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Generalised Henstock - Kurzweil Integral with Multiple Point

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Abstract:
This paper deals with a new Henstock-Kurzweil integral in Banach Space with Bilinear triple n-tuple and integrator function Ψ which depends on multiple points in partition. Finally, exhibit standard results of Generalized Henstock - Kurzweil integral in the theory of integration.

Keywords: Bilinear n tuple, Banach space, Henstock - Kurzweil Integral, Integral of Riemann, Stieltjes Integral.

Introduction:

In [1] A.G. Das and Sahu firstly introduce the definition of the generalized Henstock - Stieltjes integral, which integral also known as the HS_k integral. HS_k Integral includes the LS_k integral it shows in [2] by Sandhya Bhattacharya and A.G. Das. In the Literature, many moderations and generalizations of Riemann integral , Lebesgue integral, Denjoy - Perron integral , Stieltjes integral, Mc-shane integral and HK integral by number of authors [3-6]. By taking the motivation from that all previous literature [7 - 9], and by study of the Henstock- Kurzweil integral with multiple points and taking the integrator function Ψ in Banach space of Bilinear for n-tuples. Purpose of this paper is to introduce the Generalized HK integral that is GH_k which is the countable extension of Generalized Stieltjes integral. In this paper generalized the Henstock -Kurzweil integral $\int_p^q f d\psi$. Here took $f:[p,q]^{k+1} \to X^n$ and $\psi:[p,q]^{k+1} \to Y^n$ where, $k>0$ and X,Y are the Banach Spaces. Let p and q be fixed real numbers and k is positive integer, $f:[p,q]^{k+1} \to \mathbb{R}$ and for $k+1$ distinct points$\{\alpha_{0,}\, \alpha_1, \dots, \alpha_k\} \in [p,q]$. The function f is defined such that

$$Z_k\left(f;\ \alpha_0, \alpha_1, \dots, \alpha_k\right) = \sum_{i=0}^{k} \frac{f(\alpha_i)}{\prod_{i=0, j\neq i}^{k}(\alpha_i - \alpha_j)}.$$

f is called k - convex on $[p,q]$ if $Z_k\left(f;\ \alpha_{0,}\, \alpha_1, \dots, \alpha_k\right) \geq 0$ for all $k+1$ distinct points like $(\alpha_{0,}\, \alpha_1, \dots, \alpha_k)$ which is belongs to $[p,q]$. Purpose of this paper is to introduce the Generalized HK integral that is GH_k with bilinear map for n - tuple which is the countable extension of Generalized Stieltjes integral.

Some Definition and Elementary Properties

This section is about the study bilinear map for n – tuple and given norm on that set,

Definition 1: Bilinear Map with n Dimension

A bilinear mapping B is a mapping which is linear with respect to n – variables. A bilinear triple is a set of three Banach spaces X, Y, Z with a bilinear $B : X^n \times Y^n \to Z^n$ and define $\alpha \cdot \beta = B(\alpha, \beta)$ for $\alpha \in X^n$ and $\beta \in Y^n$ where $\alpha = (\alpha_{1,}\, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$. $B = (B, X, Y, Z)$ formed by three Banach Spaces X, Y, Z with n tuple and a bilinear map.

$B : X^n \times Y^n \to Z^n$ and denote norm on the Banach space X by $\|\cdot\|_{X^n}$ similarly can be denote $\|\cdot\|_{Y^n}$ and $\|\cdot\|_{Z^n}$ throughout all this paper consider $\|f\,\Psi\|_{Z^n} \leq \|f\|_{X^n} \cdot \|\Psi\|_{Y^n}$, where $f \in X^n$ and $\Psi \in Y^n$, where; $X^n = \{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n / \alpha \in X)\}$ and $Y^n = \{\beta = (\beta_1, \beta_2, \dots, \beta_n / \beta \in Y)\}$.

Definition 2: Generalise Riemann – Stieltjes Sum

Consider that B is a bilinear map for n – tuple. Let $f:[p,q]^{k+1} \rightarrow X^n$ and $\Psi:[p,q]^{k+1} \rightarrow Y^n$ be two Banach space – valued functions, f is a function depend on Ψ is a function of $(k+1)$ variables then the generalise Riemann – Stieltjes sum of f with respect to Ψ, for a given ξ fine partition. $\tau = \{ (\delta_i)[\alpha_i, \alpha_{i+1}]\}_{i=1}^{\tau}$ is defined as

$$S(f, \triangle \Psi; \tau) = \sum_{i=0}^{\tau-k} f(\delta_i)\, \psi\,(\alpha_i, \alpha_{i+1}, \dots, \alpha_{i+k})$$

where $[\alpha_i, \alpha_{i+1}], i = 1,2,\dots,\tau$ be a non-overlapping pairwise intervals and $\bigcup_{i=0}^{\tau}[\alpha_i, \alpha_{i+1}] \subset [p,q]$, the $\{\tau_i\}_{i=0}^{\tau}$ is said that ξ^k – fine sub – partition of $[p,q]$ for each τ_i is ξ^k – fine partition of $[\alpha_i, \alpha_{i+1}]$ corresponding Riemann – Stieltjes sum is defined by $\sum_{i=0}^{\tau} S(f, \triangle \Psi; \tau)$.

Consider p and q be fixed real numbers and $p < q$, let k be a fixed positive integer. Assume that any system of points in $[p,q]$ such that $\alpha_{1,0} < \alpha_{1,1} < \cdots < \alpha_{1,k} \leq \alpha_{2,0} < \alpha_{2,0} < \alpha_{2,1} < \cdots < \alpha_{2,k} \leq \cdots \leq \alpha_{n,0} < \alpha_{n,1} < \cdots < \alpha_{n,k}$ simply the intervals $[\alpha_{i,0}, \alpha_{i,k}], i = 1,2,\dots,n$ from given system of element. $\{(\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,k-1}) : [\alpha_{i,0}, \alpha_{i,k}], i = 1,2,\dots,n\}$ in $[p,q]$. If every $I = [\alpha_{i,0}, \alpha_{i,k}]$ tagged to $\delta_i \in [\alpha_{i,0}, \alpha_{i,k}]$ with interior points $\alpha_{i,1} < \alpha_{i,2} < \cdots < \alpha_{i,k-1}$. This system is called tagged elementary k – system and written as $\{(\delta_i, \alpha_{i,1},\ \alpha_{i,2}, \dots,\ \alpha_{i,k-1}) : [\alpha_{i,0}, \alpha_{i,k}], i = 1,2,\dots,n\}$.

Definition 3: Generalise Henstock – Kurzweil Integral of Bilinear n – Tuple Map

Consider that B is a bilinear map for n – tuple. Let $f:[p,q]^{k+1} \rightarrow X^n$ and $\Psi:[p,q]^{k+1} \rightarrow Y^n$ be two Banach space – valued functions, can say that f is generalised HK integrable with respect to Ψ on $[p,q]$ if there exist a gauge on ξ on $[p,q]$ such that $\|S(f, \triangle \Psi; \tau) - A\|_{Z^n} < \epsilon$ for any ξ- fine tagged k – partition. $\tau = \{(\delta_i, \alpha_{i,1},\ \alpha_{i,2}, \dots, \alpha_{i,k-1}) : [\alpha_{i,0}, \alpha_{i,k}], i = 1,2,\dots,n\}$ where

$$S(f, \triangle \Psi; \tau) = \sum_{i=1}^{n} f(\delta_i)\Psi(\delta_i, \alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,k}) - \Psi(\delta_i, \alpha_{i,0}, \alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,k-1})$$

The integral $A \in X^n$ is called the GH_k integral of f on $[p,q]$ with respect to Ψ and $(f,\Psi) \in GH_k\,([p,q] : Z^n)$ and the value of the integral is denoted by $(GH_k)\int_p^q f d\psi = A$.

Remark 1: If $f:[p,q]^{k+1} \rightarrow X^n$, $f = (f_1, f_2, \dots, f_n)$ and $\Psi:[p,q]^{k+1} \rightarrow Y^n$, $\Psi = (\Psi_1, \Psi_2, \dots \Psi_n)$ is generalise HK integrable if and only if each element $\Psi_m, f_m\ m = 1,2,\dots,n$ is generalised HK integrable.

Theorem 1: Consider that B is a bilinear map for n – tuple of the function $f:[p,q]^{k+1} \rightarrow X^n$ is GH_k integrable with respect to $\Psi:[p,q]^{k+1} \rightarrow Y^n$ then the integral is unique.

Proof: By considering above Remark is acceptable for function $\Psi:[p,q]^{k+1} \rightarrow Y^n$, define U be set of all $S \in Y^n$ which are in $\mathbb{R}$. In $[p,q]$ assume τ_0 is random ξ^k partition. In $[p,q]$ considering theory for each fine ξ^k partition and write

$S(f, \triangle \Psi, \tau_0) - \epsilon < S(f, \Psi; \tau) < S(f, \triangle \Psi, \tau_0) + \epsilon$ therefore $(-\infty, S(f, \triangle \Psi, \tau_0) - \epsilon) \subset U \subset (-\infty, S(f, \triangle \Psi, \tau_0) + \epsilon)$ and set of U is non-empty and bounded above, consequently $SupU$ exists and $(S(f, \triangle \Psi; \tau_0) - \epsilon) \subset SupU \subset (S(f, \triangle \Psi; \tau_0) + \epsilon)$ so that every tagged ξ – fine k – partition τ of $[p,q]$

$$\|S(f, \triangle \Psi, \tau) - SupU\|_{Z^n} \leq \|S(f, \triangle \Psi, \tau) - S(f, \triangle \Psi, \tau_0)\|_{Z^n} + \|S(f, \triangle \Psi, \tau_0) - SupU\|_{Z^n} < 2\epsilon$$

Hence $f, \Psi \in GH_k([p,q], Z^n)$ and $GH_k\ \int_p^q f d\Psi = SupU$.

Theorem 2: If $f, \Psi \in GH_k([p,q], Z^n)$ then for every $[r,t] \subset [p,q], f, \Psi \in GH_k([r,t], Z^n)$.

Proof: Assume any two ξ – fine tagged partitionsτ_1, τ_2 of $[r,t]$,

$$\tau_j = \{(\delta_{i,1}^j, \alpha_{i,1}^j, \dots, \alpha_{i,k-1}^j) : [\alpha_{i,0}^j, \alpha_{i,k}^j], i = 0,1,2,\dots,n_j\} = r\alpha_{njk}^j = t$$

for $j = 1,2$, consider $p < r < t < q$. Let $\tau_3 = \{(\delta_{i,1}^3, \alpha_{i,1}^3, \dots, \alpha_{i,k-1}^3) : [\alpha_{i,0}^3, \alpha_{i,k}^3], i = 0,1,2,\dots,n_j\}\alpha_{1,0}^3 = p\alpha_{m,k}^3 = r$, ξ^k – partition of $[p,r]$, so $\tau_4 = \{(\delta_{i,1}^4, \alpha_{i,1}^4, \dots, \alpha_{i,k-1}^4) : [\alpha_{i,0}^4, \alpha_{i,k}^4], i = 0,1,2,\dots,n_j\}\alpha_{1,0}^4 = t\alpha_{p,k}^4 = q$, be ξ – fine tagged partition of $[t,q]$. Clearly the union $\tau_3 \cup \tau_1 \cup \tau_4$ personify ξ – fine tagged k – partition τ_1' of $[p,q]$ on the same path $\tau_3 \cup \tau_2 \cup \tau_4$ personify ξ – fine tagged k – partition τ_2' of $[p,q]$, by using Theorem 2

$\|S(f, \triangle \Psi, \tau_1') - S(f, \triangle \Psi, \tau_2')\|_{Z^n} < \epsilon$,

then,

$$\|S(f, \triangle \Psi, \tau_1) - S(f, \triangle \Psi, \tau_2)\|_{Z^n} = \|S(f, \triangle \Psi, \tau_1') - S(f, \triangle \Psi, \tau_2')\|_{Z^n} < \epsilon$$

therefore by (Theorem 1) $f, \Psi \in GH_k([p,q], Z^n)$.

Theorem 3: Assume B is bilinear map for n tuple, if $f,\Psi \in GH_k([p,q],Z^n)$ and if $p<r<q$ then $f,\Psi \in GH_k([p,r],Z^n)$, $f,\Psi \in GH_k([r,q],Z^n)$ and

$$GH_k\int_p^q f d\Psi = GH_k\int_p^r f d\Psi = GH_k\int_r^q f d\Psi.$$

Proof: By considering above theorem $f,\Psi \in GH_k([p,r],Z^n)$ and $f,\Psi \in GH_k([r,q],Z^n)$ for $\epsilon>0$ define a positive function $\xi:[p,q]\to(0,\infty)$ such that for every ξ – fine tagged k – partition $\tau_1$of $[p,q]$ and τ_2 of $[r,q]$ and as a consequence $\tau = \tau_1 \cup \tau_2$ of $[p,q$,

$$\|S(f,\triangle \Psi,\tau_1) - GH_k\int_p^r f d\Psi \|_{Z^n} < \frac{\epsilon}{3}$$

$$\|S(f,\triangle \Psi,\tau_2) - GH_k\int_r^q f d\Psi \|_{Z^n} < \frac{\epsilon}{3}$$

$$\|S(f,\triangle \Psi,\tau_3) - GH_k\int_p^q f d\Psi \|_{Z^n} < \frac{\epsilon}{3}$$

then,

$\|GH_k\int_p^r f d\Psi + GH_k\int_r^q f d\Psi + GH_k\int_p^q f d\Psi \|_{Z^n} \le \|GH_k\int_p^r f d\Psi - S(f,\triangle \Psi,\tau_1)\|_{Z^n}$

$+\|GH_k\int_r^q f d\Psi - S(f,\triangle \Psi,\tau_2)\|_{Z^n}$

$+\|GH_k\int_p^q f d\Psi - S(f,\triangle \Psi,\tau_3)\|_{Z^n}$

$<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = 3$

for arbitrary $\epsilon>0$ then $GH_k\int_p^q f d\Psi = GH_k\int_p^r f d\Psi + GH_k\int_r^q f d\Psi$.

Theorem 4: Assume B is a bilinear map for n – tuple and $r\in[p,q]$ and let $J(f,r)$ exists, if $f,\Psi \in GH_k([p,r],Z^n)$ and $f,\Psi \in GH_k([r,q],Z^n)$ then $f,\Psi \in GH_k([p,q],Z^n)$ and $GH_k\int_p^q f d\Psi = GH_k\int_p^r f d\Psi + GH_k\int_r^q f d\Psi$.

Proof: Since $f,\Psi \in GH_k([p,r],Z^n)$ and $f,\Psi \in GH_k([r,q],Z^n)$, then by definition for every $\epsilon>0$ there exists $\xi_1:[p,r]\to(0,\infty)$ and $\xi_2:[r,q]\to(0,\infty)$ any fine ξ_1^k – partition τ_1 of $[p,q]$ and any fine ξ_1^k – partition τ_2 of $[r,q]$,

$$\|S(f,\triangle \Psi,\tau_1) - GH_k\int_p^q f d\Psi \|_{Z^n} < \epsilon$$

$$\|S(f,\triangle \Psi,\tau_2) - GH_k\int_p^q f d\Psi \|_{Z^n} < \epsilon$$

since; $J(f,r)$ exists there is $\eta>0$ such that for $\max_{1\le j\le k}|\sqcup_j - r|<\eta$ for $\|f(r;\sqcup_1,\sqcup_2,\dots,\sqcup_k) - J(f,r)\| < \frac{\epsilon}{6}$ 1

define a positive function ξ on $[p,q]$ by

$$\begin{cases} \min(\xi_1(\alpha), r-\alpha) & if \quad p\le\alpha\le r \\ \min(\xi_2(\alpha), \alpha-r) & if \quad p\le\alpha\le r \\ \min(\xi_1(\alpha),\xi_2(\alpha),\eta) & if \quad \alpha = r \end{cases}$$

,

consider any ξ–fine tagged partition $\tau = \{(\delta_i;\ \alpha_{i,1},\dots,\alpha_{i,k-1}):[\alpha_{i,0},\alpha_{i,k}], i=1,2,\dots,m\}$ of $[p,q]$ denote approximating sum

$S(f,\triangle \Psi,\tau) = \sum_{i=1}^{m=1}[f(\delta_i;\alpha_{i,1},\dots,\alpha_{i,k}) - f(\delta_{i,j};\alpha_{i,0},\dots,\alpha_{i,k-1})]$

$+[f(r,\alpha_{m,1},\dots,\alpha_{m,k}) - f(r;\alpha_{m,0},\dots,\alpha_{m,k-1})] + \sum_{i=m+1}^{n}[f(\delta_i;\alpha_{i,1},\dots,\alpha_{i,k}) - f(\delta_i;\alpha_{i,0},\dots,\alpha_{i,k-1})]$

constructing τ_1 and τ_2 and by using (1)

$$\|S(f,\triangle \Psi,\tau) - S(f,\triangle \Psi,\tau_1) - S(f,\triangle \Psi,\tau_2)\|_{Z^n} = \|[f(r,\alpha_{m,1},\dots,\alpha_{m,k}) - f(r;\alpha_{m,0},\dots,\alpha_{m,k-1})]$$

$-[f(r,\beta_{m,1},\dots,\beta_{m,k}) - f(r;\beta_{m,0},\dots,\beta_{m,k-1})]$

$-[f(r,\gamma_{m,1},\dots,\gamma_{m,k}) - f(r;\gamma_{m,0},\dots,\gamma_{m,k-1})]\|_{Z^n} < \epsilon,$

so,

$\|S(f,\triangle \Psi,\tau) - GH_k\int_p^r f d\Psi - GH_k\int_r^q f d\Psi \|_{Z^n} \le \|S(f,\triangle \Psi,\tau_1) - GH_k\int_p^r f d\Psi \|_{Z^n}$

$+\|S(f,\triangle \Psi,\tau_2) - GH_k\int_r^q f d\Psi \|_{Z^n}$

$+\|S(f,\triangle \Psi,\tau) - S(f,\triangle \Psi,\tau_1) - S(f,\triangle \Psi,\tau_2)\|_{Z^n}$

$<3\epsilon$

by definition τ random fine ξ –partition of $[p,q]$, then observed $f,\Psi \in GH_k([p,q],Z^n)$ and $GH_k\int_p^q f d\Psi = GH_k\int_p^r f d\Psi + GH_k\int_r^q f d\Psi$.

Main Result:

Theorem 5: Sacks Henstock Lemma

Assume B is bilinear map for n – tuple and $f:[p,q]^{k+1}\to X^n$, generalised $H-K$ integrable, $f\in GH_k([p,q],Z^n)$ as $\epsilon>0$ consider gauge ξ of $[p,q]$, $\xi:[p,q]\to(0,\infty)$ is such that for every ξ – fine tagged k – partition. $\tau = \{(\delta_i:\alpha_{i,1},\dots,\alpha_{i,k-1}):[\alpha_{i,0},\alpha_{i,k}], i=1,2,\dots,n\}$ of $[p,q]$.

$$\|\sum_{i=1}^{n}[f(\delta_i;\alpha_{i,1},\dots,\alpha_{i,k}) - f(\delta_i;\alpha_{i,0},\dots,\alpha_{i,k-1})] - GH_k\int_p^q f d\Psi \|_{Z^n} < \epsilon.$$

If $\{(\eta;\ \beta_{i,1},\dots,\beta_{i,k-1}) : [\beta_{i,0};\beta_{i,k}], i=0,1,2,\dots,m\}$ where

$p \leq \beta_{1,0}, \beta_{i-1,k} \leq \beta_{i,0} (i = 2,3, \dots, m), \beta_{m,k} \leq q$ represents a ξ – fine elementary k – system in $[p, q]$ then

$$\| \sum_{i=1}^{n} [(\eta_i;\ \beta_{i,1}, \dots, \beta_{i,k}) - f(\eta_i; \beta_{i,0}, \dots, \beta_{i,k-1})] - GH_k \int_{\beta_{i,0}}^{\beta_{i,k}} f d\Psi \|_{Z^n} < \epsilon.$$

Proof: If $\beta_{i,k} < \beta_{i+1,0}$ as $i = 1,2, \dots, m; \beta_{m+1,0} = q$ using (Theorem 2) $f \in GH_k[\beta_{i,k}, \beta_{i+1,0}]$ for arbitrary $\epsilon > 0$ exists gauge ξ of $[\beta_{i,k}, \beta_{i+1,0}]$ such that $\xi_1(\alpha) < \xi(\alpha)$ for all $\alpha \in [\beta_{i,k}, \beta_{i+1,0}]$, each fine ξ_i^k – partition τ^i on $[\beta_{i,k}, \beta_{i+1,0}]$,

$$\| S(f, \triangle\Psi, \tau^{\mathrm{i}}) - GH_k \int_{\beta_{i,k}}^{\beta_{i+1,0}} f d\Psi \|_{Z^n} = \frac{\epsilon}{m+1} \quad 2$$

If $\beta_{i,k} = \beta_{i+1,0}$, consider $S(f, \triangle\Psi, \tau^{\mathrm{i}}) = 0$. The equation

$\sum_{i=1}^{n} [(\eta_i;\ \beta_{i,1}, \dots, \beta_{i,k}) - f(\eta_i; \beta_{i,0}, \dots, \beta_{i,k-1})] + \sum_i S(f, \triangle\Psi, \tau^{\mathrm{i}})$

represents a GH_k sum correlated with certain fine ξ^k – partition on $[p, q]$ therefore

$$\| \sum_{i=1}^{n} [(\eta_i;\ \alpha_{i,1}, \dots, \alpha_{i,k}) - f(\eta_i; \beta_{i,0}, \dots, \beta_{i,k-1})] + \sum_i S(f, \triangle\Psi, \tau^{\mathrm{i}}) - GH_k \int_p^q f d\Psi \|_{Z^n} < \epsilon. \quad 3$$

Hence in the sense of (Theorem 3) and by using in equalities Eq.2 and Eq.3

$$\begin{aligned} &\| \sum_{i=1}^{n} [(\eta_i;\ \beta_{i,1}, \dots, \beta_{i,k}) - f(\eta_i; \beta_{i,0}, \dots, \beta_{i,k-1})] - GH_k \int_{\beta_{i,0}}^{\beta_{i,k}} f d\Psi \|_{Z^n} \\ &\leq \| \sum_{i=1}^{n} [(\eta_i;\ \beta_{i,1}, \dots, \beta_{i,k}) - f(\eta_i; \beta_{i,0}, \dots, \beta_{i,k-1})] + \sum_i S(f, \triangle\Psi, \tau^{\mathrm{i}}) - GH_k \int_p^q f d\Psi \|_{Z^n} \\ &+ \sum S(f, \triangle\Psi, \tau^{\mathrm{i}}) - GH_k \int_{\beta_{i,k}}^{\beta_{i+1,0}} f d\Psi \|_{Z^n} \\ &< \epsilon + \frac{m}{m+1} < 2\epsilon \end{aligned}$$

Theorem 6: Assume B is bilinear map for n – tuple and $f: [p, q]^{k+1} \to X^n$, generalised $H - K$ integrable, $f \in ([p, q], Z^n)$ then for every $\epsilon > 0$ exists gauge on ξ_i on $[p, q]$ such that for every ξ – fine tagged k – partition $\tau = \{(\delta_i;\ \alpha_{i,1}, \dots, \alpha_{i,k-1}) : [\alpha_{i,0}, \alpha_{i,k}], i = 1,2, \dots, n\}$ of $[p, q]$,

$$\sum_{i=1}^{n} \| [f(\delta_i; \alpha_{i,1}, \dots, \alpha_{i,k}) - f(\delta_i; \alpha_{i,0}, \dots, \alpha_{i,k-1})] - GH_k \int_{\alpha_{i,0}}^{\alpha_{i,k}} f d\Psi \|_{Z^n} < \epsilon.$$

Proof: Since $f \in GH_k[p, q]$, each $\epsilon > 0$ exists positive function on ξ of $[p, q]$ such that each fine ξ^k – partition $\tau = \{(\delta_i;\ \alpha_{i,1}, \dots, \alpha_{i,k-1}) : [\alpha_{i,0}, \alpha_{i,k}], i = 1,2, \dots, n\}$ of $[p, q]$, by the sense of (Theorem 3),

$$\| \sum_{i=1}^{n} [f(\delta_i; \alpha_{i,1}, \dots, \alpha_{i,k}) - f(\delta_i; \alpha_{i,0}, \dots, \alpha_{i,k-1})] - GH_k \int_{\alpha_{i,0}}^{\alpha_{i,k}} f d\Psi \|_{Z^n} < \frac{\epsilon}{4}.$$

Let $\sum^+$ denotes that part of above sum for which

$$\left[f(\delta_i; \alpha_{i,1}, \dots, \alpha_{i,k}) - f(\delta_i; \alpha_{i,0}, \dots, \alpha_{i,k-1}) - GH_k \int_{\alpha_{i,0}}^{\alpha_{i,k}} f d\Psi \right] \geq 0$$

and let $\sum^-$ denotes the left equation is less than zero, using (Theorem 5),

$$\| \sum_{i=1}^{n} [f(\delta_i; \alpha_{i,1}, \dots, \alpha_{i,k}) - f(\delta_i; \alpha_{i,0}, \dots, \alpha_{i,k-1})] - GH_k \int_{\alpha_{i,0}}^{\alpha_{i,k}} f d\Psi \|_{Z^n} = \sum^+ - \sum^-$$

$$= \sum^+ - \sum^- \leq \| \sum^+ \|_{Z^n} + \| \sum^- \|_{Z^n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

Theorem 7: Cauchy Criterion

Assume B is bilinear map for n – tuple. Let $f: [p, q]^{k+1} \to X^n$ and $\Psi: [p, q]^{k+1} \to Y^n$ be two Banach space valued functions, where $k > 1$ be such that $f, \Psi \in GH_k([p, q], Z^n)$ for every $r \in ([p, q], Z^n)$ and let $\lim_{r \to q} GH_k \int_p^r f d\Psi = \mathrm{A}$ exists finitely. If $J^-(f, q)$ exists finitely then $f, \Psi \in GH_k([p, q], Z^n)$ and $GH_k \int_p^q f d\Psi = \mathrm{A}$.

Proof: Given $\epsilon > 0$ be arbitrary. There is a number $\eta_1 > 0$ such that for each $r \in (q - \eta_1, q)$.

$$\| GH_k \int_p^r f d\Psi - \mathrm{A} \|_{Z^n} < \epsilon \quad 4$$

Let $\{r_\tau\}_{r=0}^{\infty}$ be an increasing sequence in $[p, q], r_0 = p$ with $r_\tau \to q$ so $f \in GH_k[p, r_\tau]$ each $\tau = 1,2, \dots$ so exist gauge $\xi_r: [p, r_\tau] \to (0, \infty)$ such that as fine ξ^k – partition τ_r on $[p, r_\tau]$

$$\|S(f,\triangle\Psi,\tau_\tau) - GH_k\int_p^{\tau_\tau} fd\Psi\|_{Z^n} = \frac{\epsilon}{2^{\tau+1}}, \quad \tau = 1,2,\dots \qquad 5$$

for any $\delta \in [p,q)$ there exists one $\tau(\delta) = 1,2,\dots$ for which $\delta \in [r_{p(\delta)-1}.r_p(\delta)]$. So for given $\delta \in [p,q)$ choose $\hat{\xi}(\delta) \le \xi_{\tau(\delta)}$ and $(\delta - \hat{\xi}(\delta), \delta + \hat{\xi}(\delta)) \cap [p,q) \subset [p, r_\tau(\xi)]$ consider that $r \in [p,q)$ and $\hat{\tau} = \{(\delta_i;\ \alpha_{i,1},\dots,\alpha_{i,k-1}) : [\alpha_{i,0},\alpha_{i,k}], i = 1,2,\dots,n\}$ is fine $\hat{\xi}^k$ – partition of $[p,r]$. Assume $\tau(\delta_i) = \tau$ then $[\alpha_{i,0},\alpha_{i,k}] \subset (\delta_i - \hat{\xi}(\delta_i), \delta_i + \hat{\xi}(\delta_i)) \subset [p, r_\tau]$ also $[\alpha_{i,0},\alpha_{i,k}] \subset (\delta_i - \xi_\tau(\delta_i), \delta_i + \xi_\tau(\delta_i))$. Let

$$\sum_{i=1_{\tau(\delta_i)=\tau}}^{n-1}\left[f(\delta_i;\alpha_{i,1},\dots,\alpha_{i,k}) - f(\delta_i;\alpha_{i,0},\dots,\alpha_{i,k-1}) - GH_k\int_{\alpha_{i,0}}^{\alpha_{i,k}} fd\Psi\right]$$

be the sum of those terms in

$$\sum_{i=1}^{\tau-1}\left[f(\xi_i;\alpha_{i,1},\dots,\alpha_{i,k}) - f(\delta_i;\alpha_{i,0},\dots,\alpha_{i,k-1}) - GH_k\int_{\alpha_{i,0}}^{\alpha_{i,k}} fd\Psi\right]$$

for a tag δ_i satisfy $\delta_i \in [r_{\tau-1}\cdot r_\tau]$, which hold by using (Theorem 5)

$$\|\sum_{i=1_{\tau(\delta_i)=\tau}}^{n-1}[f(\delta_i;\alpha_{i,1},\dots,\alpha_{i,k}) - f(\delta_i;\alpha_{i,0},\dots,\alpha_{i,k-1}) - GH_k\int_{\alpha_{i,0}}^{\alpha_{i,k}} fd\Psi]\|_{Z^n} < \frac{\epsilon}{2^\tau} \qquad 6$$

since $f \in GH_k([p,r],Z^n)$ for every $r \in [p,q]$ by (Theorem 3),

$$(GH_k)\int_p^r fd\Psi = \sum_{i=1}^{n-1} GH_k\int_{\alpha_i}^{\alpha_{i,k}} fd\Psi$$

so by using Eq.6

$$\|\sum_{i=1}^{n-1}[f(\delta_i;\alpha_{i,1},\dots,\alpha_{i,k}) - f(\delta_i;\alpha_{i,0},\dots,\alpha_{i,k-1}) - GH_k\int_p^r fd\Psi]\|_{Z^n}$$

$$= \|\sum_{i=1}^{n-1}[f(\delta_i;\alpha_{i,1},\dots,\alpha_{i,k}) - f(\delta_i;\alpha_{i,0},\dots,\alpha_{i,k-1}) - GH_k\int_{\alpha_{i,0}}^{\alpha_{i,k}} fd\Psi]\|_{Z^n} \qquad 7$$

$$\le \sum_{\tau=1}^{\infty}\|\sum_{i=1_{\tau(\delta_i)=\tau}}^{n-1}[f(\delta_i;\alpha_{i,1},\dots,\alpha_{i,k}) - f(\delta_i;\alpha_{i,0},\dots,\alpha_{i,k-1}) - GH_k\int_{\alpha_{i,0}}^{\alpha_{i,k}} fd\Psi]\|_{Z^n}$$

$$< \sum_{\tau=1}^{\infty}\frac{\epsilon}{2^\tau} = \epsilon$$

If $J^-(f,q)$ existential, there is $\eta_2 > 0$ such that for every $q - \eta_2 < t_1 < t_2 < \dots < t_k < q$ then

$$\|f(q;t_2,\dots,t_k,q) - \tau(q,t_1,t_2,\dots,t_k)\|_{Z^n} < \epsilon \qquad 8$$

Let $\eta = \min(\eta_1,\eta_2)$. Define gauge ξ on $[p,q]$ such that $\xi(\delta) = \min(\hat{\xi}(\delta), q - \delta)$ if $\delta \in [p,q), \xi(q) < \eta$. By using Eq.4 and Eq.8

$$\|S(f,\triangle\Psi,\tau) - A\|_{Z^n} = \|\sum_{i=1}^{n-1}[f(\delta_i;\alpha_{i,1},\dots,\alpha_{i,k}) - f(\delta_i;\alpha_{i,0},\dots,\alpha_{i,k-1})] + f(q;\alpha_{n,1},\dots,q) - f(q;\alpha_{n,0},\dots,\alpha_{n,k-1}) - I\|_{Z^n}$$

$$\le \|\sum_{i=1}^{n-1}[f(\delta_i;\alpha_{i,1},\dots,\alpha_{i,k}) - f(\delta_i;\alpha_{i,0},\dots,\alpha_{i,k-1}) - GH_k\int_p^{\alpha_{n-1,k}} fd\Psi]\|_{Z^n} + \|GH_k\int_p^{\alpha_{n-1,k}} fd\Psi - A\|_{Z^n} + \|f(q;\alpha_{n,1},\dots,q) - f(q;\alpha_{n,0},\dots,\alpha_{n,k-1})\|_{Z^n}$$

$$< \|\sum_{i=1}^{n-1}[f(\delta_i;\alpha_{i,1},\dots,\alpha_{i,k}) - f(\delta_i;\alpha_{i,0},\dots,\alpha_{i,k-1}) - GH_k\int_p^{\alpha_{n-1,k}} fd\Psi]\|_{Z^n} + 2\epsilon$$

since $\alpha_{n-1,k} < q$ and $\tau = \{(\delta_i;\ \alpha_{i,1},\dots,\alpha_{i,k-1}) : [\alpha_{i,0},\alpha_{i,k}], i = 1,2,\dots,n\}$ is a $\hat{\xi}$ – fine tagged k – partition of $[p,\alpha_{n-1,k}]$, by using Eq.7,

$$\|S(f,\triangle\Psi,\tau) - A\|_{Z^n} < 3\epsilon$$

this gives that $f \in GH_k([p,q],Z^n)$ and $GH_k\int_p^q fd\Psi = A$.

Conclusion:

In this paper generalised H-K integral of bilinear n-tupple map is defined. The elementary properties of H-K integral of bilinear n-tupple studied. Saks Henstock lemma and Cauchy criterion are generalised for H-K integral of bilinear n-tupple map.

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Author's Contribution:

This work was carried out in collaboration between T. G. Thange and S. S. Gangane. T. G. Thange provided the research paper idea and S. S. Gangane implemented the idea, generalised the Henstock – Kurzweil with bilinear n-tupple and create the manuscript. T. G. Thange edited the manuscript and made some changes in the manuscript. All authors read and approved the final manuscript.

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تكامل هينستوك -كورزويل المعمم مع نقاط متعددة

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الخلاصة:

يتعامل هذا البحث مع تكامل هينستوك -كورزويل الجديد في فضاء باناخ مع دالة التكامل الثلاثية ثنائية الخطية Ψ التي تعتمد على النقاط المتعددة في التجزئة. أخيرا, تم عرض النتائج القياسية لتكامل هينستوك -كورزويل المعمم في نظرية التكامل.

الكلمات المفتاحية: نوني ثنائي الخطية، فضاء باناخ، تكامل هينستوك -كورزويل، تكامل ريمان، تكامل سترلينج.