Odd Fibonacci edge irregular labeling for some trees obtained from subdivision and vertex identification operations

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Abstract:

Let G be a graph with p vertices and q edges and f: V(G) → {0,1,2, ..., k} be an injective function, where k is a positive integer. If the induced edge labeling f*: E(G) → \{F_2, F_4, F_5, F_7, F_8, F_{10}, ..., F_{q+\left\lfloor \frac{q}{2} \right\rfloor +1}\} defined by f*(uv) = f(u) + f(v), for each uv ∈ E(G), is a bijection, then the labeling f is called an odd Fibonacci edge irregular labeling of G. A graph which admits an odd Fibonacci edge irregular labeling is called an odd Fibonacci edge irregular graph. The odd Fibonacci edge irregularity strength ofes(G) is the minimum k for which G admits an odd Fibonacci edge irregular labeling. In this paper, the odd Fibonacci edge irregularity strength for some subdivision graphs and graphs obtained from vertex identification is determined.

Keywords: Edge irregular labeling, Edge irregularity strength, Irregular labeling, Odd Fibonacci edge irregular labeling, Odd Fibonacci sequence.

Introduction:

By a graph, it means a finite undirected graph without loops or multiple edges with p vertices and q edges. Graph labeling is an assignment of integers to the vertices or edges or both. Rosa introduced the concept of graceful labeling. The Fibonacci numbers can be defined by the linear recurrence F_n = F_{n-1}+F_{n-2}, n ≥3. This generates an infinite sequence of integers F_1=1, F_2=1, F_3=2, F_4=3, F_5=5, F_6=8, F_7=13, etc. The concept of Fibonacci graceful graphs, total edge Fibonacci irregular labeling, and odd Fibonacci mean labeling based on Fibonacci numbers are studied in. Also, some results related to irregular total labeling are obtained from. An odd Fibonacci edge irregular labeling has been introduced in as follows. An odd Fibonacci edge irregular labeling (OFEIL) which is an injective function f: V(G) → {0,1,2, ..., k}, k is a positive integer if the induced edge labeling f*: E(G) → \{F_2, F_4, F_5, F_7, F_8, F_{10}, ..., F_{q+\left\lfloor \frac{q}{2} \right\rfloor +1}\} defined by f*(uv) = f(u) + f(v), for each uv ∈ E(G), is a bijection. If such labeling exists, then G is called an odd Fibonacci edge irregular graph (OFEIG) and the minimum possible k is called the odd Fibonacci edge irregularity strength ofes(G).

Fig. 1 illustrates ofes.

![Graph G with two OFEIL and ofes(G) = 212](image-url)

Figure 1. A graph G with two OFEIL and ofes(G) = 212
In the odd Fibonacci edge irregularity strength for the graphs path, star, caterpillar, and bistar graph have been found and the nonexistence of an odd Fibonacci edge irregular labeling for the complete graph and the complete bipartite graph has been discussed. If each edge of a graph $G$ is broken into two by exactly one vertex, then the resultant graph is $S(G)$, the subdivision graph of $G$. In this paper, the odd Fibonacci edge irregularity strength for some subdivision graphs and graphs obtained from vertex identification is determined.

**Theorem 1:** Every path $P_n$, $n \geq 2$ is an odd Fibonacci irregular graph and

$$\text{ofes}(P_n) = \begin{cases} \frac{1}{2} F_{2n-2}, & \text{if } n \text{ is even} \\ F_{\frac{n+3}{2}} - 1, & \text{if } n \text{ is odd.} \end{cases}$$

**Theorem 2:** Every star graph $K_{1,n}$, $n \geq 1$ is an odd Fibonacci irregular graph and

$$\text{ofes}(K_{1,n}) = F_{\frac{q+3}{2}} - 1$$

**Main Results:**

**Theorem 3:** The subdivision of $P_n, P_n \Box K_1$, $n \geq 2$ is an OFEIG and its strength is $F_{3n}$.

**Proof:** Let $G = S(P_n, P_n \Box K_1)$. In $G$, $q = 4n - 2$. Let $V(G) = \{v_1, v_2, \ldots, v_n, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n\}$ and $E(G) = \{v_iu_i / 1 \leq i \leq n\} \cup \{u_iw_i / 2 \leq i \leq n\} \cup \{w_i / 1 \leq i \leq n\}$.

Define $f(V(G)) = \{0, 1, 2, 3, \ldots, F_{6n-3}\}$ as follows:

$f(v_i) = 0$, $f(w_i) = 1$, $f(u_i) = F_{6n-4}$, $f(x_j) = F_{6n-3}$, $f(u_i)$ is labeled by $F_{6n-4}$ or $F_{6n-3}$. Then $f^*$ is obtained as follows:

$f^*(v_i) = F_{6n-4}$, $f^*(w_i) = F_{6n-3}$, $f^*(u_i) = F_{6n-4} - (F_5 + F_7 + F_9 + \ldots + F_{6n-3})$, $2 \leq i \leq n$.

Since $q$ is even, the last two odd Fibonacci numbers are not consecutive Fibonacci numbers. In order to obtain the minimum value for $k$, the Fibonacci numbers $F_{q+3}/2 \pm 1$ and $F_{q+3}/2 - 1$ are to be obtained in the adjacent edges. If $F_{q+3}/2 \pm 1$ is an edge label of a non-pendant edge, then one adjacent edge of it will be received $F_{q+3}/2 - 1$ as an edge label but the other adjacent edge cannot have any odd Fibonacci number. Hence $f(x_i) = F_{6n-3}$ is the required minimum value for $k$. □

**Theorem 4:** The subdivision of $K_{1,n}$, $n \geq 1$ is an OFEIG and its strength is $F_{3n}$.

**Proof:** Let $G = S(K_{1,n})$. In $G$, $q = 2n$. Let $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $E(G) = \{v_i / 1 \leq i \leq n\} \cup \{u_i / 1 \leq i \leq n\}$.

For $n=1$, the subdivision of $K_{1,1}$ is a path $P_3$ and by Theorem 1, ofes($P_3$) = $F_3$.

For $n>1$, define $f(V(G)) = \{0, 1, 2, 3, \ldots, F_{3n}\}$ as follows:

$f(v_1) = 0$, $f(v_2) = F_{3n-1}$, $f(u_i) = F_{3n}$, $f(v_j) = F_{i+1} + 1$, $1 \leq i \leq n-1$ and $f(u_i) = F_{n+i} - F_{n+i+1} - F_{n+i+2}$, $1 \leq i \leq n-1$.

Then $f^*$ is obtained as follows:

$f^*(v_i) = F_{3n-1}$, $f^*(w_i) = F_{3n+1}$, $f^*(w_i) = F_{i+1} + 1$, $1 \leq i \leq n-1$ and $f^*(v_i) = F_{n+i+1} - F_{n+i+2}$, $1 \leq i \leq n-1$.

To obtain $F_3$ as an edge label, it is necessary to assign 0 and 1 to a pair of adjacent vertices.

**Case (i)** Suppose 0 (or 1) is assigned to the pendant vertex then 1 (or 0) is to be assigned as a label of a vertex whose degree is 2. Therefore, it is necessary to assign 2 (or 3) to the central vertex in order to obtain the edge label $F_4$. Then the remaining $n-1$ vertices which are adjacent to central vertices can be assigned by the labels $F_{i+1} + 1$ and $F_{i+2}$, $2 \leq i \leq n-1$. Hence to obtain the edge label $F_{3n-1}$, the pendant vertex can be labeled by $F_{3n-1} + 2$ (or $F_{3n-1} + 3$), $3 \leq i \leq n-1$.

Suppose one of the adjacent vertices of the central vertex can be labeled with $F_{3n-1} - 2$ (or $F_{3n-1} - 3$) then to obtain the edge label $F_{3n-1}$ the pendant vertex can be labeled by $F_{3n-1} + 2$ (or $F_{3n-1} + 3$).

**Case (ii)** If 0 (or 1) is assigned to the central vertex then its $n$ number of adjacent vertices can be assigned by the labels $F_{i+1} + 1$ (or $F_{i+2} + 1$), $1 \leq i \leq n$ in order to obtain the minimum label. Hence to obtain the edge label $F_{3n-1}$ the pendant vertex can be labeled by $F_{3n-1} + 2$ (or $F_{3n-1} + 3$), $3 \leq i \leq n-1$.

The Coconut Tree $T(n, m)$ was obtained by identifying the central vertex of $K_{1,m}$ with a pendant vertex of a path $P_n^{11}$.

**Theorem 5:** The graph $T(n, m)$, $n \geq 2$, $m \geq 1$ is an OFEIG and its strength is
Proof: Let G = T(n,m). In G, q = n+m−1. Let V(G) = \{v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{m}\} and E(G) = \{v_{i}v_{i+1}/ 1 \leq i \leq n-1\} ∪ \{v_{n}v_{n+1}/ 1 \leq i \leq m\}.

For n=2, graph G is a star K_{1,n} and its strength is given in Theorem 2. For n=1, G is a path P_n and its strength is given in Theorem 1.

Let n ≥ 3, m ≥ 2. To obtain F_2 as an edge label, it is necessary to assign 0 and 1 to a pair of adjacent vertices.

Case (i) n+m is odd (i.e q is even)
Let f(v_k) = 0 and f(v_{k+1}) = 1.
Define f: V(G) → \{0,1,2, \ldots, F_{n+m+\frac{n+m-1}{2}}\} − F_{n+m+\frac{n+m-1}{2}} as follows:

\[
f(v_1) = F_{n+m} + \frac{n+m-1}{2} - 2
\]

\[
f(v_3) = \begin{cases} \frac{3}{2}F_{3i-5}, & 3 \leq i \leq n \text{ and i is odd} \\ \frac{3}{2}F_{3i-8}, & 4 \leq i \leq n \text{ and i is even} \end{cases}
\]

and

\[
f(v_{n+i}) = \begin{cases} F_{n+i} + \frac{n+i-1}{2} - i, & 1 \leq i \leq m \text{ and n is odd} \\ F_{n+i} + \frac{n+i-1}{2} - i, & 1 \leq i \leq m \text{ and n is even} \end{cases}
\]

Then f is obtained as follows:

f*(v_1v_2) = F_{n+m+\frac{n+m-1}{2}},

f*(v_2v_3) = F_{n+m+\frac{n+m-1}{2}}.

Sub Case (i) n, m is even.
Define f: V(G) → \{0,1,2, \ldots, \ell\},

where \ell = \sum_{i=1}^{n-1} F_{q+\frac{q}{2}-3i+2} + F_{q+\frac{q}{3}-3\frac{n-3}{2}} - 1

Then f is defined as follows:

f*(v_1v_{i+1}) = \sum_{i=1}^{n-1} F_{q+\frac{q}{2}-3i+2} + F_{q+\frac{q}{3}-3\frac{n-3}{2}} - 1

f*(v_{n+i}v_{i+1}) = F_{\frac{1}{2}i^2} + 1, 1 \leq i \leq m.

Sub Case (ii) n, m is odd & n ≥ 5.
Define f: V(G) → \{0,1,2, \ldots, \ell\},

where \ell = \sum_{i=1}^{n-1} F_{q+\frac{q}{2}-3i+2} + F_{q+\frac{q}{3}-3\frac{n-3}{2}} - 1

Then f is defined as follows:

f*(v_1v_{i+1}) = \sum_{i=1}^{n-1} F_{q+\frac{q}{2}-3i+2} + F_{q+\frac{q}{3}-3\frac{n-3}{2}} - 1

f*(v_{n+i}v_{i+1}) = F_{\frac{1}{2}i^2} + 1, 1 \leq i \leq m.

Sub Case (iii) n = 3 & m is odd.
Define f: V(G) → \{0,1,2, \ldots, \ell\},

where \ell = \sum_{i=1}^{n-1} F_{q+\frac{q}{2}-3i+2} + F_{q+\frac{q}{3}-3\frac{n-3}{2}} - 1

Then f is defined as follows:

f*(v_1v_{i+1}) = \sum_{i=1}^{n-1} F_{q+\frac{q}{2}-3i+2} + F_{q+\frac{q}{3}-3\frac{n-3}{2}} - 1

f*(v_{n+i}v_{i+1}) = F_{\frac{1}{2}i^2} + 1, 1 \leq i \leq m.
Then \( f^* \) is defined as follows:

\[
f^*(v_1, v_2) = F_{q+1}, \quad f^*(v_2, v_3) = F_{q+1}, \quad f^*(v_3, v_{3i}) = F_{1+1}, \quad 1 \leq i \leq m.
\]

Since \( q \) is odd, the last two odd Fibonacci numbers are consecutive Fibonacci numbers. So case (i) labeling does not give the minimum value for \( k \). Hence by case (ii), if both \( n \) and \( m \) are even, \( f(v_1) \) is the required minimum value for \( k \). If both \( n \) (\( n \geq 5 \)) and \( m \) are odd, \( f(v_2) \) is the required minimum value for \( k \). If \( n = 3 \) and \( m \) is odd, \( f(v_3) \) is the required minimum value for \( k \).

Let \( K_{1, m_1}, K_{1, m_2}, \ldots, K_{1, m_n} \) be the \( n \) number of star graphs. Then graph \( T(K_{1, m_1}, K_{1, m_2}, \ldots, K_{1, m_n}) \) is obtained by identifying an end vertex of each of the \( K_{1, m_i} \) stars, \( 1 \leq i \leq n \). Let the identified vertex be \( u \).

**Theorem 6:** \( T(K_{1, m_1}, K_{1, m_2}, \ldots, K_{1, m_n}), (n \geq 2, m_i \geq 2, 1 \leq i \leq n) \) is an OFEIG and its strength is

\[
F = \sum_{i=1}^{n} m_i \left[ \frac{\varphi_{i+1} m_i + 1}{2} \right] + 2.\]

**Proof:** Let \( G = T(K_{1, m_1}, K_{1, m_2}, \ldots, K_{1, m_n}) \). In \( G \), \( q = m_1 + m_2 + \ldots + m_n \). Then \( V(G) = \{v_{1,0}/1 \leq i \leq n\} \cup \{v_{i,1}/1 \leq i \leq n, 1 \leq j \leq m_i - 1\} \cup \{u\} \) and \( E(G) = \{v_{1,0}/1 \leq i \leq n\} \cup \{v_{i,0}v_{i,j}/1 \leq i \leq n, 1 \leq j \leq m_i - 1\} \).

Define \( f: V(G) \to \{0,1,2,\ldots,F, \sum_{i=1}^{n} m_i + \left[ \frac{\varphi_{i+1} m_i + 1}{2} \right] + 2 \} \) as follows:

\[
f(V_{1,0}) = 1, f(u) = 0, \quad f(v_{1,i}) = F_{\sum_{i=1}^{n} m_i + \left[ \frac{\varphi_{i+1} m_i + 1}{2} \right] + 2} - f(v_{1,0}), 2 \leq i \leq n, 1 \leq j \leq m_i - 1.
\]

Then \( f^* \) is obtained as follows:

\[
f^*(uv_{1,0}) = 1, \quad f^*(uv_{1,0}) = F_{\sum_{i=1}^{n} m_i + \left[ \frac{\varphi_{i+1} m_i + 1}{2} \right] + 2}, 2 \leq i \leq n.
\]

In order to obtain the minimum value of \( k \), the last \( m_i \) odd Fibonacci numbers are to be given for the edge labels of the pendant edge attached at \( v_{n,0} \) and \( q + \frac{a}{2} - m_n \), the greatest odd Fibonacci number. Suppose 0 is assigned to the central vertex of any one of the stars. Then 1 is assigned either to the pendant vertex or the identified vertex \( u \). If 1 is assigned to the pendant vertex then in order to obtain the minimum value of \( k \), the successive odd Fibonacci labeling \( F_q, F_{q+3}, F_{q+7}, \ldots, F_{m_1 + \left[ \frac{m_1}{2} \right] + 1} \) are obtained in the first \( m_1 \) pendant edges of the star \( K_{1, m_n} \). Hence the identified vertex \( u \) can get the label \( F_{m_1 + \left[ \frac{m_1}{2} \right] + 1} \). Depending upon this \( u \), the central vertex of the star \( K_{1, m_n} \) can get the label. The last \( m_i \) odd Fibonacci numbers are to be given for the edge labels incident at \( v_{n,0} \) and out of them \( F_q + \frac{a}{2} - m_n \) is the odd Fibonacci number is the edge label of \( uv_{n,0} \). If \( f(u) \) is non-zero say \( x \) then \( f(v_{n,0}) \) is less than \( F_q + \frac{a}{2} - m_n \). This induces the value of \( k \) as

\[
F_q + \frac{a}{2} - m_n < F_q + \frac{a}{2} - m_n + x.\]

This will be the minimum for \( x = 0 \). So ofes(G) = \( F_{\sum_{i=1}^{n} m_i + \left[ \frac{\varphi_{i+1} m_i + 1}{2} \right] + 1} - F_{\sum_{i=1}^{n} m_i + \left[ \frac{\varphi_{i+1} m_i + 1}{2} \right] + 2}.\]

**Theorem 7:** \( P_n OmK_1, (n \geq 2, m \geq 2) \) is an OFEIG and its strength is

\[
F_q + \frac{a}{2} + 1 - F_{q+3} + \frac{a}{2} - 2m, \quad \text{if } m \text{ is even},
\]

\[
F_q + \frac{a}{2} + 1 - F_{q+3} + \frac{a}{2} - 2m, \quad \text{if } m \text{ is odd}.
\]

**Proof:** Let \( G = P_n OmK_1 \). In \( G \), \( q = (m+1)-1 \).

Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \cup \{u_{1,1}, u_{1,2}, \ldots, u_{1,m} / 1 \leq i \leq n\} \) and \( E(G) = \{v_{i,j}/1 \leq i \leq n-1\} \cup \{v_{i,j}/1 \leq i \leq n, 1 \leq j \leq m_i \}.\)

Define \( f: V(G) \to \{0,1,2,\ldots, \ell\} \) as follows:

\[
f(v_1) = \left\{ \begin{array}{ll}
F_q + \frac{a}{2} - 2m, & \text{if } m \text{ is even} \\
F_q + \frac{a}{2} - 2m, & \text{if } m \text{ is odd}
\end{array} \right.
\]

\[
f(v_1) = \left\{ \begin{array}{ll}
F_q + \frac{a}{2} - 2m, & \text{if } m \text{ is even} \\
F_q + \frac{a}{2} - 2m, & \text{if } m \text{ is odd}
\end{array} \right.
\]

Then \( f^* \) is obtained as follows:

\[
f^*(uv_{1,0}) = 1, \quad f^*(uv_{1,0}) = F_{\sum_{i=1}^{n} m_i + \left[ \frac{\varphi_{i+1} m_i + 1}{2} \right] + 2}, 2 \leq i \leq n.
\]
Theorem 8: \( P(1, 2, \ldots, n) \), \( n \geq 2 \) is an OFEIG and its strength is
\[
\begin{cases} 
  f_{q + \left[ n/2 \right]} & \text{if } n \equiv 2, 3 \pmod{4} \\
  f_{q + \left[ n/2 \right] - 5} & \text{if } n \equiv 0, 1 \pmod{4}
\end{cases}
\]

Proof: Let \( G = P(1, 2, \ldots, n) \).

In \( G, q = n - \left\lfloor \frac{n+1}{2} \right\rfloor \).

Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \cup \{u_i / 1 \leq i \leq n, 1 \leq t \leq i\} \) and \( E(G) = \{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{v_i u_{i+1} / 1 \leq i \leq n, 1 \leq t \leq i\} \).

Case (i) \( n \equiv 2, 3 \pmod{4} \)(i.e q is even).

Define \( f: V(G) \to \{0, 1, 2, \ldots, F_{q + \left[ n/2 \right]}\} \) as follows:
\[ f(v_1) = F_{q + \left[ n/2 \right]} \quad \text{and} \quad f(v_2) = 0, \quad f(u_{i+1}) = F_{q + \left[ n/2 \right]} \]
\[ f(v_i) = \sum_{t=1}^{i-2} (-1)^{i-t} F_{\left\lfloor t+\left(\frac{i-j-2}{2}\right) \right\rfloor} + 1, \quad 3 \leq i \leq n, f(u_{i+1}) = F_{q + \left[ n/2 \right]} + 1 \]

Then \( f^* \) is defined as follows:
\[ f^*(v_1 v_2) = F_{q + \left[ n/2 \right] - 1}, \quad f^*(v_1 u_{i+1}) = F_{q + \left[ n/2 \right] + 1}, \quad f^*(v_2 u_{i+1}) = F_{q + \left[ n/2 \right] + 1}, 1 \leq t \leq 2 \]
\[ f^*(v_1 v_{i+1}) = F_{\sum_{t=1}^{i+1} t+\left(\frac{i-j-2}{2}\right) + 1}, 2 \leq i \leq n-1 \]
\[ f^*(v_1 u_{i+1}) = F_{\sum_{t=1}^{i+1} t+\left(\frac{i-j+3}{2}\right) + 1}, 3 \leq i \leq n-1 \]

Since \( q \) is odd, the last two odd Fibonacci numbers are consecutive Fibonacci numbers. So case (i) labeling does not give the minimum value for \( k \). If 0 is a vertex label of the path, then the value of \( k \) is
more than $F_{q + \lfloor \frac{n}{2} \rfloor} - F_{q + \lfloor \frac{n}{2} \rfloor - 5}$ in order to obtain the OFEIL. Hence $\text{ofes}(G) = F_{q + \lfloor \frac{n}{2} \rfloor} - F_{q + \lfloor \frac{n}{2} \rfloor - 5}$. □

**Conclusion:**
In this paper, the odd Fibonacci edge irregularity strength for some subdivision graphs and graphs obtained from vertex identification is determined.

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- We hereby confirm that all the Figures in the manuscript are ours. Besides, the Figures and images, which are not ours, have been given permission for the re-publication attached with the manuscript.
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**Author’s contributions statement:**
M.UD analyzes the existence of OFEIL by collecting the papers, proposing the conjecture, and trying to prove it. M.K edited the manuscript. S.A read and approved the final manuscript by verifying it.

**References:**
تبويب حافات فيبوناتشي الفردية غير المنتظمة لبعض الابراج المستحصلة من عمليات التقسيم الفرعي

ولاقيفة الرأس

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الخلاصة:

ليكن $G$ رسم بياني بروسي $p$ وحواف $q$ دالة متباينة وشاملة، حيث $k$ عدد صحيح موجب. إذا كانت $f: V(G) \rightarrow \{0,1,2,...,k\}$ دالة متباينة وشاملة، فإن علامة التبويب $f$ تعطي وضع علامات غير منتظمة على حافة فيبوناتشي الفردية لـ $G$. الرسم البياني الذي يعترف بمتباينة $f$ في لحافة فيبوناتشي الفردية يسمى الرسم البياني غير المنتظم لحافة فيبوناتشي الفردية. قوة عدم انتظام حافة فيبوناتشي الفردية هي الحد الأدنى $k$ الذي يعترف فينوناتشي الفردية بعض الرسوم البيانية لتقييمات الفرعية والرسوم البيانية التي تم الحصول عليها من تدجين الرأس.

كلمات المفتاحية: تبويب حافات غير منتظمة، قوة عدم انتظام الحافة، وضع العلامات غير النظامية، تبويب حافات فيبوناتشي الفردية غير المستقلة.