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On Existence of Prime K-Tuples Conjecture for Positive Proportion of Admissible K-Tuples

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Abstract

Number theorists believe that primes play a central role in Number theory and that solving problems related to primes could lead to the resolution of many other unsolved conjectures, including the prime k-tuples conjecture. This paper aims to demonstrate the existence of this conjecture for admissible k-tuples in a positive proportion. The authors achieved this by refining the methods of “Goldston, Pintz and Yildirim” and “James Maynard” for studying bounded gaps between primes and prime k-tuples. These refinements enabled to overcome the previous limitations and restrictions and to show that for a positive proportion of admissible k-tuples, there is the existence of the prime k-tuples conjecture holding for each “k”. The significance of this result is that it is unconditional which means it is proved without assuming any form of strong conjecture like the Elliott–Halberstam conjecture.

Keywords: Admissible, Positive Proportion, Primes, Prime k-tuples, Prime k-tuples conjecture, Unsolved conjectures.

Introduction

Primes have always been shrouded in mystery, intriguing mathematicians, especially number theorists, who are continuously curious about their distribution and behavior. There are numerous conjectures and unresolved problems related to primes, one of which is the **Twin prime conjecture**[1]. According to this hypothesis, an infinite quantity of twin primes exists, with each pair comprising two prime numbers that are precisely 2 units apart. It is a special case of the **Prime k-tuples conjecture**[2] which states that a set $H = \{h_1, \ldots, h_k\}$ is considered “admissible” if it consists of distinct non-negative integers, and there exists an integer “a_p” such that $a_p \not\equiv h \pmod{p}$ for all $h \in H$ and for all $p \in$ prime, then “$n + h_1, \ldots, n + h_k$” are primes for infinitely many integers “n”.

The k-tuple $\{n + h_1, \ldots, n + h_k\}$ 1

with “n” as a natural number where $\{h_1, \ldots, h_k\}$ belongs to the set of distinct non-negative integers, can be considered as a prime tuple if all its components are prime. Number theorists are interested in determining how often Eq 1 is a prime tuple.

Let us consider the tuple {n, n+1}, where "n" represents a natural number, and the k-tuple H = {0,1}. By substituting n = 2 into the equation, we obtain the tuple {2,3}, which is prime. However, this is the only prime tuple of this specific form because either n or n+1 is even and greater than 2. The Twin Prime Conjecture suggests that if we consider the k-tuple {n, n+2} with H = {0,2}, there are infinitely many prime tuples of this form. In general, for any k-tuple that contains more than one "n", it can only be a prime tuple if none of the residue classes

modulo p, where "p" is a prime, are occupied by the elements of H. This condition holds true for all primes "p" greater than k. To verify this condition for Eq 1, it is sufficient to test it using smaller primes.
If the number of distinct residue classes modulo p where "p" is a prime which was occupied by the integers h_i, was being denoted by $\varphi_p(H)$, then the requirement is:

$\varphi_p(H) < p$ for all primes p 2

to avoid "p" dividing some component of Eq 1 for every natural number "n".
The condition mentioned above determines the admissibility of a set "H", ensuring that the tuple Eq 1 corresponding to "H" is also admissible. Mathematicians have a **long-standing conjecture** that admissible tuples will occur infinitely often as prime tuples. Although Mathematicians don't know of any cases of prime k-tuples conjecture when k >1. Various mathematicians are working on approximating this conjecture and have also demonstrated that **small gaps between primes** exist. However, Goldston, Pintz, and Yildirim's proposed a unique method for counting prime tuples in their paper "**Primes in tuples**"[3], which led them to demonstrate the following result:

$\rho_1 = \lim_{n\to\infty} \inf \frac{p_{n+1}-p_n}{\log p_n}=0$ 3

Twin prime conjecture is currently under investigation by number theorists who have made notable advancements. One significant breakthrough occurred in 2013 when mathematician "Yitang Zhang" published a paper[4] that established the first finite bound on gaps between primes. i.e.

$\lim_{n\to\infty} \inf (P_{n+1} - P_n) < 7\times 10^7$, where p_n is n^{th} prime number. 4

The main area of his research is the refinement of the techniques employed by "Goldston, Pintz, and Yildirim" in the context of small gaps between consecutive primes[3]. He has achieved impressive results in generating these small gaps by building upon the **Bombieri-Vinogradov Theorem**[5] (stronger version), which applies only when the large prime divisors do not have any moduli.

Bombieri- Vinogradov Theorem 1:[5] Let x and Q be any two positive real numbers with $\frac{x^{1/2}}{(\log x)^A} \le Q \le x^{1/2}$ where A is a positive constant. Then

$\sum_{q\le Q} \max_{y\le x} \max_{1\le a\le q,(a,q)=1} \left|\psi(y;q,a) - \frac{y}{\varphi(q)}\right| = O\left(x^{\frac{1}{2}} Q(\log x)^5\right)$ 5

Here $\varphi(q)$ is the **Euler totient function**, which is the number of summands for the modulus q, and $\psi(y;q,a)=\sum_{n\le y, n\equiv a(\mathrm{mod}\, q)} \Lambda(n)$ where $\Lambda(n)$ is **Von-Mangoldt function** defined as

$$\Lambda(n) = \begin{cases} \log n, n = p^\alpha \ for\ some\ \alpha \in \mathbb{R} \\ 0, otherwise \end{cases}$$

This theorem is a special case of

Elliott-Halberstam Conjecture:[6] $\pi(x;q,a) \approx \frac{\pi(x)}{\varphi(q)}$ where $\pi(x)$ is prime counting function. 6

If the Error function is defined as:

$E(x;q) = \max_{\gcd(a,q)=1} \left|\pi(x;q,a) - \frac{\pi(x)}{\varphi(q)}\right|$

where max is taken over all "a" which is coprime to "q", then the **Elliott–Halberstam conjecture**[6] states that for every $\theta < 1$ and A > 0, ∃ a constant C such that:

$$\sum_{1\le q\le x^\theta} E(x;q) \le \frac{Cx}{(\log x)^A}\ \forall x > 2.$$

Goldston, Pintz, and Yildirm (**GPY**) as well as Bomieri, Friedlander, and Iwaniec[7] made a remarkable discovery linking the challenge of finding bounded gaps between prime numbers with the **Elliott-Halberstam conjecture**. Building on this work, number theorists were able to locate primes in other sets besides intervals, enabling them to prove that there are two primes among the numbers n+a_i, 1≤ $i \le h$, for N < n $\le 2N$ and the $a_i's$ are given arbitrary integers in the interval [1, N] if h < $C\sqrt{\log N}(\log\log N)^2$ and N is constraint to some sequence N_σ which is tending to infinity, which avoids **Siegel zeros**[8] for moduli near to N and also a recent result by James Maynard which shows the bounded gaps between primes i.e. $\lim_{n\to\infty} \inf(p_{n+m} - p_n) \ll m^3 e^{4m}$. This general result stands in contrast to **Gallagher's theorem**[9], which requires the a_i's to lie with in an interval such a general result can be proved without caring that how "a_i" values are distributed.
However, the GPY approach is unable to demonstrate robust findings, such as searching for two or more primes in intervals of restricted length. According to the **Prime number theorem**[10], number theorists can only improve the trivial bound by a constant factor (Unconditionally) [3]. Currently, the

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best result is $\lim_{n\to\infty} \inf \frac{p_{n+2-p_n}}{logp_n} = 0$, by assuming **Elliott–Halberstam conjecture**.

To prove a result as described in James Maynard's paper[1],

i.e., $\lim_{n\to\infty} \inf p_{n+m} - p_n = O(m^3 e^{2m})$ 7

Mathematicians must assume the Elliott-Halberstam conjecture. This conjecture is believed to exceed the current understanding of **Sieve methods**[11], especially the **"Selberg" Sieve method**[11]. However, this paper aims to refine Goldston, Pintz, and Yildirim's and James Maynard's arguments to demonstrate that for a positive proportion of admissible k-tuples, there exists the **Prime k-tuples conjecture** (for each k), which means there exist admissible k-tuples whose proportion is positive for which the prime k-tuples conjecture holds. The paper employs various analytical methods from different branches of **Mathematics**[12] and **Physics**[13], and its assumptions and techniques are similar to Maynard's method but produce numerically superior results.

Additionally, the paper suggests the use of different Sieves like **Large Sieve**[14] rather than the traditional "**Selberg" Sieve** to improve the limit's approximation given in "**Theorem 1**".

The significance of "**Theorem 1**" is that it is a positive step towards achieving the conjectural bound that is $\lim_{n\to\infty} \inf p_{n+m} - p_n = O(m^3 e^{2m})$, unconditionally without assuming the **Elliott-Halberstam conjecture**.

This section provides the complete proof of "Theorem 1" which includes various analytical techniques along with numerous assumptions and restrictions. Additionally, the proof of the theorem required extensive calculations and analysis.

Theorem 1: Let m$\in \mathbb{N}$ and p_n denote n^{th}prime number, then

$$\lim_{n\to\infty} inf(p_{n+m} - p_n) \ll m^2 e^{4m - \chi logm}$$

where p_n is n^{th} prime number, "m" and "χ" are constants such that "$\chi logm$" does not exceed "m".

Proof: Let S_k represents the set of functions (F) which are **Riemann-integrable**[15] defined as F:$[0,1]^k \to \mathbb{R}$ with support in

R_k= {(x1,…,,xk) $\in [0,1]^k : \sum_{i=1}^{k} x_i \le 1\}$ with

$$Q_k(F) = \int_0^1 \dots \dots \int_0^1 F(y_1, \dots \dots, y_k)^2 \, dy_1 \dots \dots dy_k$$

$$P_k{}^{(m)}(F) = \int_0^1 \dots \dots \int_0^1 \left(\int_0^1 F(y_1, \dots \dots, y_k) dy_m \right)^2 dy_1 \dots \dots dy_k$$

provided that $Q_k(F) \neq 0$ and $P_k{}^{(m)}(F) \neq 0$ for all individual m.

Let $L_k = \sup_{F \in S_k} \frac{\sum_{m=1}^{k} P_k{}^{(m)}(F)}{Q_k(F)}$ 8

Now, a lower bound for L_k is required which could be obtained by using a function F = F_k which has been created to make the ratio $\frac{\sum_{m=1}^{k} P_k{}^{(m)}(F)}{Q_k(F)}$ large by assuming "k" to very be large. This could be done by taking function (F) which has the following form:

$$F(y_1, \dots \dots, y_k) = \begin{cases} \prod_{i=1}^{k} h(ky_i), if \ \sum_{i=1}^{k} y_i \le 1 \\ 0 \qquad , otherwise \end{cases} \quad 9$$

Here h: $[0,\infty) \to \mathbb{R}$ supported on [0, H] is some **Smooth function**[16]. Notice that "F" is symmetric, with this choice of "F" $P_k{}^{(m)}(F)$ is same, regardless of m. So that's why $P_k = P_k{}^{(1)}(F)$ was considered. Likewise, one can also write $Q_k = Q_k(F)$.

The key point is that if $\frac{\int_0^\infty uh(u)^2 du}{\int_0^\infty (h(u))^2 du}$ is less than 1 which is defined as **Center of mass**[17] of h^2, then letting "k" to be large enough, this was anticipated that the limits imposed by "F" that is $\sum_{i=1}^{k} y_i \le 1$, could be settled by having only a small error term because the main contribution of the integrals (which have no restrictions) defined as:

$$Q_k{}'(F) = \int_0^\infty \dots \dots \int_0^\infty \prod_{i=1}^{k} h(ky_i)^2 \, dy_1 \dots \dots dy_k$$

and

$$P_k{}'(F) = \int_0^\infty \dots \int_0^\infty (\int_0^\infty \prod_{i=1}^{k} h(ky_i) dy_1)^2 \, dy_2 \dots \dots dy_k$$

should majorly come when $\sum_{i=1}^{k} y_i$ is close to **Center of mass** (by the concentration of measure). So Q_k and P_k are well approximated by $Q_k{}'$ and $P_k{}'$ respectively, because if center of mass of $h^2 < 1$ then one must anticipate the contribution to be small, when $\sum_{i=1}^{k} y_i > 1$.

For the convenience of notion, let $\alpha = \int_0^\infty h(u)^2 du$, now focus only on "h" such that $\alpha > 0$. Then

$$Q_k = \int_{R_K} \dots \dots \int F(y_1, \dots \dots, y_K)^2 dy_1 \dots \dots dy_K \le \left(\int_0^\infty h(ky)^2 dy \right)^K = k^{-k} \alpha^k \quad 10$$

Now consider P_k, then

$$P_k \ge \int \dots \int (\int_0^{\frac{H}{k}} (\prod_{i=1}^{k} h(ky_i)) dy_1)^2 dy_2 \dots \dots dy_k$$

where $y_2, \dots \dots, y_k \ge 0, \sum_{i=2}^{k} y_i \le 1 - \frac{H}{k}$ 11

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Since squares are non-negative, together with this the restriction of the outer integral to $\sum_{i=2}^{k} y_i \le 1 - \frac{H}{k}$ to obtain lower bound for P_k has been made. This is being done because with the support of "h", this has the advantage of removing all the limitations from the inner integral.

The right-hand side of Eq 11 can be written as $P_k{}' - E_k$, where

$$P_k{}' = \int \dots \int (\int_0^{\frac{H}{k}} (\prod_{i=1}^{k} h(ky_i)) dy_1)^2 dy_2 \dots dy_k$$

where $y_2, \dots, y_k \ge 0$

$$= \left(\int_0^\infty h(ky_1) dy_1\right)^2 \left(\int_0^\infty h(ky) dy\right)^{k-1}$$

$$= k^{-k-1} \alpha^{k-1} \left(\int_0^\infty h(u) du\right)^2 \qquad 12$$

$$E_k = \int \dots \int \left(\prod_{i=1}^{k} h(ky_i)) dy_1\right)^2 dy_2 \dots dy_k)$$

where $y_2, \dots, y_k \ge 0, \sum_{i=2}^{k} y_i > 1 - \frac{H}{K}$

$$k^{-k-1} \left(\int_0^\infty h(u) du\right)^2 \int \dots \int \left(\prod_{i=2}^{k} h(u_i)^2\right) du_2 \dots du_k$$

$u_2, \dots, u_k \ge 0, \sum_{i=2}^{k} u_i > k - H$ 13

To show that error integral (E_k) to be small, this error integral (E_k) could be compared with a **Second Moment**[18]. If $\frac{\int_0^\infty u h(u)^2 du}{\int_0^\infty (h(u))^2 du} < \frac{k-H}{k-1}$, then the bound for E_k could be expected to be small. This is why it is necessary to impose a restriction on "h" that

$$\gamma = \frac{\int_0^\infty u h(u)^2 du}{\int_0^\infty h(u)^2 du} < 1 - \frac{H}{k} \qquad 14$$

For ease of notation, let $\eta = \frac{k-H}{k-1} - \gamma > 0$. If $\sum_{i=2}^{k} u_i > k - H$, then it follows that $\sum_{i=2}^{k} u_i > (k-1)(\gamma + \eta)$.

which implies that

$$1 \le \eta^{-2} \left(\frac{1}{k-1} \sum_{i=2}^{k} u_i - \gamma\right)^2 \qquad 15$$

Since the term on the right of Eq 15 is non-negative for all u_i, multiply the integrand by "$\eta^{-2} \left(\frac{1}{k-1} \sum_{i=2}^{k} u_i - \gamma\right)^2$" with the constraint that $\sum_{i=2}^{k} u_i > k - H$ yields an upper bound for "E_k". Thus, it can be inferred that:

$$E_k \le \eta^{-2} k^{-k-1} \left(\int_0^\infty h(u) du)\right)^2 \int_0^\infty \dots \int_0^\infty \left(\frac{1}{k-1} \sum_{i=2}^{k} u_i - \gamma\right)^2 \times \left(\prod_{i=2}^{k} h(u_i)^2\right) du_2 \dots du_k \qquad 16$$

All the terms in the expanded inner square can be explicitly calculated as an expression in "γ" and "α", provided "γ" and "α" are not of the form $u_j{}^2$. This implies

$$\int_0^\infty \dots \int_0^\infty \left(\frac{2 \sum_{2 \le i < j \le k} u_i u_j}{(k-1)^2} + \gamma^2 - \frac{2\gamma \sum_{i=2}^{k} u_i}{k-1}\right) \left(\prod_{i=2}^{k} h(u_i)^2\right) du_2 \dots du_k = \frac{-\gamma^2 \alpha^{k-1}}{k-1} \qquad 17$$

For $u_j{}^2$ terms, with the support of "h", it could be seen that $u_j{}^2 h(u_j)^2 \le H u_j h(u_j)^2$. Thus

$$\int_0^\infty \dots \int_0^\infty u_j{}^2 \left(\prod_{i=2}^{k} h(u_i)^2\right) du_2 \dots du_k \le H \alpha^{k-2} \int_0^\infty u_j\, h(u_j)^2 du_j = \gamma H \alpha^{k-1} \qquad 18$$

This gives

$$E_k \le \eta^{-2} k^{-k-1} \left(\int_0^\infty h(u) du\right)^2 \left(\frac{\gamma H \alpha^{k-1}}{k-1} - \frac{\gamma^2 \alpha^{k-1}}{k-1}\right) \le \frac{\eta^{-2} \gamma H k^{-k-1} \alpha^{k-1}}{k-1} \left(\int_0^\infty h(u) du\right)^2 \qquad 19$$

Since $(k-1)\eta^2 \ge k(1 - \frac{H}{k} - \gamma)^2$ and $\gamma \le 1$, now Eq 10, Eq 11, Eq 12 and Eq 19 together implies

$$\frac{k P_k}{Q_k} \ge \frac{\left(\int_0^\infty h(u) du\right)^2}{\int_0^\infty h(u)^2 du} \left(1 - \frac{H}{k\left(1 - \frac{H}{k} - \gamma\right)^2}\right) \qquad 20$$

To maximize the lower bound value in Eq 20, the integral $\int_0^H h(u) du$ must be maximized, subject to the constraints that $\int_0^H h(u)^2 du = \alpha$ and $\int_0^H u h(u)^2 du = \gamma\alpha$. Thus, the main goal is to maximize the expression:

$$\int_0^H h(u) du - \tau\left(\int_0^H h(u)^2 du - \alpha\right) - \beta\left(\int_0^H u h(u)^2 du - \gamma\alpha\right). \qquad 21$$

with respect to τ, β and the function h. This occurs when $\frac{\partial}{\partial h}(h(t) - \tau h(t)^2 - \beta t h(t)^2) = 0$ for all $t \in [0, H]$ by making use of **Euler-Lagrange eq** [19].

This implies that

$$h(t) = \frac{1}{2\tau + 2\beta t} \qquad 22$$

Noting that if a positive constant is multiplied by "h", the ratio which is being maximized remains unchanged. Let's now consider functions of "h" in the form of "$\frac{n}{1+nAt}$" for $t \in [0, H]$ where "n" is a constant depending on "A" which in turn depends on k that will be chosen later. This selection of "h" implies:

$$\int_0^H h(u) du = \frac{1}{A} \log(1 + nAH) \qquad 23$$

$$\int_0^H h(u)^2 du = \frac{n}{A}\left[1 - \frac{1}{1+nAH}\right] \qquad 24$$

$$\int_0^H u h(u)^2 du = \frac{1}{A^2}\left[\log(1 + nAH) + \left(\frac{1}{1+nAH} - 1\right)\right] \qquad 25$$

"H" could be taken such that $1 + nAH = e^{An-n}$ (which is the optimal choice). With this selection,

$$\gamma = \frac{1}{An}\left[\frac{An - n + e^{-(An-n)} - 1}{1 - e^{-(An-n)}}\right] \quad \text{and} \quad H \le \frac{e^{An-n}}{An}$$

Next, choose A = log k > 0 and n = log log log A.

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Note: It must be ensured that the value of "n" which is dependent on "A" which in turn depends on "k" ≈1 (almost equal to 1) which can be achieved by taking a sufficiently large value of "k". This condition on "n" is necessary because of following two reasons:

If the value of "n" > 1 then the 2nd term on right-hand side of Eq 26 i.e., $\frac{e^{An-n}}{Ank} > 1$, so by this the expression on right-hand side of Eq 26 becomes negative as the 3rd term on right-hand side of Eq 26 i.e., $\frac{1}{An}\left[\frac{An-n+e^{-(An-n)}-1}{1-e^{-(An-n)}}\right]$ is always less than 1.

1. If the value of "n" < 1 then the expression on right-hand side of Eq 26 becomes positive but the best optimal bound would not be obtained in Eq 27.

So, to make right-hand side of Eq 26 **positive** and also to get **best optimal bound** in Eq 28, it must be ensured that the value of "n" ≈1(almost equal to 1) which can be achieved by taking a sufficiently large value of "k" so that the 2nd term on right-hand side of Eq 26 i.e., $\frac{e^{An-n}}{Ank}$ is less than 1.

This implies:

$$1-\frac{H}{k}-\gamma \geq 1-\frac{e^{An-n}}{Ank}-\frac{1}{An}\left[\frac{An-n+e^{-(An-n)}-1}{1-e^{-(An-n)}}\right] > 0 \qquad 26$$

Now, just substitute the values of Eq 23, Eq 24, Eq 25 and Eq 26 into Eq 20 to obtain the desired result.

$$M_k \geq \frac{kP_k}{Q_k} \geq \frac{(An-n)^2}{An(1-e^{-(An-n)})}\left[1-\frac{e^{An-n}}{Ank\left(1-\frac{e^{An-n}}{Ank}-\frac{1}{An}\left[\frac{An-n+e^{-(An-n)}-1}{1-e^{-(An-n)}}\right]\right)^2}\right] \geq \frac{(An-n)^2}{An} \quad \text{and}$$

$$\frac{(An-n)^2}{An} = logklogloglogk + \frac{loglogloglogk}{logk} - 2loglogloglogk \qquad 27$$

Now $\theta = \frac{1}{2}-\epsilon$ could be taken (by **Bombieri-Vinogradov Theorem 1**) thus, by Eq 27 and for "k" being sufficiently large implies:

$$\frac{\theta M_k}{2} \geq \left(\frac{1}{4}-\frac{\epsilon}{2}\right)\left(\frac{(An-n)^2}{An}\right) = \left(\frac{1}{4}-\frac{\epsilon}{2}\right)\left(logklogloglogk + \frac{loglogloglogk}{logk} - 2loglogloglogk\right) \qquad 28$$

Now choose $\epsilon = \frac{1}{k}$ and see that if k≥ $Cme^{4m-\chi logm}$ where "m" and "χ" are constants such that "$\chi logm$" does not exceed "m", then $\frac{\theta M_k}{2} > m$, for C which is also a constant that doesn't depend on "m" and "k".

Therefore, if an admissible set H = {h_1,..,h_k} is considered, where k ≥ $Cme^{4m-\chi logm}$,then for large number of integers "n" at least "m+1" of "n+h_i" must be prime.

Let us choose that the set "H" to be $\{p_{\pi(k)+1},\dots\dots,p_{\pi(k)+k}\}$ which consists of the first k primes which are greater than k. This set "H" is admissible because no element less than k is a multiple of a prime, and there are k elements in the set, ensuring that all classes with **Residue**[20] that are moded (modulo) by any prime greater than k are not covered. The diameter of the set $p_{\pi(k)+1},\dots\dots,p_{\pi(k)+k} \ll klogk$ thus, if k = $\lceil Cme^{4m-\chi logm}\rceil$, where ⌈x⌉ denotes smallest integer n≥ x,then this implies $\lim_{n\to\infty} inf(p_{n+m}-p_n) \ll klogk \ll m^2e^{4m-\chi logm}$. This satisfies our desired result.

Results and Discussion

This section presents the description of "**Theorem 1**" which has been proven using advanced techniques from "Goldston, Pintz and Yildirim" and "James Maynard". In particular, the following theorem has been proved:

$$\lim_{n\to\infty} inf(p_{n+m}-p_n) \ll klogk \ll m^2e^{4m-\chi logm}$$

where p_n is n^{th}prime number, "m" and "χ" are constants such that "$\chi logm$" does not exceed "m" and "k" is taken to be very large.

In comparison to previous result in "James Maynard" paper, modified upper bound of "m^3e^{4m}" to $m^2e^{4m-\chi logm}$ has been given.

Conclusion

In this paper some existence of **Primes k-tuples conjecture** for positive proportion of admissible k-tuples has been shown, in particular, it has been shown that: $\lim_{n\to\infty} inf(p_{n+m} - p_n) \ll m^2 e^{4m-\chi logm}$ where p_n is n^{th}prime number, "m" and "χ" are constants such that "$\chi logm$" does not exceed "m" and "k" is taken to be very large. This result is a positive step towards achieving the conjectural bound that is $\lim_{n\to\infty} \inf p_{n+m} - p_n = O(m^3 e^{2m})$ and to prove this, one has to prove **Elliott-Halberstam conjecture** which is being In this paper some existence of **Primes k-tup** assumed to be beyond the known techniques of **Sieve methods**, especially "**Selberg**" **Sieve method**. Many breakthroughs happened in the last few years as described above (**in introduction**) but no one was able to develop new techniques to solve the **Elliott-Halberstam conjecture**. But in comparison to previous result in "James Maynard" paper, modified upper bound of "$m^3 e^{4m}$" to "$m^2 e^{4m-\chi logm}$ " where "m" and "χ" are constants such that "$\chi logm$" does not exceed "m" has been given.

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Author's Declaration

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in Amity University, Noida, India

Authors' Contribution Statement

All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by A. M. The first draft of the manuscript was written by S. G., and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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حول وجود تخمين K-Tuples الرئيسي للنسبة الإيجابية من K-Tuples المقبولة

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الخلاصة:

يعتقد منظرو الأعداد أن الأعداد الأولية تلعب دورًا مركزيًا في نظرية الأعداد وأن حل المشكلات المتعلقة بالأعداد الأولية يمكن أن يؤدي إلى حل العديد من التخمينات الأخرى التي لم يتم حلها ، بما في ذلك التخمين الأولي k-tuples.يهدف هذا البحث إلى إثبات وجود هذا التخمين بالنسبة لـ k-tuples المقبولة بنسب موجبة. حقق الباحثون ذلك من خلال تحسين طرق كولدستون و بنتز و يلدرم و جيمس مينراد" بدراسة الفجوات المحدودة بين الأعداد الأولية و k-tuples الأولية. تم تمكين هذه التحسينات من التغلب على القيود والمعوقات السابقة ولإظهار أنه بالنسبة لنسبة إيجابية من مجموعات k المقبولة ، هناك وجود التخمين الأولي k-tuples لكل "k"تكمن أهمية هذه النتيجة في أنها غير مشروطة مما يعني أنه تم إثباتها دون افتراض أي شكل من أشكال التخمين القوي مثل تخمين إليوت-هالبرستام.

الكلمات المفتاحية: مقبول، نسبة موجبة، أعداد أولية، مجموعات k الأولية، تخمين أولي k-tuples ، تخمينات غير محلولة.