Open Access | Baghdad Science Journal | P-ISSN: 2078-8665
2023, 20(1 Special Issue) ICAAM: 350-358 | E-ISSN: 2411-7986

DOI:https://dx.doi.org/10.21123/bsj.2023.8432

Topological Structures on Vertex Set of Digraphs

K.Lalithambigai[1] ***P.Gnanachandra****[*2]

[1]Department of Mathematics, Sri Kaliswari College (Autonomous-affiliated to Madurai Kamaraj University), Sivakasi, Tamil Nadu, India.
[2]Centre for Research and Post Graduate Studies in Mathematics,AyyaNadarJanakiAmmal College(Autonomous-affiliated to Madurai Kamaraj University), Sivakasi, Tamil Nadu, India.
*Corresponding author: pgchandra07@gmail.com
E-mail address:lalithambigaimsc@gmail.com
ICAAM= International Conference on Analysis and Applied Mathematics 2022.

Received 21/1/2023, Revised 13/2/2023, Accepted 14/2/2023, Published 1/3/2023

Abstract:

Relation on a set is a simple mathematical model to which many real-life data can be connected. A binary relation R on a set X can always be represented by a digraph. Topology on a set Xcan be generated by binary relations on the set X. In this direction, the study will consider different classical categories of topological spaces whose topology is defined by the binary relations adjacency and reachability on the vertex set of a directed graph. This paper analyses some properties of these topologies and studies the properties of closure and interior of the vertex set of subgraphs of a digraph. Further, some applications of topology generated by digraphs in the study of biological systems are cited.

Keywords: Basis for a topology,Closure,Digraph,Interior,Subbasis for a topology.

Introduction:

Topology is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, rumpling, and bending. For a very long time, it was believed that abstract topological structures have limited applications in the generalization of the real line and complex plane or some connections to Algebra and other branches of Mathematics. Further, it seems that there is a large gap between these structures and real-life applications. Generating topologies by relations and the representation of topological concepts through binary relations will narrow the gap between Topology and its applications. Binary relations are used in the construction of topological structures in many fields such as dynamics, rough set theory and approximation space, digital topology, biochemistry, and biology. In 1969,Smithson[1], generated topologic al structures via relations on a set. In 1993, Slapal[2] studied the methods of generating topologies through binary relations. In 2008, Salama[3] introduced different approaches for obtaining topologies by similarity and pre-order relations; and Allam, et al.[4] obtained quasi-discrete topology from symmetric relations. Khalifa and Jasim.[5] introduced a nano-topological space via graph theory which depends on the neighborhood between the vertices based on an undirected graph. The motivating insight behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way, they are put together.

In Mathematics, a manifold is a topological space that locally resembles Euclidean space near each point. One-dimensional manifolds include lines and circles. Two-dimensional manifolds are also called surfaces. Manifolds naturally arise as solution sets of systems of equations and as graphs of functions. The concept has applications in computer graphics given the need to associate pictures with co-ordinates. Aboodand Abass[6]have determined the components of the covariant derivative of the Riemannian curvature tensor which are mostly applicable in manifolds.

In the very first research on Topology, Leonhard Euler demonstrated that it was impossible to find a route through the town of Konigsberg that would cross each of its seven bridges exactly once. This result did not depend on the lengths of the bridges or their distance from one another, but only on connectivity properties: which bridges connect to which islands or river banks. These seven bridges of the Konigsberg problem led to the branch of Mathematics known as Graph Theory. Graphs can be regarded as a one-dimensional topological space.

The metric dimension and dominating set are the concepts of graph theory that can be developed in terms of the concept and its application in graph operations. One of some concepts in graph theory that combine these two concepts is resolving dominating numbers. Adirasari RP, Suprajitno H, and Susilowati L[7] studied resolvent dominating numbers as the dominant metric dimension and explored some of their properties.

Taha and Awad[8] introduced a new class of separation axioms via the concepts of graphs. Lalithambigai and Gnanachandra[9] introduced the method of generating topologies on a vertex set of graphs using the relations adjacency, non-adjacency, incidence, and non-incidence on the vertex set of graphs and studied the characterizations of closure and interior of vertex induced subgraphs of the graphs. They introduced graph grills and studied the topological structures induced by graph grills. Abdelmonen et al.[10]introduced the technique to construct a new type of topological structure by graphs called topological graphs.

Iman and Hassan[11] have studied the independent incompatible edge topology on digraphs. In this paper, topological structures induced by the relations adjacency and reachability of vertices on the vertex set of digraphs, various neighborhoods of vertices in digraphs and their approximations are defined and some properties of closure and interior of sub-digraphs with respect to the topologies induced are studied.

Preliminaries:

In this section, basic definitions of digraphs that are required for the study are presented. The definitions cited in this Preliminaries section are taken from the textbooks listed in References[12]. The notions of topology are taken from the textbook listed in References[13].

A directed graph or a digraph D is a pair (V, A) where V is a finite nonempty set and A is a subset of $V \times V\{(x,x): x \in V\}$.The elements of V and A are respectively called vertices and arcs. If $(u, v) \in A$ then the arc (u, v) is said to have u as its initial vertex and v as its terminal vertex. Also the arc (u, v) is said to join u to v. The in-degree (or in-valence) $d^-(v)$ of a vertex v in a digraph,D is the number of arcs having v as its terminal vertex. The out-degree (or out-valence) $d^+(v)$ of v is the number of arcs having v as its initial vertex. Digraphs in which for every edge (a, b) there is also an edge (b, a) are called symmetric digraphs. A digraph $D' = (V', A')$ is called a subdigraph of $D = (V, A)$ if $V' \subseteq V$and $A' \subseteq A$. D' is called an induced subgraph of D if D' is the maximal subgraph of D with vertex set V'.The converse digraph $D'$of a digraph, D is obtained from D by reversing the direction of each arc. A digraph $D = (V, A)$ is called complete if for every pair of distinct points v and w in V, both (v, w) and (w, v) are in A. A walk in a digraph is a finite alternating sequence of $v_0, x_1, v_1 \dots x_n v_n$ of vertices and arcs in which $x_i = (v_{i-1}, v_i)$for everyarc x_i. The vertices $v_0$and $v_n$arecalled origin and terminus of the walk respectively and $v_1, v_2, \dots, v_{n-1}$ are called internal vertices.A path is a walk in which all the vertices are distinct. If there is a path from u to $v$then$v$ is said to be reachable from u. A digraph is called strongly connected if every pair of points are mutually reachable.

Topologies induced by vertex adjacency on the Vertex Set of a digraph

This section aims to present the methodology of generating topologies on a vertex set of digraphs based on the notion of the valence of vertices in digraphs. The main properties of the induced topology and various approximations of different neighborhood sets are studied. Further, the basic properties of closure and interior of subgraphs with respect to these induced topological spaces are analyzed.

Definition. 1:

Let $D = (V, A)$ be a digraph with $d^+(v) \geq 1$ and $d^-(v) \geq 1$ for every $v \in V$. For every $u \in V$,define$u^+(D) = \{v : uv \in A(D)\}, u^-(D) = \{v : vu \in A(D)\}$.Let $S_1(D) = \{u^+(D) : u \in V(D)\}$, $S_2(D) = \{u^-(D) : u \in V(D)\}$. Then $S_1(D)$ and $S_2(D)$ form a subbase for topologies $T_1(D)$ and $T_2(D)$ on $V(D)$ and the pairs $(V(D), T_1(D)), (V(D), T_2(D))$ are called out-valence topological space and in-valence topological space respectively.

Example.1:

Consider the digraph in Fig.1.

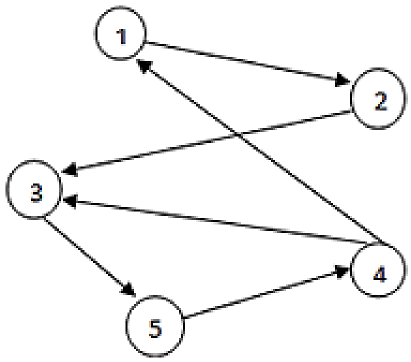

Figure 1. Example of Valence Topological Space

$1^+(D)$ = {2}, $2^+(D)$ = {3}, $3^+(D)$ = {5}, $4^+(D)$ = {1,3}, $5^+(D)$ = {4}.
T_1 = {ϕ, {2}, {3}, {5}, {1,3}, {4}, {2,3}, {2,5}, {1,2,3}, {2,4}, {3,5}, {3,4}, {1,3,5}, {4,5},{1,3,4}, {2,3,5}, {1,3,5,4}, {1,2,3,5}, {2,3,4}, {2,4,5}, {1,2,4,5}, {1,2,3,4,5}}.
$1^-(D)$ = {4}, $2^-(D)$ = {1}, $3^-(D)$ = {2,4}, $4^-(D)$ = {5}, $5^-(D)$ = {3}.
T_2 = {ϕ,{4}, {1}, {2,4}, {5}, {3}, {1,4}, {4,5}, {3,4}, {1,2,4}, {4}, {1,5}, {1,3}, {2,4,5}, {2,3,4},{3,5}, {1,2,4}, {1,2,4,5}, {2,3,4,5}, {3,4,5}, {1,3,4,5}, {1,2,3,4,5}}.

Definition.2:

Let T be a topology on a set X. If $T^c = \{G^c : G \in T\}$ is also a topology on X, then T^c is the dual of T.

From the definitions cited above the following observations are derived:

Observation.1:

1. For any digraph D, $T_1(D)$ is not the dual of $T_2(D)$.
2. In a symmetric digraph D, $T_1(D) = T_2(D)$.
3. If D is a complete digraph, then $T_1(D) = T_2(D)$.
4. If $D'$is a converse digraph of a digraph D, then $u^+(D) = u^-(D)$and $u^-(D) = u^+(D)$.So $T_1(D) = T_2(D)$and $T_2(D) = T_1(D)$.

Definition.3:

For any vertex $v$in $V(D)$, define
$\langle v\rangle^+ = \{\cap_{v\in u^+(D)} u^+(D)$, if there exists $u \in V(D)$ such that $v \in u^+(D)$ and ϕ otherwise} and
$\langle v\rangle^- = \{\cap_{v\in u^-(D)} u^-(D)$, if there exists $u \in V(D)$ such that $v \in u^-(D)$ and ϕ otherwise}.
The families $B_1 = \{\langle v\rangle^+ : v \in V(D)\}$ and $B_2 = \{\langle v\rangle^- : v \in V(D)\}$ form a basis for topologies $\tau^+_{\langle v\rangle}(D)$ and $\tau^-_{\langle v\rangle}(D)$ respectively. The pair $(V(D),\tau^+_{\langle v\rangle}(D))$,$(V(D),\tau^-_{\langle v\rangle}(D))$ are called min out-valence topological space and min in-valence topological space respectively.

Example.2:
Consider the digraph in Fig.2.

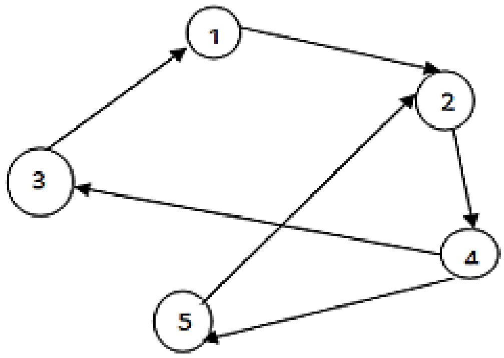

Figure 2. Example of min Valence Topological Space

$1^+(D)$ = {2}, $2^+(D)$ = {4}, $3^+(D)$ = {1}, $4^+(D)$ = {5,3}, $5^+(D)$ = {2}.
$1^-(D)$ = {3}, $2^-(D)$ = {1,5}, $3^-(D)$ = {4}, $4^-(D)$ = {2},$5^-(D)$ = {4}.
$\langle 1\rangle^+$ = {1}, $\langle 2\rangle^+$ = {2}, $\langle 3\rangle^+$ = {5,3}, $\langle 4\rangle^+$ = {4}, $\langle 5\rangle^+$ = {5,3}
$\langle 1\rangle^-$ = {1,5}, $\langle 2\rangle^-$ = {2}, $\langle 3\rangle^-$ = {3}, $\langle 4\rangle^-$ = {4}, $\langle 5\rangle^-$ = {1,5}
$S_1(D)$ = {{2},{4},{1},{5,3},{2}}.
Basis generated by $S_1(D)$ = {{2},{4},{1},{5,3}}.
B_1 = {{2},{4},{1},{5,3}}.
Basis generated by $S_1(D) = B_1$ and so $T_1(D) = \tau^+_{\langle v\rangle}(D)$.
$S_2(D)$ = {{3},{1,5},{4},{2},{4}}
Basis generated by $S_2(D)$ = {{3},{1,5},{4},{2}}.
B_2 = {{3},{1,5},{4},{2}}.
Basis generated by $S_2(D) = B_2$ and so $T_2(D) = \tau^-_{\langle v\rangle}(D)$.

There may exist digraphs such that $T_1(D) \neq \tau^+_{\langle v\rangle}(D)$ and $T_2(D) \neq \tau^-_{\langle v\rangle}(D)$.
For instance:

Example.3
Consider the digraph in Fig.3

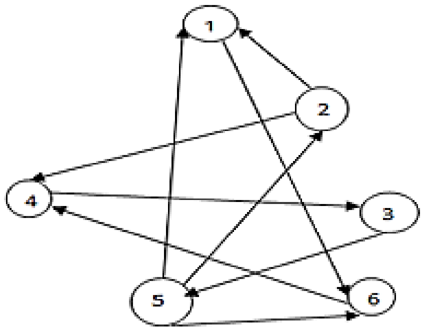

Figure 3. Example for $T_1(D) \neq \tau^+_{\langle v\rangle}(D)$ and $T_2(D) \neq \tau^-_{\langle v\rangle}(D)$

$1^+(D)$ = {6}, $2^+(D)$ = {1,4}, $3^+(D)$ = {5},
$4^+(D)$ = {3}, $5^+(D)$ = {1,2}, $6^+(D)$ = {4,5}.
$1^-(D)$ = {2,5}, $2^-(D)$ = {5}, $3^-(D)$ = {4},
$4^-(D)$ = {2,6}, $5^-(D)$ = {3,6}, $6^-(D)$ = {1}.
$\langle 1\rangle^+ = \{1\}$, $\langle 2\rangle^+ = \{1,2\}$, $\langle 3\rangle^+ = \{3\}$, $\langle 4\rangle^+ = \{4\}$, $\langle 5\rangle^+ = \{5\}$, $\langle 6\rangle^+ = \{6\}$.
$\langle 1\rangle^- = \{1\}$, $\langle 2\rangle^- = \{2\}$, $\langle 3\rangle^- = \{3,6\}$, $\langle 4\rangle^- = \{4\}$, $\langle 5\rangle^- = \{5\}$, $\langle 6\rangle^- = \{6\}$.
$S_1(D)$ = {{6}, {1,4}, {5}, {3}, {1,2}, {4,5}}.
Basis generated by $S_1(D)$ =
{{6}, {1,4}, {5}, {3}, {1,2}, {4,5}, {1}, {4}}.
$B_1 = \{\{1\}, \{1,2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$.
Basis generated by $S_1(D) \neq B_1$ and so $T_1(D) \neq \tau^+_{\langle v\rangle}(D)$.
$S_2(D)$ = {{2,5}, {5}, {4}, {2,6}, {3,6}, {1}}.
Basis generated by $S_2(D)$ =
{{2,5}, {5}, {4}, {2,6}, {3,6}, {1}, {2}, {6}}.
$B_2 = \{\{1\}, \{2\}, \{3,6\}, \{4\}, \{5\}, \{6\}\}$.
Basis generated by $S_2(D) \neq B_2$ and so $T_2(D) \neq \tau^-_{\langle v\rangle}(D)$.

The following definition depicts the method of determining closure and interior of subgraphs of digraphs with respect to the valence of vertices.

Definition.4:

Let H be a subgraph of D. In the out-valence topological space ($V(D)$,$T_1(D)$), the closure of V(H), cl_1(V(H)), is defined as cl_1(V(H)) = V(H) ∪ {v ∈ V(D) : v^+(D)∩ V(H) ≠ ϕ} and the interior of V(H),int_1(V(H)),is defined as int_1(V(H)) = {v ∈V(D) : v^+(D) ⊆ V(H)}.In the in-valence topological space ($V(D)$,$T_2(D)$), the closure of V(H),cl_2(V(H)), is defined as cl_2(V(H)) = V(H) ∪ {v ∈ V(D) : v^-(D)∩ V(H) ≠ ϕ} and the interior of V(H), int_2(V(H)), is defined as int_2(V(H)) = {v∈ V(D) : v^-(D) ⊆ V(H)}. In the min out-valence topological space ($V(D)$, $\tau^+_{\langle v\rangle}(D)$), the closure of V(H), cl_3(V(H)), is defined as cl_3(V(H)) = V(H) ∪ {v ∈ V(D): $\langle v\rangle^+$ ∩ V(H) ≠ ϕ} and the interior of V(H), int_3(V(H)), is defined asint_3(V(H))= {v ∈ V(D) : $\langle v\rangle^+$ ⊆ V(H)}.In the min in-valence topological space ($V(D)$, $\tau^-_{\langle v\rangle}(D)$), the closure of V(H), cl_4(V(H)), is defined as cl_4(V(H)) = V(H) ∪ {v∈ V(D): $\langle v\rangle^-$ ∩ V(H) ≠ ϕ} and the interior of V(H), int_4(V(H)), is defined as int_4(V(H)) ={v∈ V(D) : $\langle v\rangle^-$ ⊆ V(H)}.

The following definition defines a new topology generated from the old one.

Definition.5:

For i = 1,2,3,4, $\tau_i{}^*$ is defined as the topology generated by the closure cl_i.(ie), the topology consisting of complements of cl_i-closed sets. (i.e.) $\tau_i{}^*$= {A∈ τ_i: cl_i(A) = A}

The following theorems relate the new topologies $\tau_1{}^*$and $\tau_2{}^*$.

Theorem 1:$\tau_1{}^*$ is the dual of $\tau_2{}^*$.
Proof: To prove, if A ∈ $\tau_1{}^*$ then $A^c \in \tau_2{}^*$.
(i.e.) if cl_1(A)=A then to prove $cl_2(A^c)=A^c$.
By definition, $A^c \subseteq cl_2(A^c)$.
If y ∈ $cl_2(A^c)$ then y ∈ A^c or y ∈ {v : v^-(D) ∩ A^c ≠ ϕ}.
If y ∈A^c then the result follows.
Let y ∈ {v : v^-(D) ∩ A^c ≠ ϕ}. So y^-(D) ∩ A^c ≠ ϕ.
Hence there exists x ∈ A^c such that x∈y^-(D).
x∈ y^-(D) ⇒ xy is an arc in D.
But x ∈ A^c ⇒ x ∉ A ⇒ x ∉ cl_1(A) ⇒ x^+(D)∩ A = ϕ ⇒ no element of A is in x^+(D).
Hence y ∉ A.So y ∈ A^c.
Hence $cl_2(A^c) = A^c$ and so $\tau_1{}^*$ is the dual of $\tau_2{}^*$.

Theorem.2:$A \in \tau_1^*$ if and only if $\cup_{v\in A} v^-(D) \subseteq A$.
Proof: By Theorem1, A∈ $\tau_1{}^*$ ⇔ $A^c \in \tau_2{}^*$
⇔A is $\tau_2{}^*$-closed
⇔cl_2(A) = A
⇔ A ∪ {v : v^-(D) ∩ A ≠ ϕ} = A
⇔$\cup_{v\in A} v^-(D) \subseteq A$.

Theorem. 3: A ∈ $\tau_2{}^*$ if and only if $\bigcup_{v\in A} v^+(D) \subseteq A$.
Proof: Proof is similar to that of Theorem 2.

Theorem.4: If D is a symmetric digraph, then $\tau_1{}^*= \tau_2{}^*$.

Proof: Now, $cl_1(V(H) = V(H) \cup \{v \in V(D): v^+(D) \cap V(H) \neq \phi\}$.
$= V(H) \cup \{v \in V(D): v^-(D) \cap V(H) \neq \phi\}$.
$= cl_2(V(H))$.
Also, $int_1(V(H) = \{v \in V(D): v^+(D) \subseteq V(H)\}$.
$= \{v \in V(D): v^-(D) \subseteq V(H)\}$.
$= int_2(V(H))$.
Hence $\tau_1{}^*= \tau_2{}^*$.

The basic properties of closure and interior with respect to the valence of vertices are presented in the following lemmas.

Lemma.1: If D is a symmetric digraph, then the following are equivalent:
(i) $V(H) = int_1(V(H)) = int_2(V(H))$
(ii)$V(H) = cl_1(V(H)) = cl_2(V(H))$
Proof: Let $V(H)$ be τ_1^* open. Then by Theorem2, $\cup_{v\in V(H)} v^-(D) \subseteq V(H)$.
So $\cup_{v\in V(H)} v^+(D) \subseteq V(H)$.

Now,

$$cl_1(V(H)) = V(H) \cup \{v \in V(D): v^+(D) \cap V(H) \neq \phi\} = V(H) \cup \{v \in V(D): v^-(D) \cap V(H) \neq \phi\} = V(H)$$

So V(H) is ${\tau_1}^*$ closed.

Conversely, let $V(H)$ be ${\tau_1}^*$ closed. Then $cl_1(V(H)) = V(H)$.

So $V(H) \cup \{v \in V(D): v^+(D) \cap V(H) \neq \phi\} = V(H)$.

Hence $\cup_{v\in V(H)} v^+(D) \subseteq V(H)$.

So $\cup_{v\in V(H)} v^-(D) \subseteq V(H)$.

By Theorem3, $V(H)$ is τ_1^* open.

Lemma.2: In the min out-valence topological space $(V(D), \tau^+_{\langle v\rangle}(D))$, $cl_3\left(cl_3(V(H))\right) = cl_3(V(H))$.

Proof: By definition, $cl_3(V(H)) \subseteq cl_3\left(cl_3(V(H))\right)$.

Now, $v \in cl_3\left(cl_3(V(H))\right) \Rightarrow v \in cl_3(V(H))$or $v \in \{v \in V(D): \langle v\rangle^+ \cap cl_3(V(H)) \neq \phi\}$.

If $v \in cl_3(V(H))$, then there is nothing to prove.

Now, $v \in \{v \in V(D): \langle v\rangle^+ \cap cl_3(V(H)) \neq \phi\} \Rightarrow \langle v\rangle^+ \cap cl_3(V(H)) \neq \phi$.

$\Rightarrow \langle v\rangle^+ \cap [V(H) \cup \{x \in V(D): \langle x\rangle^+ \cap V(H) \neq \phi\}] \neq \phi$.

$\Rightarrow [\langle v\rangle^+ \cap V(H)] \cup [\langle v\rangle^+ \cap \{x \in V(D): \langle x\rangle^+ \cap V(H) \neq \phi\}] \neq \phi$.

$\Rightarrow$either$[\langle v\rangle^+ \cap V(H)] \neq \phi$ or $[\langle v\rangle^+ \cap \{x \in V(D): \langle x\rangle^+ \cap V(H) \neq \phi\}] \neq \phi$.

$\Rightarrow$either$v \in cl_3(V(H))$ or there exists $z \in [\langle v\rangle^+ \cap \{x \in V(D): \langle x\rangle^+ \cap V(H) \neq \phi\}]$.

Now, $z \in \langle v\rangle^+ \Rightarrow \langle z\rangle^+ \subseteq \langle v\rangle^+$.

$z \in \{x \in V(D): \langle x\rangle^+ \cap V(H) \neq \phi\} \Rightarrow \langle z\rangle^+ \cap V(H) \neq \phi$.

$\Rightarrow v \in cl_3(V(H))$.

Hence $cl_3\left(cl_3(V(H))\right) \subseteq cl_3(V(H))$.

The following definition defines some more neighborhood sets on the vertex set of a digraph.

Definition.6:

For a vertex $u \in V(D)$, the neighborhood sets are defined by:

$$OA(u) = \{v \in V(D): u^+(D) = v^+(D)\}$$
$$IA(u) = \{v \in V(D): u^-(D) = v^-(D)\}$$
$$minOA(u) = \{v \in V(D): \cap_{u\in y^+(D)} y^+(D) = \cap_{v\in x^+(D)} x^+(D)\}$$
$$minIA(u) = \{v \in V(D): \cap_{u\in y^-(D)} y^-(D) = \cap_{v\in x^-(D)} x^-(D)\}$$

Fromthe above neighborhood sets, the valence set $A(D)$ and min-valence set $minA(D)$ are defined by:

$$A(D) = \{v \in V(D): OA(v) = IA(v)\}$$
$$minA(D) = \{v \in V(D): minOA(v) = minIA(v)\}$$

Also, the approximations of the neighborhood sets can be defined as follows:

If H is a subgraph of D,then

$$\overline{\mathrm{OA}}(V(H)) = V(H) \cup \{u \in V(D): OA(u) \cap V(H) \neq \phi\}$$
$$\underline{\mathrm{OA}}(V(H)) = \{u \in V(D): OA(u) \subseteq V(H)\}$$
$$\overline{\mathrm{IA}}(V(H)) = V(H) \cup \{u \in V(D): IA(u) \cap V(H) \neq \phi\}$$
$$\underline{\mathrm{IA}}(V(H)) = \{u \in V(D): IA(u) \subseteq V(H)\}$$
$$\overline{\mathrm{minOA}}(V(H)) = V(H) \cup \{u \in V(D): minOA(u) \cap V(H) \neq \phi\}$$
$$\underline{\mathrm{minOA}}(V(H)) = \{u \in V(D): minOA(u) \subseteq V(H)\}$$
$$\overline{\mathrm{minIA}}(V(H)) = V(H) \cup \{u \in V(D): minIA(u) \cap V(H) \neq \phi\}$$
$$\underline{minIA}\,(V(H)) = \{u \in V(D): minIA(u) \subseteq V(H)\}$$

Example.4:

Consider the digraph in Fig.4

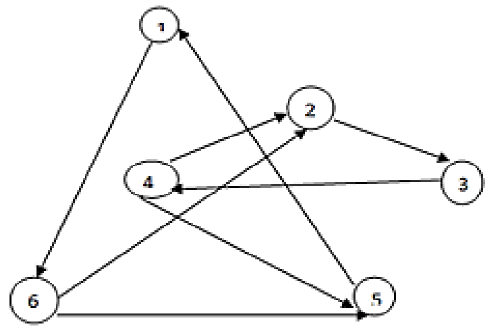

Figure 4. Example of NeighborhoodSets

$$1^+(D) = \{6\}, 2^+(D) = \{3\}, 3^+(D) = \{4\}, 4^+(D) = \{2,5\}, 5^+(D) = \{1\}, 6^+(D) = \{2,5\}$$
$$1^-(D) = \{5\}, 2^-(D) = \{4,6\}, 3^-(D) = \{2\}, 4^-(D) = \{3\}, 5^-(D) = \{4,6\}, 6^-(D) = \{1\}$$
$$OA(1) = \phi, OA(2) = \phi, OA(3) = \phi, OA(4) = \{6\}, OA(5) = \phi, OA(6) = \{4\}$$
$$IA(1) = \phi, IA(2) = \{5\}, IA(3) = \phi, IA(4) = \phi, IA(5) = \{2\}, IA(6) = \phi$$
$$A(D) = \{1,3\}$$
$$minOA(1) = \phi, minOA(2) = \{5\}, minOA(3) = \phi, minOA(4) = \phi, minOA(5) = \{2\}, minOA(6) = \phi$$
$$minIA(1) = \phi, minIA(2) = \phi, minIA(3) = \phi, minIA(4) = \{6\}, minIA(5) = \phi, minIA(6) = \{4\}.$$

$$minA(D) = \{1,3\}$$

For $V(H) = \{1,4\}$,
$\overline{\mathrm{OA}}(V(H)) = \{1,4,6\}$, $\underline{\mathrm{OA}}\,(V(H)) = \{6\}$
$\overline{\mathrm{IA}}(V(H)) = \{1,4\}$, $\underline{IA}(V(H)) = \phi$
$\overline{\mathrm{minOA}}(V(H)) = \{1,4\}$, $\underline{minOA}(V(H)) = \phi$
$\overline{\mathrm{minIA}}(V(H)) = \{1,4,6\}$, $\underline{\mathrm{minIA}}(V(H)) = \{6\}$.
So, A(D) = minA(D)

Now, seeing an example in which $A(D) \neq minA(D)$.

Example.5:
Consider the digraph in Fig.5

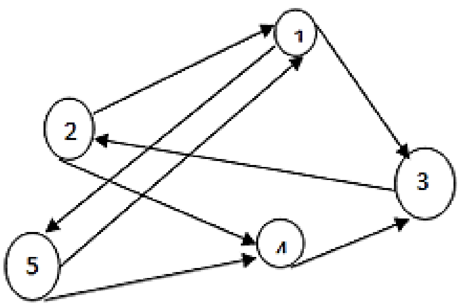

Figure 5. Example for $A(D) \neq min\ A(D)$

$1^+(D) = \{3,5\}, 2^+(D) = \{1,4\}, 3^+(D) = \{2\}, 4^+(D) = \{3\}, 5^+(D) = \{1,4\}$
$1^-(D) = \{5,2\}, 2^-(D) = \{3\}, 3^-(D) = \{1,4\}, 4^-(D) = \{2,5\}, 5^-(D) = \{1\}$ $OA(1) = \phi, OA(2) = \{5\}, OA(3) = \phi, OA(4) = \phi, OA(5) = \{2\}$ $IA(1) = \{4\}, IA(2) = \phi, IA(3) = \phi, IA(4) = \{1\}, IA(5) = \phi$
$A(D) = \{1,3\}$
$minOA(1) = \{4\}, minOA(2) = \phi, minOA(3) = \phi, minOA(4) = \{1\}, minOA(5) = \phi$ $minIA(1) = \phi, minIA(2) = \{5\}, minIA(3) = \phi, minIA(4) = \phi, minIA(5) = \{2\}, minA(D) = \{3\}$
So $A(D) \neq minA(D)$.

Note that, for any digraph $D, minA(D) \subseteq A(D)$.
The following proposition is a consequence of the respective definitions:

Proposition.1: (i) $\underline{OA}(V(D)) = V(D)$ and $\overline{OA}(V(D)) = V(D)$
(ii) $\underline{OA}(V(H)) \subseteq V(H)$ and $V(H) \subseteq \overline{OA}(V(H))$
(iii) If $V(H) \subseteq V(K)$, then $\underline{OA}(V(H)) \subseteq \underline{OA}(V(K))$ and $\overline{OA}(V(H)) \subseteq \overline{OA}(V(K))$
(iv) $\underline{OA}\left(\underline{\mathrm{OA}}(V(H))\right) = \underline{OA}(V(H))$ and $\overline{OA}\left(\overline{\mathrm{OA}}(V(H))\right) = \overline{OA}(V(H))$
(v) $\underline{OA}(V(H) \cap V(K)) = \underline{OA}(V(H)) \cap \underline{OA}(V(K))$ and $\overline{\mathrm{OA}}(V(H) \cap V(K)) \subseteq \overline{OA}(V(H)) \cap \overline{OA}(V(K))$
(vi) $\underline{OA}(V(H)) \cup \underline{OA}(V(K)) \subseteq \underline{\mathrm{OA}}\,(V(H) \cup V(K))$ and $\overline{OA}(V(H) \cup V(K)) = \overline{OA}(V(H)) \cup \overline{OA}(V(K))$
(vii) $\underline{OA}(V(H)) = \left(\underline{\mathrm{OA}}(V(H))^c\right)^c$ and $\overline{OA}(V(H)) = \left(\overline{\mathrm{OA}}(V(H))^c\right)^c$

The above result also holds for other neighborhood sets.

Topologies induced by reachability on Vertex Set of a digraph:

Reachability in digraphs is one of the most common queries in a graph database. In many applications where graphs are used as the basic data structure, reachability is one of the fundamental operations. The efficient processing of reachability queries is critical in the graph database. This section aims to present the methodology of generating topologies on a vertex set of digraphs based on the notion of the reachability of vertices in digraphs. The main properties of the induced topology and the basic properties of closure and interior of subgraphs with respect to those induced topological spaces are analyzed.

Definition.7:

Let $D = (V, A)$ be a digraph with $d^+(v) \geq 1$ and $d^-(v) \geq 1$ for every $v \in V$. For every $u \in V$, define ${u_R}^+(D)$ ={v∈ V(D) : v is reachable from u},${u_R}^-(D)$ ={v ∈ V(D) : u is reachable from v}. Since every vertex is assumed to be reachable from itself, u $\in {u_R}^+(D)$ and u $\in {u_R}^-(D)$.Let ${S_R}^+(D)$ = $\{{u_R}^+(D) : u \in V(D)\}$,${S_R}^-(D)$ =$\{{u_R}^-(D) : u \in V(D)\}$. Then ${S_R}^+(D)$ and ${S_R}^-(D)$ form a subbasefor topologies ${T_R}^+(D)$ and ${T_R}^-(D)$ on $V(D)$ and the pairs $(V(D), {T_R}^+(D))$,$(V(D),{T_R}^-(D))$are called out-reachable topological space and in-reachable topological space respectively.

Example.6:
Consider the digraph in Fig.6.

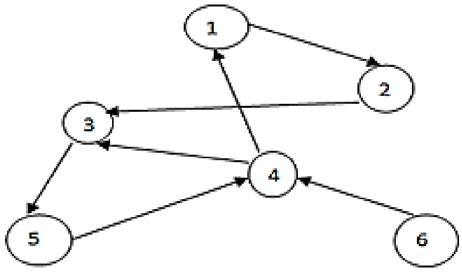

Figure 6. Example of Reachability Topological Space

$1_R{}^+(D)$ = {1,2,3,4,5}, $2_R{}^+(D)$ = {1,2,3,4,5}, $3_R{}^+(D)$ = {1,2,3,4,5}, $4_R{}^+(D)$ = {1,2,3,4,5},
$5_R{}^+(D)$ = {1,2,3,4,5}, $6_R{}^+(D)$ = {1,2,3,4,5,6}
$T_R{}^+(D) = \{\phi$,{ 1,2,3,4,5},{1,2,3,4,5,6}}
$1_R{}^-(D)$ = {1,2,3,4,5,6}, $2_R{}^-(D)$ = {1,2,3,4,5,6}, $3_R{}^-(D)$ = {1,2,3,4,5,6}, $4_R{}^-(D)$ = {1,2,3,4,5,6},
$5_R{}^-(D)$ = {1,2,3,4,5,6}, $6_R{}^-(D)$ = {6}
$T_R{}^-(D)$={ϕ,{1,2,3,4,5,6},{6}}

Observation.2:
If D is a strongly connected digraph, then $T_R{}^+(D)$= $T_R{}^-(D)$.

Theorem.5:i. $A \in T_R^+(D)$ if and only if $A = \cup_{u\in A} u_R^+(D)$
ii. $A \in T_R^-(D)$ if and only if $A = \cup_{u\in A} u_R^-(D)$

Proof:i. Assume that $A \in T_R^+(D)$.
If $y \in A$, then by definition, $y \in y_R^+(D)$.
So, $y \in \cup_{u\in A} u_R^+(D)$ and $A \subseteq \cup_{u\in A} u_R^+(D)$
Assume that $y \in \cup_{u\in A} u_R^+(D)$.
Hence $y \in u_R^+(D)$ for some $u \in A$ and y is reachable from u.
If $z \in y_R^+(D)$, then z is reachable from y and hence z is reachable from u.
So, $z \in u_R^+(D)$and $y_R^+(D) \subseteq u_R^+(D)$.
Since $y \in y_R^+(D)$and $A \in T_R^+(D)$, $y \in A$.
Therefore, $\cup_{u\in A} u_R^+(D) \subseteq A$.
Hence $A =\cup_{u\in A} u_R^+(D)$.
Conversely, assume that $A =\cup_{u\in A} u_R^+(D)$.
For each $u \in A, u_R^+(D)$ belongs to the subbasis of $T_R^+(D)$ and so $\cup_{u\in A} u_R^+(D) \in T_R^+(D)$.
So $A \in T_R^+(D)$.
ii. Proof is similar to that of (i).

The following theorem relates $T_R^+(D)$and $T_R^-(D)$.

Theorem.6:$T_R^+(D)$ is the dual of $T_R^-(D)$.

Proof: It is enough to prove, $A \in T_R^+(D) \Rightarrow A^c \in T_R^-(D)$
By Theorem5, it is enough to prove $A = \cup_{u\in A} u_R^+(D) \Rightarrow A^c =\cup_{u\in A^c} u_R^-(D)$.
Suppose that $A = \cup_{u\in A} u_R^+(D)$and $A^c \neq \cup_{u\in A^c} u_R^-(D)$.
Hence there exists $y \in \cup_{u\in A^c} u_R^-(D)$and $y \notin A^c$.
So, there exists $u \in A^c$ such that $y \in u_R^-(D)$and $y \in A$.
Therefore, u is reachable from $y$and $y \in A$.
Hence $u \in \cup_{y\in A} y_R^+(D)$
So $u \in A$this is a contradiction.
Thus $A^c =\cup_{u\in A^c} u_R^-(D)$.

The following definition depicts the method of determining the closure and interior of subgraphs of digraphs with respect to topologies induced by the reachability of vertices.

Definition.8:
Let H be a subgraph of D. In the out-reachable topological space ($V(D)$,$T_R{}^+(D)$), the closureof V(H), $cl_1^R(V(H))$, is defined as $cl_1^R(V(H)) = V(H) \cup \{v \in V(D): v_R^+(D) \cap V(H) \neq \phi\}$ and the interiorof V(H), $int_1^R(V(H))$, is defined as $int_1^R(V(H)) = \{v \in V(D): v_R^+(D) \subseteq V(H)\}$.In the in-reachable topological space ($V(D)$,$T_R{}^-(D)$), the closure of V(H), $cl_2^R(V(H))$, is defined as $cl_2^R(V(H)) = V(H) \cup \{v \in V(D): v_R^-(D) \cap V(H) \neq \phi\}$ and the interior of V(H), $int_2^R(V(H))$, is defined as $int_2^R(V(H)) = \{v \in V(D): v_R^-(D) \subseteq V(H)\}$.

The basic properties of closure and interior of subgraphs of digraphs with respect to topologies induced by the reachability of vertices are presented in the following proposition.

Proposition.2: (i). If ($V(D)$,$T_R{}^+(D)$) is an out-reachable topological space and H is a subgraph of D, then $cl_1^R\left(cl_1^R(V(H))\right) = cl_1^R(V(H))$.
(ii). If ($V(D)$,$T_R{}^-(D)$) is an in-reachable topological space and H is a subgraph of D, then $l_2^R\left(cl_2^R(V(H))\right) = cl_2^R(V(H))$.
Proof: (i). By definition, $cl_1^R(V(H)) \subseteq cl_1^R\left(cl_1^R(V(H))\right)$.
To prove $cl_1^R\left(cl_1^R(V(H))\right) \subseteq cl_1^R(V(H))$.
Let $v \in cl_1^R\left(cl_1^R(V(H))\right)$.
Hence $v \in cl_1^R(V(H))$or $v \in \{u \in V(D): u_R^+(D) \cap cl_1^R(V(H)) \neq \phi\}$.
If $v \in cl_1^R(V(H))$, then there is nothing to prove.
If $v \in \{u \in V(D): u_R^+(D) \cap cl_1^R(V(H)) \neq \phi\}$, then $v_R^+(D) \cap cl_1^R(V(H)) \neq \phi$.

So there is some $x \in cl_1^R(V(H))$ such that x is reachable from v.
Since $x \in cl_1^R(V(H))$,$x \in V(H)$ or $x \in \{y: y_R^+(D) \cap V(H) \neq \phi\}$.
Now, $x_R^+(D) \cap V(H) \neq \phi$.
⇒there exists $z \in V(H)$ such that z is reachable from x.
⇒there exists $z \in V(H)$ such that z is reachable from v.
⇒there exists $z \in V(H)$ such that $z \in v_R^+(D)$.
$\Rightarrow z \in V(H) \cap v_R^+(D)$.
$\Rightarrow V(H) \cap v_R^+(D) \neq \phi$.
$\Rightarrow v \in cl_1^R(V(H))$.
Hence $cl_1^R\left(cl_1^R(V(H))\right) \subseteq cl_1^R(V(H))$ and so $cl_1^R\left(cl_1^R(V(H))\right) = cl_1^R(V(H))$.
(ii). Proof is similar to that of (i).

Applications:

Complex network theory plays a vital role in the bio-chemical and bio-medical fields. Such networks, electrical circuits, and information systems can be modeled using Graph Theory notion by representing vertices and edges as the nature of the trend of study. The most important feature of the hydrogen bond is that it possesses direction and hence hydrogen bond networks along with co-operativity and antico-opertativity can be modeled as directed graphs. Hydrogen bond networks can be represented by digraphs where vertices correspond to the donor and acceptor group, and arcs correspond to hydrogen bonds from proton-donor to proton-acceptor. Protein functioning can be shown graphically. Interactions between entities such as proteins, chemicals, or macro molecules can be represented using directed graphs and it can also be used to describe biological pathways. The most important issue in our biological system is the process of blood circulation and the functioning of the kidney. Medical tests play an important role in the life of rights to make sure that the retreat of diseases, perhaps the most prominent of those analyzes macro-economic analysis functions. Through the medical application, the system can be modeled graphically. By considering the parts of the heart/kidney as vertices and the flow of blood/liquid between the parts as edges, the systems can be modeled as directed graphs. Interior and closure of induced subgraphs under the topology generated from the resulting directed graph of the system will detect and predict the diseases of the heart/kidney.

Conclusion:

Based on different types of binary relation on a set and the topologies induced by them,subbasis and basis for different topologies on the vertex set of directed graphs are introduced. Using the binary relations,adjacency, and reachability on a vertex set of digraphs different topologies were generated. The different topologies generated were compared under different contexts. Some basic properties of closure and interior of subgraphs of a digraph are studied. The results discussed in this paper will be helpful in further study of some other topological structures and their properties. Also, the results and properties discussed in this paper can be studied further with respect to other binary relations on the vertex set of a digraph. The study of complex networks in the biological field can be done effectively using the mathematical division Graph Theory. The topologies generated using the digraphs can be used to solve problems on digraphs thatfocus on directed paths. This paper can be regarded as an initial stage of studying topological structure on digraphs which could lead to significant applications in real life.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures in the manuscript are ours. Besides, the Figures and images, which are not ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in AyyaNadar Janaki Ammal College.

Authors' Contributions statement:

This work was carried out in collaboration between all authors. K.Lalithambigai and P.Gnanachandra developed the idea of inducing topology on a vertex set of digraphs using binary relations on a vertex set of digraphs. K.Lalithambigai and P.Gnanachandra derived some of the observations and methods of generating topology. K.Lalithambigai wrote the manuscript. P.Gnanachandra edited the manuscript with the revised idea. All authors read and approved the final manuscript.

References:

1. Smithson RE. Topologies Generated by Relations. Bull Austral Math Soc. 1969; 1(3): 297-306. https://doi.org/10.1017/S0004972700042167
2. Slapal J. Relations and Topologies. Czech Math J. 1993; 43(1): 141-150.http://dml.cz/bitstream/handle/10338.dmlcz/128381/CzechMathJ_43-1993-1_12.pdf
3. Salama AS. Topologies Induced by Relations with Applications. J Computer Sci. 2008; 4(10): 877-887.https://thescipub.com/pdf/jcssp.2008.877.887.pdf

4. Allam AA, Bakeir MY, Abo-Tabl EA. Some Methods for Generating Topologies by Relations. BullMalaysian MathSci SocSer. 2. 2008; 31(1): 1-11.http://www.emis.de/journals/BMMSS/pdf/v31n1/v31n1p4.pdf
5. Khalifa WR,Jasim TH. On Study of Some Concepts in Nano Continuity via Graph Theory. Open Access Libr J. 2021; 8:e7568.https://doi.org/10.4236/oalib.1107568
6. Abood HM, Abass MY. On Generalized φ-Recurrent of Kenmotsu Type Manifolds. Baghdad SciJ. 2022 Apr. 1; 19(2): 0304.https://doi.org/10.21123/bsj.2022.19.2.0304
7. Adirasari RP, Suprajitno H, Susilowati L. The Dominant Metric Dimension of Corona Product Graphs. Baghdad SciJ. 2021 Jun. 1; 18(2): 0349.https://doi.org/10.21123/bsj.2021.18.2.0349
8. Jasim TH, Awad AI. Separation Axioms via Graph Theory. JPhys.:ConfSer.2020;1530: 012114.https://doi.org/10.1088/1742-6596/1530/1/012114
9. LalithambigaiK, Gnanachandra P. Topologies Induced by Graph Grills on Vertex Set of Graphs. Neuroquantology. 2022; 20(19): 426-435.
10. Abdelmonen Kozae, Abd El Fattah El Atik, Ashraf Elrokh, Mohamed Atef. New Types of Graphs Induced by Topological Spaces. JIntellFuzzy Syst. 2019 ; 36(4) : 1-10. https://doi.org/10.3233/JIFS-171561
11. Hassan AF, Ali IA. The Independent Incompatible Edges Topology on Digraphs. Multicult Educ. 2021; 7(12): 1-6. https://doi.org/10.5281/zenodo.6078158
12. Chartrand G,Zhang P. Chromatic Graph Theory. 2ndedition. UK: Chapman & Hall/CRC Press; 2019. p. 525.
13. Munkres JR.Topology. 2nd edition. UK: Pearson Education Limited; 2021. p. 556.

الانشاءات التبولوجية للرسومات الثنائية لمجموعة الرؤس

ك. لاليثامبيجاي[1] ب. جناناشاندرا[2] *

[1]قسم الرياضيات، كلية سري كاليسواري (التابعة لجامعة مادوراي كاماراج المستقلة)، سيفاكاسي، تاميل نادو، الهند.
[2]مركز البحوث والدراسات العليا في الرياضيات، كلية أيا نادار جاناكي أمل (التابعة لجامعة مادوراي كاماراج المستقلة)، سيفاكاسي، تاميل نادو، الهند.

الخلاصة:
العلاقة على مجموعة هي نموذج رياضي بسيط يمكن توصيل العديد من بيانات الحياة الواقعية به. يمكن دائما تمثيل العلاقة الثنائية R على المجموعة X بواسطة الرسومات الثنائية. يمكن إنشاء التبولوجيا على المجموعة X بواسطة العلاقات الثنائية على المجموعة X. في هذا الاتجاه ، ستنظر الدراسة في فئات كلاسيكية مختلفة من المساحات التبولوجية التي يتم تعريف تبولوجيتها من خلال تجاور العلاقات الثنائية وإمكانية الوصول على مجموعة الرؤس في الرسم البياني الموجه. يحلل هذا البحث بعض خصائص هذه التبولوجيا ويدرس خصائص الإغلاق والداخلية لمجموعة الرؤوس من الرسوم البيانية الفرعية للرسم البياني. علاوة على ذلك ، يتم الاستشهاد ببعض تطبيقات التبولوجيا الناتجة عن للرسومات الثنائية في دراسة النظم البيولوجية.

الكلمات المفتاحية:أساس الطوبولوجيا، إنهاء، ديجراف، الداخلية، سوباسيس لطوبولوجيا.