Open Access **Baghdad Science Journal** **P-ISSN: 2078-8665**
2023, 20(1 Special Issue) ICAAM: 300-304 **E-ISSN: 2411-7986**

DOI: https://dx.doi.org/10.21123/bsj.2023.8417

Stability of Complement Degree Polynomial of Graphs

Safeera K* ***Anil Kumar V***

Department of Mathematics, University of Calicut, Kerala, India.
*Corresponding author: safeerakoralatil@gmail.com
E-mail address: anil@uoc.ac.in
ICAAM= International Conference on Analysis and Applied Mathematics 2022.

Received 21/1/2023, Revised 18/2/2023, Accepted 19/2/2023, Published 1/3/2023

Abstract:

A graph is a structure amounting to a set of objects in which some pairs of the objects are in some sense related. The objects correspond to mathematical abstractions called vertices (also called nodes or points) and each of the related pairs of vertices is called an edge (also called link or line). A directed graph is a graph in which edges have orientation. A simple graph is a graph that does not have more than one edge between any two vertices and no edge starts and ends at the same vertex. For a simple undirected graph G with order n, and let $\bar{G}$ denotes its complement. Let δ(G), Δ(G) denotes the minimum degree and maximum degree of G respectively. The complement degree polynomial of G is the polynomial CD[G,x]=$\sum_{\delta(\bar{G})}^{\Delta(\bar{G})} Cd_i$(G)xi , where Cd_i(G) is the cardinality of the set of vertices of degree i in $\bar{G}$. A multivariable polynomial f(x_1,...,x_n) with real coefficients is called stable if all of its roots lie in the open left half plane. In this paper, investigate the stability of complement degree polynomial of some graphs.

2020 Mathematics Subject Classification: 05C31, 05C50

Keywords: Complement degree polynomial, Complement of a graph, Degree of a vertex, Graph polynomial, Root of polynomial, Stability of polynomial.

Introduction:

For a simple undirected graph G of order n, CD(G,i) be the set of vertices of degree i in its complement graph $\bar{G}$ and Cd_i(G) =|CD(G,i)| [1]. Then complement degree polynomial of G is defined as CD[G,x]=$\sum_{\delta(\bar{G})}^{\Delta(\bar{G})} Cd_i$(G)xi , where δ(G), Δ(G) denotes respectively the minimum degree and maximum degree of G [2]. The complement degree polynomial of some graphs and some graph operations are investigated. A multivariable polynomial f(x_1,...,x_n) with real coefficients is called stable if all of its roots lie in the open left half plane [3]. The Hurwitz polynomial is the one variable variant of our concept. Schur polynomials are different sort of multivariable polynomials that have all of their roots in the open unit disc [4]. In this paper, study the stability of complement degree polynomial of some graphs. Let $f(x) = x^n+b_1x^{n-1} + \dots + b_{n-1}x + b_0$ be a polynomial with real coefficients [5]. For i=1,2,…,n define the n Hurwitz matrices as follows:

$$H_1 = [b_1]$$

$$H_2 = \begin{bmatrix} b1 & 1 \\ b3 & b2 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} b1 & 1 & 0 \\ b3 & b2 & b1 \\ b5 & b4 & b3 \end{bmatrix}$$

$$\vdots$$

$$H_n = \begin{bmatrix} b1 & 1 & 0 & 0 & \cdots & 0 \\ b3 & b2 & b1 & 1 & \cdots & 0 \\ b5 & b4 & b3 & b2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & bn \end{bmatrix}$$

$$\vdots$$

where b_j=0 if j>n. Then all the roots of the polynomial f(x) are negative or have negative real part if and only if det(H_j) > 0, j=1, 2, n.

Main Results:

Theorem 1: If G is a regular graph, then CD[G,x] is stable.

Proof: Let G be a r-regular graph, then CD[G,x]=nx^{n-1-r} . Note that x=0 is the only root of this polynomial and hence CD[G,x] is stable.

Corollary 1:
(1) If $n \geq 3$, then CD[C_n,x] is stable,
(2) CD[P,x] is stable, where P is the Peterson graph,
(3) CD[$K_{n,n}$,x] is stable,
(4) CD[Cr_n,x] is stable, where Cr_n is the crown graph,
(5) CD[B_n,x] is stable, where B_n is the bipartite cocktail graph,
(6) CD[CL_n,x] is stable, where CL_n is the circular ladder graph,
(7) CD[ML_n,x] is stable, where ML_n is the Mobius ladder graph.

Corollary 2: If S(G) is the splitting graph of a regular graph G, then CD[S(G),x] is stable

Theorem 2: If CS(G) is a cosplitting graph of r-regular graph G, then CD[CS(G),x] is stable if and only if r=1.

Proof: Let G be a r-regular graph. Note that CD[CS(G),x]= $nx^{n-1}(1+x^r)$. Observe that CD[CS(G),x] is stable if r =1. Also note that when r =2, CD[CS(G),x] is not stable. For $r \geq 2$, this polynomial has real roots and complex roots with positive and negative real parts. Hence CD[CS(G),x] is not stable. A plot of roots of x^{50}+1 in the complex plane is shown in Fig. 1. □

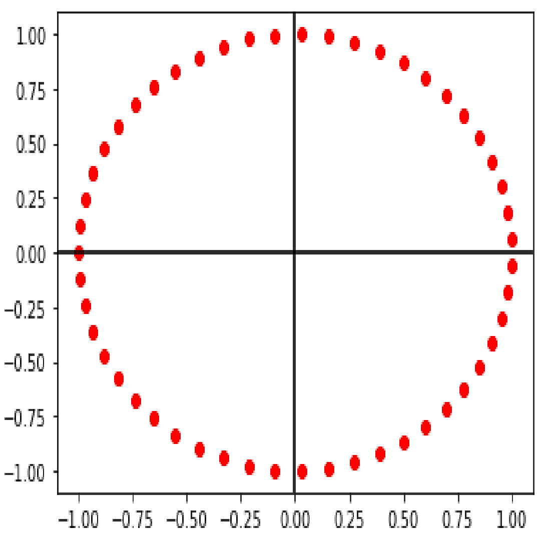

Figure 1. Plot of roots of $x^{50} + 1$ in the complex plane

Theorem. 3: Let G be a graph of order n, then CD[G,x] is stable if and only if CD[mG,x] is stable.

Proof: Note that CD[mG,x] = mCD[G,x]. This implies that CD[G,x] is stable if and only if CD[mG,x] is stable. □

Theorem. 4: Let G be a graph with order n and **G** = G∪G∪ ⋯ ∪G (m times). Then CD[G,x] is stable if and only if CD[**G**,x] is stable.

Proof: Let G be a graph with order n and **G** = G∪G∪ ⋯ ∪G (m times), then CD[**G**,x] = $mx^{(m-1)n}$CD[G,x]. This implies that CD[G,x] is stable if and only if CD[**G**,x] is stable. □

Theorem. 5: If G′ is a graph obtained by duplication of a vertex of regular graph G. Then CD[G′,x] is stable.

Proof: Let G be the r-regular graph. Note that CD[G′,x] = $(n+1-r)x^{n-r}+rx^{n-r-1}$. Note that the roots of CD[G′,x] are x = 0 and x = $-r/(n+1-r)$ which lie in the open left half plane, hence the result follows. □

Corollary. 3: If $K_{n,n}'$ is a graph obtained by duplication of a vertex of complete bipartite graph $K_{n,n}$. Then $K_{n,n}'$ is stable.

Theorem. 6: For a path P_n ($n \geq 2$), CD[P_n,x] is stable unless n =2.

Proof: The authors proved that

$$\text{CD}[P_n,x]= \begin{cases} 2x^{n-2} & n \leq 2 \\ (n-2)x^{n-3}+2x^{n-2} & n \geq 3 \end{cases}$$

If $n \neq 2$, CD[P_n,x]= $(n-2)x^{n-3}+2x^{n-2}=x^{n-3}(n-2+2x)$. In this case x = 0 and x = $-(n-2)/2$ are the roots of CD[P_n,x] which lie in the left half plane, it follows that CD[P_n,x] is stable. If n =2, CD[P_2,x] = 2. Hence CD[P_2,x] is not stable. □

Theorem. 7: If L_n is the ladder graph with $n \geq 2$ vertices, then CD[L_n,x] is stable.

Proof: Note that CD[L_n,x] = $4x^{2n-3}+ (2n-4)\, x^{2n-4}$ = $x^{2n-4}(4x+2n-4)$. Observe that the roots of CD[L_n,x] are x=0, $(n-2)/2$ which lie in the open left half plane. Hence the result follows. □

Theorem. 8: Let K_n' be the graph obtained by the duplication of one of the vertices of the complete graph K_n with $n \geq 2$, then CD[K_n',x] is stable.

Proof: Note that CD[K_n',x] = 2x + n-1. Also observe that CD[K_n',x] has a single root x = $-(n-1)/2$ Which lie in the open left half plane and hence the result follows. □

Theorem. 9: If $T_{m,n}$ is a tadpole graph with $m \geq 3$ and $n \geq 1$ vertices, then CD[$T_{m,n}$,x] is stable.

Proof: Note that $CD[T_{m,n},x] = x^{m+n-4}(x^2 + (m + n-2) x +1)$. The roots of $x^2 + (m + n-2) x +1$ are $(-(m+n-2) \pm \sqrt{(m+n-2)^2-4})/2$. For $m \geq 3$ and $n > 1$, observe that $(m+n-2) > 0$, $(m +n - 2) - 4 > 0$ and and $\sqrt{(m+n-2)^2-4} - (m+n-2) < 0$, the result follows. □

Theorem. 10: If $A(Q_n)$ is an alternating quadrilateral snake graph with $n \geq 3$ vertices, then $CD[A(Q_n),x]$ is stable.
Proof: The authors proved that,
$$CD[A(Q_n),x]= \begin{cases} (n-2)x^{2n-5}+nx^{2n-4}+x^{2n-3} & if\ n\ is\ odd \\ (n-2)x^{2n-4}+(n+2)x^{2n-3} & if\ n\ is\ even \end{cases}$$
Case (i) (If n is odd): In this case $CD[A(Q_n),x]= x^{2n-5}(x^2+ nx + n -2)$. The roots of $x^2+ nx + n-2$ are $(-n\pm.\sqrt{n^2-4(n-2)})/2$ Since $\sqrt{n^2-4(n-2)} < n,$ it follows that $CD[A(Q_n)$ is stable.
Case(ii) (If n is even): In this case, $CD[A(Q_n),x] = x^{2n-4} (n-2 + (n + 2)x)$. Observe that the roots of $CD[A(Q_n),x]$ are $x =0$ and $x= -(n-2)/(n+2)$ which lie in the open left half plane. Hence for n is even, $CD[A(Q_n),x]$ is stable.
This completes the proof. □
Theorem. 11: If CP_n is a cocktail party graph with $n \geq 2,$ then $CD[CP_n,x]$ is stable.
Proof: Note that $CD[CP_n,x] = 4x^2 + 2(n-2) x^3 = 2x^2(2 + (n-2) x)$. Then the roots of $CD[CP_n,x]$ are $x= 0$ and $x= -2/(n-2)$ which lie in the open left half plane. Hence the proof. □

Theorem. 12: If $n \geq 3$, G_n is the gear graph of order 2n+1, then $CD[G_n,x]$ is stable if and only if n=3,4.
Proof: Note that $CD[G_n,x] = nx^{2n-2}+ nx^{2n-3}+ x^n = x^n (nx^{n-2} + nx^{n-3} +1)$. When n=3, the roots of $CD[G_3,x]$ are x=0 and $x= -4/3$. When n = 4, the roots of $CD[G_4,x]$ are x= 0 and $x=1/2$. Hence $CD[G_n,x]$ is stable for n=3,4. When n > 4, the result follows from the fact that the determinant of second Hurwitz matrix of $CD[G_n,x]$ is zero or negative. □

Theorem. 13: If Sh_n denotes the shell graph then $CD[Sh_n,x]$ is stable if and only if n=4,5.
Proof: Note that when $n \geq 4$, $CD[Sh_n,x] = 2x^{n-3} + (n-3) x^{n-4} +1$. For n = 4,5 the result is trivial. When n >5, consider the polynomial $x^{n-3}+ ((n-3)/2) x^{n-4} + (1/2)$. Observe that second Hurwitz matrix is $|H_2| = 0$. Hence the result follows from the fact that the determinant of second Hurwitz matrix is zero. □

Theorem. 14: If $L_{n,n}$ is the lollipop graph, then $CD[L_{m,n},x]$ is stable if and only if n = 2,3,4.

Proof: Note that $CD[L_{n,n},x] = x^{2n-2} + (n-1) x^{2n-3} + (n-1) x^n + x^{n-1}$. For n=1, $CD[L_{1,1},x] = 2$ is a constant polynomial and for n=2, the roots of $CD[L_{2,2},x]$ are x=0,-1, for n =3, the roots of $CD[L_{3,3},x]$ are $(-4 \pm \sqrt{12})/2$ and for n=4, the roots of $CD[L_{4,4},x]$ are x=0,-1. Thus, for n=2,3,4, $CD[L_{n,n},x]$ is stable. When n=5, $CD[L_{5,5},x] = x^8 + 4x^7 + 4x^5+x^4$,the determinant of Hurwitz matrices are $|H_1|=4$ and $|H_2|= - 4$. It follows that $CD[L_{5,5},x]$ is not stable. When n> 5 the result follows from the fact that the determinant of second Hurwitz matrix of $CD[L_{n,n},x]$ is zero. □

Theorem. 15: If $F_{1,n}$ is the fan graph ($n\geq 3$), then $CD[F_{1,n},x]$ is stable if and only if n= 3,4.
Proof: Note that $CD[F_{1,n},x] = 2x^{n-2}+ (n-2) x^{n-3}+ 1$. The result is trivial for n=3 and n=4. When n > 4, the result follows from the fact that the determinant of second Hurwitz matrix of $CD[F_{1,n},x]$ is zero. □

Theorem. 16: If $F_{2,n}$ ($n \geq 3$) is the double fan graph, then $CD[F_{2,n},x]$ is stable if and only if n=3,4,5.
Proof: Note that $CD[F_{2,n},x] = 2x^{n-2} + (n-2) x^{n-3}+ 2x$. For n= 3,4 the result follows from elementary algebra. For n=5, the roots of $CD[F_{2,5},x]$ are x =0 and $x= (-3 \pm i\sqrt{7})/4$ which lie in the open left half plane. It follows that $CD[F_{2,5},x]$ is stable. When n > 5, the result follows from the fact that the determinant of second Hurwitz matrix of $CD[F_{2,n},x]$ is zero. □

Theorem. 17: For an armed crown graph $C_n \odot P_m$ ($n \geq 3$, $m \geq 1$), $CD[C_n \odot P_m ,x]$ is stable unless m =1.
Proof: Note that $CD[C_n \odot P_m ,x] = nx^{n(m+1)-4}(x^2 + (m-1)x +1)$. Observe that Hurwitz matrices of the polynomial $x^2+(m-1)x+1$ are $|H_1|= m-1$ and $|H_2|= m-1$. Since all the determinants are positive except when m =1, the result follows. □

Theorem. 18: If Bk_n is the book graph with 2n+2 vertices, then $CD[Bk_n,x]$ is stable if and only if n =1,2.
Proof: Note that $CD[Bk_n,x] = 2nx^{2n-1} + 2x^n$. When n=1, $Cd[Bk_1,x] = 4x$ and when n = 2, $CD[Bk_2,x]$
Are lie in the open left half plane, It follows that for n=1,2, $CD[Bk_n,x]$ is stable. When n > 2, the result follows from the fact that the determinant of first Hurwitz matrix of the polynomial $x^{2n-1}+ (1/n)x^n$ are zero.□

Theorem. 19: If Bl is the bull graph, then $CD[Bl,x]$ is stable.
Proof: Note that $CD[Bl, x] = 2x^3 + x^2+2x = x(2x^2 + x+2)$ and its roots are x=0, $(-1 \pm i\sqrt{15})/4$, which

lie in the open left half plane. Hence CD[Bl,x] is stable. □

Theorem. 20: If Fr is the fork graph, then CD[Fr,x] is stable.
Proof: Obviously, CD[Fr,x] = $3x^3 + x^2 + x= x(3x^2 + x+1)$. The roots of CD[Fr,x] are x=0, $(-1 \pm i\sqrt{11})/6$. Hence CD[Fr,x] is stable. □

Theorem. 21: If TL_n is the triangular ladder graph, then CD[TL_n,x] is stable for n ≥ 2.
Proof: Note that CD[TL_n,x] = $2x^{2n-3} + 2x^{2n-4} + (2n-4)x^{2n-5} = 2x^{2n-5}(x^2 + x + n -2)$. For n =2, the result follows from simple elementary algebra. For n > 2, consider the polynomial x^2 + x+ n-2. The determinants of Hurwitz matrices are $|H_1|$= 1 and $|H_2|$+ n-2 and so on. Since all the determinants are positive when n > 2, the result follows. □

Theorem. 22: If DSL_n is the double-sided step ladder graph, then CD[DSL:$_n$,x] is stable.
Proof: Note that CD[DSL_n,x]= $(2n+2)x^{3n-3+n^2}$+$+(2n-2)x^{3n-4+n^2}$ + $+(n^2-n)x^{3n-5+n^2}$

$= x^{3n-5+n^2}$ ((2n + 2)x^2 + (2n-2)x + n^2 – n).

For n=1, the result follows from elementary algebra. For n > 1, consider the polynomial x^2+$((2n-2)/(2n+2))$ x+ $(n^2-n)/(2n+2)$. Observe that the determinants of Hurwitz matrices are

$|H_1|$= $(2n-2)/(2n+2)$ and $|H_2|$=$((2n-2)(n^2-2))/(2n+2)^2$. Since all the determinants are positive when n > 1, it follows that CD[DSL_n,x] is stable. □

Conclusion:

In this paper, the stability of complement degree polynomial of some graphs has been discussed. The primary goal of this paper is to launch research on the stability of complement degree polynomials in graphs.

Acknowledgment:

The author is indebted to the Council of Scientific and Industrial Research (CSIR), Government of India for granting and timely disbursement of the research fellowship.

Author's Declaration:

- Conflict of interest: None
- We hereby confirm that all the Figures in the manuscript are ours. Besides, the Figures and images, which are not ours, have been given permission for the re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Calicut.

Author's contribution statement:

This work was carried out in collaboration between all authors. S K evaluated the stability of complement degree polynomial of some graphs and verified the results. A K V edited the manuscript with revisions idea. All authors read and approved the final manuscript.

References:

1. Omran AA, Oda HH. Hn-Domination in Graphs. Baghdad Sci J. 2019; 16(1): 242-247. https://doi.org/10.21123/bsj.2019.16.1(Suppl.).0242
2. Safeera K, Kumar VA. Vertex cut polynomial of graphs. Adv Appl Discrete Math. 2022; 32(1): 1-12. https://ds.doi.org/10.17654/0974165822028
3. Juma ARS, Abdulhussain MS, Al-khafaji SN. Faber Polynomial Coefficient Estimates for Subclass of Analytic Bi-Bazilevic Functions Defined by Differential Operator. Baghdad Sci J. 2019; 16(1(Suppl.)): 248-253. https://doi.org/10.21123/bsj.2019.16.1(Suppl.).0248
4. Shikhi M, Kumar AV. On the stability of common neighbor polynomial of some graphs. South East Asian J Math. Math Sci. 2018; 14(1): 95-102.
5. Fuhrmann PA. A polynomial approach to linear algebra. Springer-Berlin Heidelberg. New York; 1996. 422 p.

ثبات للمتمم متعدد حدود الدرجة للرسومات البيانية

سفيرة ك.* **أنيل كومار**

قسم الرياضيات، جامعة كاليكوت، كيرالا، الهند.

الخلاصة:
البيان عبارة عن هيكل يصل إلى مجموعة من الكائنات ترتبط فيها بعض أزواج الكائنات ببعض المعاني. تتوافق الكائنات مع التجريدات الرياضية التي تسمى الرؤوس (وتسمى أيضًا العقد أو النقاط) ويسمى كل زوج من الأزواج ذات الصلة بالحافة (وتسمى أيضًا الرابط أو الخط). البيان الموجه هو رسم بياني يكون للحواف فيه اتجاه. الرسم البياني البسيط هو رسم لا يحتوي على أكثر من حافة بين أي رأسين ولا تبدأ أي حافة وتنتهي عند نفس الرأس. للبيان البسيط غير موجه **G** بالترتيب **n**، وليكن $\bar{G}$ تدل على مكمله. ليكن **δ (G)**، **Δ (G)** تشير إلى الحد الأدنى للدرجة والحد الأقصى لدرجة **G** على التوالي. متعددة حدود الدرجة المتممة لـ **G** هي متعددة الحدود CD[G,x] $=\sum_{\delta(\bar{G})}^{\Delta(\bar{G})} Cd_i(G)x^i$ حيث ان CD[G,x] هي أصل مجموعة من القمم من الدرجة **i** في $\bar{G}$. تسمى متعدد المتغيرات $f(x_1,\dots,x_n)$ مع معاملات حقيقية مستقرة إذا كانت جميع جذورها تقع في مستوى النصف الأيسر المفتوح. في هذا البحث، تحقق من استقراريه متعددة حدود الدرجة المتممة لبعض البيانات.

الكلمات المفتاحية: متمم متعدد حدود الدرجة، متمم الرسم البياني، درجة الرأس، رسم بياني متعدد الحدود، جذر متعدد الحدود، ثبات متعدد الحدود.