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On the Split Mersenne and Mersenne-Lucas Hybrid Quaternions

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Abstract:

In this communication, introduce the split Mersenne and Mersenne-Lucas hybrid quaternions, also obtaining generating functions and Binet formulas for these hybrid quaternions and investigating some properties among them.

Keywords: Binet formula, Hybrid numbers, Mersenne sequence, Mersenne-Lucas sequence, Quaternions.

Introduction:

Number theory is one of the most fascinating fields in Mathematics. One can see a variety of problems[1,2]. In 1644, a French Mathematician, Marin Mersenne defined a number of the form $M_n \equiv 2^n - 1$, n is an integer. Mersenne numbers are binary repunits. There have been many studies on the Mersenne sequences [3]. The Mersenne - Lucas sequences are defined as $ML_n = 2^n + 1,\ n \geq 2$ with $ML_0 = 2,\ ML_1 = 3$ [4]. The hybrid numbers were introduced by Ozdemir [5] in 2018 and it is a composition of dual, complex, hyperbolic numbers satisfying the relation $ih = -hi = i + \varepsilon$ [6,7]. It is of the form

$$\mathcal{H} = z_0 + z_1 i + z_2 \varepsilon + z_3 h,$$

where $z_0, z_1, z_2, z_3 \in \mathbb{R}$ and $i,\ \varepsilon,\ h$ are operators such that $i^2 = -1, \varepsilon^2 = 0, h^2 = 1$.

In 1843, the Irish Mathematician, William Rowan Hamilton described quaternions as the quotient of two vectors in three-dimensional space represented as $Q = a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = k^2 = -1,\ ijk = -1,$ $ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$ Quaternions are non-commutative. For more details on quaternions, one can see [8-10]. Note that, the split quaternions have been introduced by James Cockle [11] in 1849 and they form a four-dimensional non-commutative, associative algebra over the real numbers with bases 1, i, j, k has the form $q = a_0 + a_1 \mathrm{i} + a_2 \mathrm{j} + a_3 \mathrm{k}$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and $\mathrm{i}^2 = -1,\ \mathrm{j}^2 = k^2 = ijk = 1,\ \mathrm{ij} = -\mathrm{ji} = \mathrm{k},$ $\mathrm{jk} = -\mathrm{kj} = -\mathrm{i},\ \mathrm{ki} = -\mathrm{ik} = \mathrm{j}$ [12]. Our interest centers on associating split quaternions and hybrid numbers on the Mersenne and Mersenne-Lucas sequences.

Preliminaries:

Definition. 1: The Mersenne quaternions and Mersenne-Lucas quaternions are defined by

$$\widetilde{M_n} = M_n + iM_{n+1} + jM_{n+2} + kM_{n+3}$$

$$\widetilde{ML_n} = ML_n + iML_{n+1} + jML_{n+2} + kML_{n+3}$$

Definition. 2: The split Mersenne quaternions and split Mersenne-Lucas quaternions are defined as

$$\widetilde{SM_n} = \textstyle\sum_{s=0}^{3} M_{n+s} e_s$$

and

$$\widetilde{SML_n} = \sum_{s=0}^{3} ML_{n+s} e_s$$

Definition. 3: Binet's formula for split Mersenne quaternions and split Mersenne-Lucas quaternions have the form

$$\widetilde{SM_n} = 2^n \mathcal{A} - \mathcal{B} \text{ and } \widetilde{SML_n} = 2^n \mathcal{A} + \mathcal{B}$$

where $\mathcal{A} = \sum_{s=0}^{3} 2^s e_s\ , \mathcal{B} = \sum_{s=0}^{3} e_s$

Definition. 4: The Mersenne hybrid numbers and Mersenne-Lucas hybrid numbers are defined as

$$M\mathcal{H}_n = M_n + iM_{n+1} + \varepsilon M_{n+2} + hM_{n+3}$$

$$ML\mathcal{H}_n = ML_n + iML_{n+1} + \varepsilon ML_{n+2} + hML_{n+3}$$

where i, ε, h are hybrid units.

Definition. 5: For $n \geq 0$, the nth split Mersenne hybrid quaternion sequence $\{\widetilde{SM\mathcal{H}_n}\}$ is defined by

$$\widetilde{SM\mathcal{H}}_n = M\mathcal{H}_n e_0 + M\mathcal{H}_{n+1} e_1 + M\mathcal{H}_{n+2} e_2 + M\mathcal{H}_{n+3} e_3$$
$$= (M_n + iM_{n+1} + \varepsilon M_{n+2} + hM_{n+3})e_0$$
$$+(M_{n+1} + iM_{n+2} + \varepsilon M_{n+3} + hM_{n+4})e_1$$
$$+(M_{n+2} + iM_{n+3} + \varepsilon M_{n+4} + hM_{n+5})e_2$$
$$+(M_{n+3} + iM_{n+4} + \varepsilon M_{n+5} + hM_{n+6})e_3$$

where i, ε, h are hybrid units and e_0, e_1, e_2, e_3, are split quaternion basis.

The split Mersenne hybrid quaternions can be rewritten by

$$\widetilde{SM\mathcal{H}}_n = \widetilde{SM_n} + i\widetilde{SM_{n+1}} + \varepsilon\widetilde{SM_{n+2}} + h\widetilde{SM_{n+3}}$$

Definition. 6: The nth split Mersenne-Lucas hybrid quaternion sequence $\{\widetilde{SML\mathcal{H}_n}\}$ is defined as

$$\widetilde{SML\mathcal{H}}_n = ML\mathcal{H}_n e_0 + ML\mathcal{H}_{n+1} e_1 + ML\mathcal{H}_{n+2} e_2 + ML\mathcal{H}_{n+3} e_3$$
$$= (ML_n + iML_{n+1} + \varepsilon ML_{n+2} + hML_{n+3})e_0$$
$$+(ML_{n+1} + iML_{n+2} + \varepsilon ML_{n+3} + hML_{n+4})e_1$$
$$+(ML_{n+2} + iML_{n+3} + \varepsilon ML_{n+4} + hML_{n+5})e_2$$
$$+(ML_{n+3} + iML_{n+4} + \varepsilon ML_{n+5} + hML_{n+6})e_3$$

where i, ε, h are hybrid units and e_0, e_1, e_2, e_3 are split quaternion basis.

The split Mersenne hybrid quaternions can be rewritten by

$$\widetilde{SML\mathcal{H}}_n = \widetilde{SML_n} + i\widetilde{SML_{n+1}} + \varepsilon\widetilde{SML_{n+2}} + h\widetilde{SML_{n+3}}$$

Definition. 7: For n ≥ 0, Binet's formulas for the split Mersenne hybrid quaternions and split Mersenne-Lucas hybrid quaternions are

$$\widetilde{SM\mathcal{H}}_n = 2^n\alpha^*\mathcal{A} - \beta^*\mathcal{B} \quad \text{and}$$
$$\widetilde{SML\mathcal{H}}_n = 2^n\alpha^*\mathcal{A} + \beta^*\mathcal{B}$$

where $\mathcal{A} = \sum_{s=0}^{3} 2^s e_s$, $\mathcal{B} = \sum_{s=0}^{3} e_s$,

$\alpha^* = (1 + 2i + 2^2\epsilon + 2^3h)$,

$\beta^* = (1 + i + \epsilon + h)$.

Theorem. 1: The generating functions for the split Mersenne hybrid quaternions and the split Mersenne-Lucas hybrid quaternions are

$$f(t) = \frac{\widetilde{SM\mathcal{H}}_0(1-3t) + \widetilde{SM\mathcal{H}}_1 t}{1-3t+2t^2}$$

and

$$g(t) = \frac{\widetilde{SML\mathcal{H}}_0(1-3t) + \widetilde{SML\mathcal{H}}_1 t}{1-3t+2t^2}$$

Proof: Let $f(t) = \sum_{n=0}^{\infty} \widetilde{SM\mathcal{H}}_n t^n$

Multiplying this equation by $1, 3t, 2t^2$ respectively and summing these equations,

$$(1-3t+2t^2)f(t)$$
$$= \widetilde{SM\mathcal{H}}_0 + (\widetilde{SM\mathcal{H}}_1 - 3\widetilde{SM\mathcal{H}}_0)t + (\widetilde{SM\mathcal{H}}_2 - 3\widetilde{SM\mathcal{H}}_1 + 2\widetilde{SM\mathcal{H}}_0)t^2 + (\widetilde{SM\mathcal{H}}_3 - 3\widetilde{SM\mathcal{H}}_2 + 2\widetilde{SM\mathcal{H}}_1)t^3 + \cdots + (\widetilde{SM\mathcal{H}}_n - 3\widetilde{SM\mathcal{H}}_{n-1} + 2\widetilde{SM\mathcal{H}}_{n-2})t^n$$
$$= \widetilde{SM\mathcal{H}}_0 + (\widetilde{SM\mathcal{H}}_1 - 3\widetilde{SM\mathcal{H}}_0)t + \sum_{n=2}^{\infty}(\widetilde{SM\mathcal{H}}_n - 3\widetilde{SM\mathcal{H}}_{n-1} + 2\widetilde{SM\mathcal{H}}_{n-2})t^n$$

$f(t) = \frac{\widetilde{SM\mathcal{H}}_0(1-3t) + \widetilde{SM\mathcal{H}}_1 t}{1-3t+2t^2}$ is the generating function for the split Mersenne hybrid quaternions.

And let $g(t) = \sum_{n=0}^{\infty} \widetilde{SML\mathcal{H}}_n t^n$

Multiplying this equation by $1, 3t, 2t^2$ respectively and summing these equations,

$$(1-3t+2t^2)g(t)$$
$$= \widetilde{SML\mathcal{H}}_0 + (\widetilde{SML\mathcal{H}}_1 - 3\widetilde{SML\mathcal{H}}_0)t + \sum_{n=2}^{\infty}(\widetilde{SML\mathcal{H}}_n - 3\widetilde{SML\mathcal{H}}_{n-1} + 2\widetilde{SML\mathcal{H}}_{n-2})t^n$$

$g(t) = \frac{\widetilde{SML\mathcal{H}}_0(1-3t) + \widetilde{SML\mathcal{H}}_1 t}{1-3t+2t^2}$ is the generating function for split Mersenne-Lucas hybrid quaternions.

Theorem. 2: Let m, n be any positive integers and $m \geq n$ then

i. $\widetilde{SM\mathcal{H}}_m\widetilde{SML\mathcal{H}}_n + \widetilde{SML\mathcal{H}}_m\widetilde{SM\mathcal{H}}_n = 2[2^{m+n}(\alpha^*)^2(\mathcal{A})^2 - (\beta^*)^2(\mathcal{B})^2]$

ii. $\widetilde{SM\mathcal{H}}_m\widetilde{SML\mathcal{H}}_n - \widetilde{SML\mathcal{H}}_m\widetilde{SM\mathcal{H}}_n = 2^{n+1}[2^{m-n}\alpha^*\beta^*\mathcal{A}\mathcal{B} - \beta^*\alpha^*\mathcal{B}\mathcal{A}]$

Proof:

i. $\widetilde{SM\mathcal{H}}_m\widetilde{SML\mathcal{H}}_n + \widetilde{SML\mathcal{H}}_m\widetilde{SM\mathcal{H}}_n$

$$= (2^m\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} + \beta^*\mathcal{B}) + (2^m\alpha^*\mathcal{A} + \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} - \beta^*\mathcal{B})$$

$$= 2^{m+n}(\alpha^*)^2(\mathcal{A})^2 - 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A} + 2^m\alpha^*\beta^*\mathcal{A}\mathcal{B} - (\beta^*)^2(\mathcal{B})^2 + 2^{m+n}(\alpha^*)^2(\mathcal{A})^2 - (\beta^*)^2(\mathcal{B})^2 + 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A} - 2^m\alpha^*\beta^*\mathcal{A}\mathcal{B}$$

$$= 2[2^{m+n}(\alpha^*)^2(\mathcal{A})^2 - (\beta^*)^2(\mathcal{B})^2]$$

ii. $\widetilde{SM\mathcal{H}}_m\widetilde{SML\mathcal{H}}_n - \widetilde{SML\mathcal{H}}_m\widetilde{SM\mathcal{H}}_n$

$$= (2^m\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} + \beta^*\mathcal{B}) - (2^m\alpha^*\mathcal{A} + \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} - \beta^*\mathcal{B})$$

$$= 2^{m+n}(\alpha^*)^2(\mathcal{A})^2 - 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A} + 2^m\alpha^*\beta^*\mathcal{A}\mathcal{B} - (\beta^*)^2(\mathcal{B})^2 - 2^{m+n}(\alpha^*)^2(\mathcal{A})^2 + (\beta^*)^2(\mathcal{B})^2 - 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A} + 2^m\alpha^*\beta^*\mathcal{A}\mathcal{B}$$

$$= 2^{n+1}[2^{m-n}\alpha^*\beta^*\mathcal{A}\mathcal{B} - \beta^*\alpha^*\mathcal{B}\mathcal{A}]$$

Theorem. 3: Let m, n be any positive integers then

$$\widetilde{SM\mathcal{H}}_m\widetilde{SML\mathcal{H}}_n + \widetilde{SM\mathcal{H}}_n\widetilde{SML\mathcal{H}}_m = 2[2^{m+n}(\alpha^*)^2(\mathcal{A})^2 - (\beta^*)^2(\mathcal{B})^2] - 2^n ML_{m-n}(\beta^*\alpha^*\mathcal{B}\mathcal{A} - \alpha^*\beta^*\mathcal{A}\mathcal{B})$$

Proof:

$\widetilde{SM\mathcal{H}}_m\widetilde{SML\mathcal{H}}_n + \widetilde{SM\mathcal{H}}_n\widetilde{SML\mathcal{H}}_m$

$$= (2^m\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} + \beta^*\mathcal{B}) + (2^n\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^m\alpha^*\mathcal{A} + \beta^*\mathcal{B})$$

$$= 2^{m+n+1}(\alpha^*)^2(\mathcal{A})^2 - 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A}(1 - 2^{m-n}) + 2^n\alpha^*\beta^*\mathcal{A}\mathcal{B}(1 + 2^{m-n}) - 2(\beta^*)^2(\mathcal{B})^2$$

$$= 2[2^{m+n}(\alpha^*)^2(\mathcal{A})^2 - (\beta^*)^2(\mathcal{B})^2] - 2^n ML_{m-n}(\beta^*\alpha^*\mathcal{B}\mathcal{A} - \alpha^*\beta^*\mathcal{A}\mathcal{B})$$

Theorem. 4: Let n be any positive integer then

$$\widetilde{SML\mathcal{H}}_n^{\ 2} - \widetilde{SM\mathcal{H}}_n^{\ 2} = 2^{n+1}[\alpha^*\beta^*\mathcal{A}\mathcal{B} + \beta^*\alpha^*\mathcal{B}\mathcal{A}]$$

Proof:

$\widetilde{SML\mathcal{H}}_n^{\ 2} - \widetilde{SM\mathcal{H}}_n^{\ 2}$

$$= (2^n\alpha^*\mathcal{A} + \beta^*\mathcal{B})^2 - (2^n\alpha^*\mathcal{A} - \beta^*\mathcal{B})^2$$

$$= (2^n\alpha^*\mathcal{A} + \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} + \beta^*\mathcal{B}) - (2^n\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} - \beta^*\mathcal{B})$$

$$= 2^{2n}(\alpha^*)^2(\mathcal{A})^2 - 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A} + 2^n\alpha^*\beta^*\mathcal{A}\mathcal{B} + (\beta^*)^2(\mathcal{B})^2 - 2^{2n}(\alpha^*)^2(\mathcal{A})^2 - (\beta^*)^2(\mathcal{B})^2 + 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A} + 2^n\alpha^*\beta^*\mathcal{A}\mathcal{B}$$

$$= 2^{n+1}\alpha^*\beta^*\mathcal{A}\mathcal{B} - 2^{n+1}\beta^*\alpha^*\mathcal{B}\mathcal{A}$$

$$= 2^{n+1}[\alpha^*\beta^*\mathcal{A}\mathcal{B} + \beta^*\alpha^*\mathcal{B}\mathcal{A}]$$

Theorem. 5: Let n, i, j be any positive integers then

i. $\widetilde{SM\mathcal{H}}_{n+i}\widetilde{SM\mathcal{H}}_{n+j} - \widetilde{SM\mathcal{H}}_n\widetilde{SM\mathcal{H}}_{n+i+j} = 2^n M_i(2^j\beta^*\alpha^*\mathcal{B}\mathcal{A} - \alpha^*\beta^*\mathcal{A}\mathcal{B})$

ii. $\widetilde{SML\mathcal{H}}_{n+i}\widetilde{SML\mathcal{H}}_{n+j} - \widetilde{SML\mathcal{H}}_n\widetilde{SML\mathcal{H}}_{n+i+j} = 2^n M_i(\alpha^*\beta^*\mathcal{A}\mathcal{B} - 2^j\beta^*\alpha^*\mathcal{B}\mathcal{A})$

Proof:

i. $\widetilde{SM\mathcal{H}}_{n+i}\widetilde{SM\mathcal{H}}_{n+j} - \widetilde{SM\mathcal{H}}_n\widetilde{SM\mathcal{H}}_{n+i+j}$

$$= \left(2^{n+i}\alpha^*\mathcal{A} - \beta^*\mathcal{B}\right)\left(2^{n+j}\alpha^*\mathcal{A} - \beta^*\mathcal{B}\right) - (2^n\alpha^*\mathcal{A} - \beta^*\mathcal{B})\left(2^{n+i+j}\alpha^*\mathcal{A} - \beta^*\mathcal{B}\right)$$

$$= -2^n\alpha^*\beta^*\mathcal{A}\mathcal{B}(2^i - 1) + 2^{n+j}\beta^*\alpha^*\mathcal{B}\mathcal{A}(2^i - 1)$$
$$= 2^n M_i(2^j\beta^*\alpha^*\mathcal{B}\mathcal{A} - \alpha^*\beta^*\mathcal{A}\mathcal{B})$$

ii. $\widetilde{SML\mathcal{H}}_{n+i}\widetilde{SML\mathcal{H}}_{n+j} - \widetilde{SML\mathcal{H}}_n\widetilde{SML\mathcal{H}}_{n+i+j}$

$$= \left(2^{n+i}\alpha^*\mathcal{A} + \beta^*\mathcal{B}\right)\left(2^{n+j}\alpha^*\mathcal{A} + \beta^*\mathcal{B}\right) - (2^n\alpha^*\mathcal{A} + \beta^*\mathcal{B})\left(2^{n+i+j}\alpha^*\mathcal{A} + \beta^*\mathcal{B}\right)$$

$$= 2^n\alpha^*\beta^*\mathcal{A}\mathcal{B}\left(2^i - 1\right) - 2^{n+j}\beta^*\alpha^*\mathcal{B}\mathcal{A}(2^i - 1)$$

$$= 2^n M_i(\alpha^*\beta^*\mathcal{A}\mathcal{B} - 2^j\beta^*\alpha^*\mathcal{B}\mathcal{A})$$

Theorem. 6: Let m, n be any positive integers then

i. $\widetilde{SM\mathcal{H}}_{m-1}\widetilde{SM\mathcal{H}}_n + \widetilde{SM\mathcal{H}}_m\widetilde{SM\mathcal{H}}_{n+1} = 2^{m+n-1}ML_2(\alpha^*)^2(\mathcal{A})^2 - 2^n ML_1\beta^*\alpha^*\mathcal{B}\mathcal{A} - 2^{m-1}ML_1\alpha^*\beta^*\mathcal{A}\mathcal{B} + 2(\beta^*)^2(\mathcal{B})^2$

ii. $\widetilde{SML\mathcal{H}}_{m-1}\widetilde{SML\mathcal{H}}_n + \widetilde{SML\mathcal{H}}_m\widetilde{SML\mathcal{H}}_{n+1} = 2^{m+n-1}ML_2(\alpha^*)^2(\mathcal{A})^2 + 2^n ML_1\beta^*\alpha^*\mathcal{B}\mathcal{A} + 2^{m-1}ML_1\alpha^*\beta^*\mathcal{A}\mathcal{B} + 2(\beta^*)^2(\mathcal{B})^2$

Proof:

i. $$\widetilde{SM\mathcal{H}}_{m-1}\widetilde{SM\mathcal{H}}_n + \widetilde{SM\mathcal{H}}_m\widetilde{SM\mathcal{H}}_{n+1}$$
$$= (2^{m-1}\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} - \beta^*\mathcal{B}) + (2^m\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^{n+1}\alpha^*\mathcal{A} - \beta^*\mathcal{B})$$
$$= 2^{m+n-1}(\alpha^*)^2(\mathcal{A})^2(2^2+1) - 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A}(2+1) - 2^{m-1}\alpha^*\beta^*\mathcal{A}\mathcal{B}(2+1) + 2(\beta^*)^2(\mathcal{B})^2$$
$$= 2^{m+n-1}ML_2(\alpha^*)^2(\mathcal{A})^2 - 2^n ML_1\beta^*\alpha^*\mathcal{B}\mathcal{A} - 2^{m-1}ML_1\alpha^*\beta^*\mathcal{A}\mathcal{B} + 2(\beta^*)^2(\mathcal{B})^2$$

ii. $$\widetilde{SML\mathcal{H}}_{m-1}\widetilde{SML\mathcal{H}}_n + \widetilde{SML\mathcal{H}}_m\widetilde{SML\mathcal{H}}_{n+1}$$
$$= (2^{m-1}\alpha^*\mathcal{A} + \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} + \beta^*\mathcal{B}) + (2^m\alpha^*\mathcal{A} + \beta^*\mathcal{B})(2^{n+1}\alpha^*\mathcal{A} + \beta^*\mathcal{B})$$
$$= 2^{m+n-1}(\alpha^*)^2(\mathcal{A})^2(2^2+1) - 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A}(2+1) - 2^{m-1}\alpha^*\beta^*\mathcal{A}\mathcal{B}(2+1) + 2(\beta^*)^2(\mathcal{B})^2$$
$$= 2^{m+n-1}ML_2(\alpha^*)^2(\mathcal{A})^2 + 2^n ML_1\beta^*\alpha^*\mathcal{B}\mathcal{A} + 2^{m-1}ML_1\alpha^*\beta^*\mathcal{A}\mathcal{B} + 2(\beta^*)^2(\mathcal{B})^2$$

Theorem. 7: Let n, r, s be any positive integers then

$$\widetilde{SM\mathcal{H}}_{n+r}\widetilde{SML\mathcal{H}}_{n+s} - \widetilde{SM\mathcal{H}}_{n+s}\widetilde{SML\mathcal{H}}_{n+r} = 2^n(M_r - M_s)[\alpha^*\beta^*\mathcal{A}\mathcal{B} + \beta^*\alpha^*\mathcal{B}\mathcal{A}]$$

Proof:

$$\widetilde{SM\mathcal{H}}_{n+r}\widetilde{SML\mathcal{H}}_{n+s} - \widetilde{SM\mathcal{H}}_{n+s}\widetilde{SML\mathcal{H}}_{n+r}$$
$$= (2^{n+r}\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^{n+s}\alpha^*\mathcal{A} + \beta^*\mathcal{B}) - (2^{n+s}\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^{n+1}\alpha^*\mathcal{A} + \beta^*\mathcal{B})$$
$$= 2^{2n+r+s}(\alpha^*)^2(\mathcal{A})^2 - 2^{n+s}\beta^*\alpha^*\mathcal{B}\mathcal{A} + 2^{n+r}\alpha^*\beta^*\mathcal{A}\mathcal{B} - (\beta^*)^2(\mathcal{B})^2 - 2^{2n+r+s}(\alpha^*)^2(\mathcal{A})^2 + (\beta^*)^2(\mathcal{B})^2 + 2^{n+r}\beta^*\alpha^*\mathcal{B}\mathcal{A} - 2^{n+s}\alpha^*\beta^*\mathcal{A}\mathcal{B}$$
$$= 2^n[\alpha^*\beta^*\mathcal{A}\mathcal{B}(2^r - 2^s) + \beta^*\alpha^*\mathcal{B}\mathcal{A}(2^r - 2^s)]$$
$$= 2^n[\alpha^*\beta^*\mathcal{A}\mathcal{B} + \beta^*\alpha^*\mathcal{B}\mathcal{A}](2^r - 2^s - 1 + 1)$$
$$= 2^n(M_r - M_s)[\alpha^*\beta^*\mathcal{A}\mathcal{B} + \beta^*\alpha^*\mathcal{B}\mathcal{A}]$$

Theorem. 8: (Catalan's Identity) Let $n \geq 0, r \geq 0$ be integers such that $r \leq n$ then

$$\widetilde{SM\mathcal{H}}_{n+r}\widetilde{SM\mathcal{H}}_{n-r} - \widetilde{SM\mathcal{H}}_n^{\,2} = 2^{n-r}M_r[\beta^*\alpha^*\mathcal{B}\mathcal{A} - 2^r\alpha^*\beta^*\mathcal{A}\mathcal{B}]$$

$$\widetilde{SML\mathcal{H}}_{n+r}\widetilde{SML\mathcal{H}}_{n-r} - \widetilde{SML\mathcal{H}}_n^{\,2} = 2^{n-r}M_r[2^r\alpha^*\beta^*\mathcal{A}\mathcal{B} - \beta^*\alpha^*\mathcal{B}\mathcal{A}]$$

where $\mathcal{A} = \sum_{s=0}^{3} 2^s e_s$, $\mathcal{B} = \sum_{s=0}^{3} e_s$

Proof:

$$\widetilde{SM\mathcal{H}}_{n+r}\widetilde{SM\mathcal{H}}_{n-r} - \widetilde{SM\mathcal{H}}_n^{\,2}$$
$$= (2^{n+r}\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^{n-r}\alpha^*\mathcal{A} - \beta^*\mathcal{B}) - (2^n\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} - \beta^*\mathcal{B})$$
$$= 2^n\alpha^*\beta^*\mathcal{A}\mathcal{B}(1 - 2^r) - 2^{n-r}\beta^*\alpha^*\mathcal{B}\mathcal{A}(1 - 2^r)$$
$$= 2^{n-r}\beta^*\alpha^*\mathcal{B}\mathcal{A}M_r - 2^n\alpha^*\beta^*\mathcal{A}\mathcal{B}M_r$$
$$= 2^{n-r}M_r[\beta^*\alpha^*\mathcal{B}\mathcal{A} - 2^r\alpha^*\beta^*\mathcal{A}\mathcal{B}]$$

$$\widetilde{SML\mathcal{H}}_{n+r}\widetilde{SML\mathcal{H}}_{n-r} - \widetilde{SML\mathcal{H}}_n^{\,2}$$
$$= (2^{n+r}\alpha^*\mathcal{A} + \beta^*\mathcal{B})(2^{n-r}\alpha^*\mathcal{A} + \beta^*\mathcal{B}) - (2^n\alpha^*\mathcal{A} + \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} + \beta^*\mathcal{B})$$
$$= -2^n\alpha^*\beta^*\mathcal{A}\mathcal{B}(1 - 2^r) + 2^{n-r}\beta^*\alpha^*\mathcal{B}\mathcal{A}(1 - 2^r)$$
$$= 2^n\alpha^*\beta^*\mathcal{A}\mathcal{B}M_r - 2^{n-r}\beta^*\alpha^*\mathcal{B}\mathcal{A}M_r$$
$$= 2^{n-r}M_r[2^r\alpha^*\beta^*\mathcal{A}\mathcal{B} - \beta^*\alpha^*\mathcal{B}\mathcal{A}]$$

Theorem. 9: (Cassini's Identity) For any integer $n \geq 0$,

$$\widetilde{SM\mathcal{H}}_{n+1}\widetilde{SM\mathcal{H}}_{n-1} - \widetilde{SM\mathcal{H}}_n^{\,2} = 2^{n-1}[\beta^*\alpha^*\mathcal{B}\mathcal{A} - 2\alpha^*\beta^*\mathcal{A}\mathcal{B}]$$

$$\widetilde{SML\mathcal{H}}_{n+1}\widetilde{SML\mathcal{H}}_{n-1} - \widetilde{SML\mathcal{H}}_n^{\,2} = 2^{n-1}[2\alpha^*\beta^*\mathcal{A}\mathcal{B} - \beta^*\alpha^*\mathcal{B}\mathcal{A}]$$

Proof: By substituting $r = 1$ in Catalan's identity, these results are obtained.

Theorem. 10: (d'Ocagne's Identity) Let m, n be any positive integers then

$$\widetilde{SM\mathcal{H}}_m\widetilde{SM\mathcal{H}}_{n+1} - \widetilde{SM\mathcal{H}}_{m+1}\widetilde{SM\mathcal{H}}_n = 2^m\alpha^*\beta^*\mathcal{A}\mathcal{B} - 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A}$$

$$\widetilde{SML\mathcal{H}}_m\widetilde{SML\mathcal{H}}_{n+1} - \widetilde{SML\mathcal{H}}_{m+1}\widetilde{SML\mathcal{H}}_n = 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A} - 2^m\alpha^*\beta^*\mathcal{A}\mathcal{B}$$

Proof:

$$\widetilde{SM\mathcal{H}}_m\widetilde{SM\mathcal{H}}_{n+1} - \widetilde{SM\mathcal{H}}_{m+1}\widetilde{SM\mathcal{H}}_n$$

$$= (2^m\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^{n+1}\alpha^*\mathcal{A} - \beta^*\mathcal{B}) - (2^{m+1}\alpha^*\mathcal{A} - \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} - \beta^*\mathcal{B})$$

$$= 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A}(1-2) - 2^m\alpha^*\beta^*\mathcal{A}\mathcal{B}(1-2)$$

$$= 2^m\alpha^*\beta^*\mathcal{A}\mathcal{B} - 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A}$$

$$\widetilde{SML\mathcal{H}}_m\widetilde{SML\mathcal{H}}_{n+1} - \widetilde{SML\mathcal{H}}_{m+1}\widetilde{SML\mathcal{H}}_n$$

$$= (2^m\alpha^*\mathcal{A} + \beta^*\mathcal{B})(2^{n+1}\alpha^*\mathcal{A} + \beta^*\mathcal{B}) - (2^{m+1}\alpha^*\mathcal{A} + \beta^*\mathcal{B})(2^n\alpha^*\mathcal{A} + \beta^*\mathcal{B})$$

$$= 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A}(2-1) - 2^m\alpha^*\beta^*\mathcal{A}\mathcal{B}(2-1)$$

$$= 2^n\beta^*\alpha^*\mathcal{B}\mathcal{A} - 2^m\alpha^*\beta^*\mathcal{A}\mathcal{B}$$

Theorem. 11: The recurrence relation for nth split Mersenne hybrid quaternions and split Mersenne-Lucas hybrid quaternions are

$$\widetilde{SM\mathcal{H}}_n = 3\widetilde{SM\mathcal{H}}_{n-1} - 2\widetilde{SM\mathcal{H}}_{n-2}$$

and

$$\widetilde{SML\mathcal{H}}_n = 3\widetilde{SML\mathcal{H}}_{n-1} - 2\widetilde{SML\mathcal{H}}_{n-2}$$

Proof:

$$3\widetilde{SM\mathcal{H}}_{n-1} - 2\widetilde{SM\mathcal{H}}_{n-2}$$

$$= 3(M\mathcal{H}_{n-1}e_0 + M\mathcal{H}_ne_1 + M\mathcal{H}_{n+1}e_2 + M\mathcal{H}_{n+2}e_3) - 2(M\mathcal{H}_{n-2}e_0 + M\mathcal{H}_{n-1}e_1 + M\mathcal{H}_ne_2 + M\mathcal{H}_ne_3)$$

$$= (3M\mathcal{H}_{n-1} - 2M\mathcal{H}_{n-2})e_0 + (3M\mathcal{H}_n - 2M\mathcal{H}_{n-1})e_1 + (3M\mathcal{H}_{n+1} - 2M\mathcal{H}_n)e_2 + (3M\mathcal{H}_{n+2} - 2M\mathcal{H}_{n+1})e_3$$

$$= M\mathcal{H}_ne_0 + M\mathcal{H}_{n+1}e_1 + M\mathcal{H}_{n+2}e_2 + M\mathcal{H}_{n+3}e_3$$

$$= \widetilde{SM\mathcal{H}}_n$$

Similarly, $\widetilde{SML\mathcal{H}}_n = 3\widetilde{SML\mathcal{H}}_{n-1} - 2\widetilde{SML\mathcal{H}}_{n-2}$

Theorem. 12: Let $\widetilde{SM\mathcal{H}}_n$ be the nth split Mersenne hybrid quaternions then

i. $\sum_{m=2}^{n}\widetilde{SM\mathcal{H}}_n = 2\left(\widetilde{SM\mathcal{H}}_{n-1} - \widetilde{SM\mathcal{H}}_0\right) + \sum_{m=1}^{n-1}\widetilde{SM\mathcal{H}}_m$
ii. $3\sum_{m=1}^{n}\widetilde{SM\mathcal{H}}_{2m-1} = 2\left(\widetilde{SM\mathcal{H}}_0 - \widetilde{SM\mathcal{H}}_{2n}\right) + 3\sum_{m=1}^{n}\widetilde{SM\mathcal{H}}_{2m}$
iii. $3\sum_{m=1}^{n}\widetilde{SM\mathcal{H}}_{2m} = 2\left(\widetilde{SM\mathcal{H}}_1 - \widetilde{SM\mathcal{H}}_{2n+1}\right) + 3\sum_{m=1}^{n}\widetilde{SM\mathcal{H}}_{2m+1}$

Proof:

i. From the recurrence relation for the split Mersenne hybrid quaternions,

$$\widetilde{SM\mathcal{H}}_2 = 3\widetilde{SM\mathcal{H}}_1 - 2\widetilde{SM\mathcal{H}}_0$$

$$\widetilde{SM\mathcal{H}}_3 = 3\widetilde{SM\mathcal{H}}_2 - 2\widetilde{SM\mathcal{H}}_1$$

$$\widetilde{SM\mathcal{H}}_4 = 3\widetilde{SM\mathcal{H}}_3 - 2\widetilde{SM\mathcal{H}}_2$$

$$\vdots$$

$$\widetilde{SM\mathcal{H}}_{n-1} = 3\widetilde{SM\mathcal{H}}_{n-2} - 2\widetilde{SM\mathcal{H}}_{n-3}$$

$$\widetilde{SM\mathcal{H}}_n = 3\widetilde{SM\mathcal{H}}_{n-1} - 2\widetilde{SM\mathcal{H}}_{n-2}$$

$$\widetilde{SM\mathcal{H}}_2 + \widetilde{SM\mathcal{H}}_3 + \cdots + \widetilde{SM\mathcal{H}}_n = \widetilde{SM\mathcal{H}}_1 - 2\widetilde{SM\mathcal{H}}_0 + \widetilde{SM\mathcal{H}}_2 + \cdots + \widetilde{SM\mathcal{H}}_{n-2} + 3\widetilde{SM\mathcal{H}}_{n-1}$$

$$\sum_{m=2}^{n}\widetilde{SM\mathcal{H}}_n = 2\left(\widetilde{SM\mathcal{H}}_{n-1} - \widetilde{SM\mathcal{H}}_0\right) + \sum_{m=1}^{n-1}\widetilde{SM\mathcal{H}}_m$$

ii. From the recurrence relation for the split Mersenne hybrid quaternions,

$$3\widetilde{SM\mathcal{H}}_1 = \widetilde{SM\mathcal{H}}_2 + 2\widetilde{SM\mathcal{H}}_0$$

$$3\widetilde{SM\mathcal{H}}_3 = \widetilde{SM\mathcal{H}}_4 + 2\widetilde{SM\mathcal{H}}_2$$

$$3\widetilde{SMH}_5 = \widetilde{SMH}_6 + 2\widetilde{SMH}_4$$
$$\vdots$$
$$3\widetilde{SMH}_{2n-3} = \widetilde{SMH}_{2n-2} + 2\widetilde{SMH}_{2n-4}$$
$$3\widetilde{SMH}_{2n-1} = \widetilde{SMH}_{2n} + 2\widetilde{SMH}_{2n-2}$$
$$3\sum_{m=1}^{n}\widetilde{SMH}_{2m-1} = 2\widetilde{SMH}_0 + 3\widetilde{SMH}_2 + 3\widetilde{SMH}_4 + \cdots + 3\widetilde{SMH}_{2n-2} + \widetilde{SMH}_{2n}$$
$$3\sum_{m=1}^{n}\widetilde{SMH}_{2m-1} = 2\left(\widetilde{SMH}_0 - \widetilde{SMH}_{2n}\right) + 3\sum_{m=1}^{n}\widetilde{SMH}_{2m}$$

iii. $$3\widetilde{SMH}_2 = \widetilde{SMH}_3 + 2\widetilde{SMH}_1$$
$$3\widetilde{SMH}_4 = \widetilde{SMH}_5 + 2\widetilde{SMH}_3$$
$$3\widetilde{SMH}_6 = \widetilde{SMH}_7 + 2\widetilde{SMH}_5$$
$$\vdots$$
$$3\widetilde{SMH}_{2n-2} = \widetilde{SMH}_{2n-1} + 2\widetilde{SMH}_{2n-3}$$
$$3\widetilde{SMH}_{2n} = \widetilde{SMH}_{2n+1} + 2\widetilde{SMH}_{2n-1}$$
$$3\sum_{m=1}^{n}\widetilde{SMH}_{2m} = 2\widetilde{SMH}_1 + 3\widetilde{SMH}_3 + 3\widetilde{SMH}_5 + \cdots + 3\widetilde{SMH}_{2n-1} + \widetilde{SMH}_{2n+1}$$
$$3\sum_{m=1}^{n}\widetilde{SMH}_{2m} = 2\left(\widetilde{SMH}_1 - \widetilde{SMH}_{2n+1}\right) + 3\sum_{m=1}^{n}\widetilde{SMH}_{2m+1}$$

Theorem. 13: Let $\widetilde{SMLH}_n$ be the nth split Mersenne-Lucas hybrid quaternions, then

i. $\sum_{m=2}^{n}\widetilde{SMLH}_n = 2\left(\widetilde{SMLH}_{n-1} - \widetilde{SMLH}_0\right) + \sum_{m=1}^{n-1}\widetilde{SMLH}_m$

ii. $3\sum_{m=1}^{n}\widetilde{SMLH}_{2m-1} = 2\left(\widetilde{SMLH}_0 - \widetilde{SMLH}_{2n}\right) + 3\sum_{m=1}^{n}\widetilde{SMLH}_{2m}$

iii. $3\sum_{m=1}^{n}\widetilde{SMLH}_{2m} = 2\left(\widetilde{SMLH}_1 - \widetilde{SMLH}_{2n+1}\right) + 3\sum_{m=1}^{n}\widetilde{SMLH}_{2m+1}$

Proof: The proof is obtained by using the definition and recurrence relation of the split Mersenne-Lucas hybrid quaternions.

Conclusions:

The present work focuses on split Mersenne and Mersenne-Lucas hybrid quaternions. Further results in this paper explicit that split Mersenne and Mersenne-Lucas hybrid octonions.

Author's declaration:

- Conflicts of Interest: None
- Ethical Clearance: The project was approved by the local ethical committee at Sri Meenakshi Government Arts College for Women (A).

Author's contributions:

B.M. and S.D. contributed to the analysis of the results and the writing of the manuscript. All authors read and approved the final manuscript

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في تقسيم رباعيات مرسين و مرسين-لوكس الهجينة

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الخلاصة:

في هذا الاتصال ، تم تقديم رباعيات مرسين و مرسين-لوكس الهجينة المنقسمة ، وكذلك تم الحصول على دوال توليد وصيغ بينت لهذه الرباعيات الهجينة والتحقيق في بعض الخصائص فيما بينها.

الكلمات المفتاحية: صيغة بينية، أرقام هجينة، تسلسل ميرسين، تسلسل ميرسين لوكاس، الرباعية.