(Tri)_\tau + (\tau + 85)][(Mer)_\tau+1]^2 - [7(Gno)_\tau+19][(Mer)_\tau+1] + 3$$

Proof: From Theorem 4,

$$\omega^2(\tau+1)+\omega^2(\tau+2) = 2^{2\tau}[20(\tau^2+2\tau+1)+32(\tau + 1)+16]-2^\tau[12\tau+12]+2$$

$$\omega^2(\tau+1)+\omega^2(\tau+2) = 4(2^{2\tau})[(5\tau^2+5\tau+2)+(13\tau + 15)]-12(2^\tau)[\tau+1]+2$$

$$= 8(X^2)C(5,\tau)+4X^2[6(Gno)_\tau+(\tau+1)+20]-12(X)(\tau+1)+2 \quad 2$$

where $X = 2^\tau$

Adding the first term in the LHS of Eq.2,

$$\omega^2(\tau)+\omega^2(\tau+1)+\omega^2(\tau+2) = [8C(5,\tau)+25(Gno)_\tau + 2(Tri)_\tau+(\tau+85)]X^2 - [7(Gno)_\tau+19]X+3$$

It is easy to prove that, $X = [(Mer)_\tau + 1]$,

Therefore, the above equation modified as follows,

$$\omega^2(\tau)+\omega^2(\tau+1)+\omega^2(\tau+2) = [8C(5,\tau)+25(Gno)_\tau + 2(Tri)_\tau + (\tau + 85)][(Mer)_\tau+1]^2 - [7(Gno)_\tau+19][(Mer)_\tau+1] + 3$$

Hence the proof of the theorem.

Conclusions:

In this paper, authors have observed the sums of squares of Woodall Numbers and their matrix representation, which are exactly the Woodall Numbers and other specialization numbers. And it is interesting to see that the researcher can also proceed for further results in the problems. A similar study can be extended for other special numbers also.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee at National College, India.

Authors' contributions:

This work was carried out in collaboration between all authors. T.D. wrote and edited the manuscript with new ideas. P.S. reviewed the results with suggestions for corrections. All authors read and approved the final manuscript.

Reference:

1. Friedberg R. An Adventurer's Guide to Number Theory. USA: Dover Publications; 2012. p. 282.
2. Burton D. Elementary Number Theory. 7th edition. USA: McGraw Hill; 2010. p. 448.
3. Griffiths M. Sums of Squares of Integers. In Moreno CJ, Wagstaff SS. Chapman & Hall/CRC; 2006. p. 354. Math Gaz. Cambridge University Press. 2008; 92(524):377-379. https://doi.org/10.1017/S002555720018355X
4. Jackson TH. From Polynomials to Sums of Squares. USA: CRC Press; 1995. p. 194.
5. Prodinger H, Selkirk SJ. Sums of Squares of Tetranacci Numbers: A Generating Function Approach. arXiv preprint arXiv: 1906.08336. 2019 Jun 19; 1-6. https://arxiv.org/pdf/1906.08336
6. Wamiliana, Suharsono, Kristanto PE. Counting the Sum of Cubes for Lucas and Gibonacci Numbers. Sci Technol. Indonesia. 2019; 4(2): 31-35. https://doi.org//10.26554sti.2019.4.2.31-35
7. Soykan Y. A Closed Formula for the Sums of Squares of Generalized Tribonacci Numbers. J Progress Res Math. 2020; 16(2): 2932-2941. https://dergipark.org.tr/tr/download/article-file/1588784
8. Soykan Y. On the Sum of Squares of Generalized Mersenne Numbers: The Sum of formula $\sum_{k=0}^{n} x^k w^2$ m k + j. Int J Adv Appl Math Mech. 2021; 9(2): 28 – 37. http://www.ijaamm.com/uploads/2/1/4/8/21481830/v9n2p3_28-37.pdf
9. Johari MAM, Sapar SH, Zaini NA. Relation Between Sums of Squares and Sums of Centred Pentagonal Numbers Induced by Partitions of 8. Malaysian J Math Sci. 2021; 15(1): 21-31.
10. Soykan Y. Closed Formulas for the Sums of Squares of Generalized Fibonacci Numbers. Asian J Adv Res Rep. 2020; 9(1): 23-39. https://doi.org/10.9734/ajarr/2020/v9i130212
11. Soykan Y. On the Sums of Squares of Generalized Tribonacci Numbers: Closed Formulas of $\sum_{k=0}^{n} x^k W_k^2$. Arch Curr Res Int. 2020; 20(4): 22-47. https://doi.org/10.9734/acri/2020/v20i430187
12. Soykan Y. Generalized Fibonacci Numbers: Sum Formulas of the Squares of Terms. MathLAB J. 2020; 5(5): 46-62. http://www.purkh.com/index.php/mathlab
13. Adirasari RP, Suprajitno H, Susilowati L. The Dominant Metric Dimension of Corona Product Graphs. Baghdad Sci J. 2021; 18(2): 349-356. https://doi.org/10.21123/bsj.2021.18.2.0349
14. Hussein LH, Abed SS. Fixed Point Theorems in General Metric Space with an Application. Baghdad Sci J. 2021; 18(1(Suppl.)): 812-815. https://doi.org/10.21123/bsj.2021.18.1(Suppl.).0812

مجموع مربعات "m" أرقام وودال المتتالية

بي شانموغاناندهام[1] تي ديبيكا[2]

[1]الكلية الوطنية (التابعة لجامعة بهاراتيداسان) ، تيروتشيرابالي، الهند.
[2]الكلية الوطنية، تيروتشيرابالي، الهند

الخلاصة:
في هذه الورقة سوف نناقش مجموع مربعات "م" أرقام وودال المتتالية. يتم إجراء هذه المناقشات من تعريف أرقام وودال. نتعلم أيضا قابلية مقارنة أرقام وودال والأرقام الخاصة الأخرى. كما نحاول هنا توصيل صيغة مجموع مربعات أرقام وودال "m" وشكل المصفوفة الخاصة بها. علاوة على ذلك ، نعبر عن بعض الارتباطات بين أرقام وودال والأرقام الخاصة الأخرى.

الكلمات المفتاحية: رقم مضلع مركزي، أرقام جنومية، رقم ميرسين، مجموع المربعات، مجموع مربعات رقمين وودال، أرقام وودال.