Orthogonal Functions Solving Linear functional Differential EquationsUsing Chebyshev Polynomial

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Abstract:

A method for Approximated evaluation of linear functional differential equations is described. where a function approximation as a linear combination of a set of orthogonal basis functions which are chebyshev functions .The coefficients of the approximation are determined by (least square and Galerkin's) methods. The property of chebyshev polynomials leads to good results , which are demonstrated with examples.

Introduction:

Functional Differential Equations are differential equations in which function appears with delay argument and which has been developed over twenty year, only in the last few years has much effort been devoted to study nonlinear functional differential equations

Of the form

$$F(t, y(t), y(t-\tau_1),\ldots, y(t-\tau_k), y'(t), y'(t-\tau_1),\ldots, y'(t-\tau_k), y^{(n)}(t),\ldots, y^{(n)}(t-\tau_k)) = 0 \quad \ldots (1)$$

Where F is a given function and $\tau_1, \tau_2, \ldots, \tau_k$ are given and called the "time 1, 2].

The differential equation (1) is categorized into three types: -Equation (1) is called a Retarded type, if the highest-order derivative of unknown function appears for just one value of the argument. Equation (1) is called a Neutral type, if the highest- order derivative of unknown function appears both with and without difference argument. All other differential equation (1) with an advanced types. [3,4,5]

Weighted Residual Methods:

We present these methods by considering the following functional equation.

$$L\,[y\,(x)] = g\,(x) \quad x \in D \qquad \ldots(2)$$

Where L denotes an operator which maps a set of functions Y in to a set G such that $y \in Y$, $g \in G$ and D is a prescribed domain.

The epitome of the expansion method is to approximate the solution y(x) of equ. (2) By the form $y \cong y_N(x) = \sum_{i=0}^{N} c_i \psi_i(x)$ …. (3)

Where the parameters c_i and the functions $\psi_i(x)$ are chosen in our work as:

The Choice of Basis Functions

In this work, the choice of basis functions $\psi_i(x)$ are:

Orthogonal polynomials:

1. Chebyshev Polynomials: [6]

The Chebyshev polynomials $T_n(x)$ are an important set of orthogonal function over the interval [-1, 1].

The general form of these polynomials is

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad n \geq 1$$

Where $T_0(x) = 1$ and $T_1(x) = x$

An approximate solution $y_N(x)$ given by (3) , in general, satisfy equ.(2) exactly and associated with such an approximate solution is the residual defined by

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$$E(x) = L[y_N(x)] - g(x) \qquad \ldots (4)$$

The residual E(x) depends on x as well as on the way that the parameters c_i are chosen.

It is obvious that when E(x) = 0, then the exact solution is obtained which is difficult to be achieved, therefore we shall try to minimize E(x) in some sense.

In the weighted residual method the unknown parameters c_i are chosen to minimize the residual E(x) by setting its weighted integral equal to zero, i.e.

$$\int_D w_j E(x)dx = 0 \qquad j = 0,1,\ldots.,N \qquad \ldots (5)$$

Where w_j is prescribed weighting function, the technique based on (5) is called weighted residual method. Different choices of w_j yield different methods with different approximate solutions.

The weighted residual methods that will be discussed in this work are:

1.1 Galerkin's Method
1.2 Least Square Method

1.1 Galerkin' s Method: [7]

Galerkin method is the most efficient of the weighted residual methods. This method makes the residual E(x) of equ.(4) orthogonal to (N+1) given linearly independent function on the domain D.

The weighting function w_j is defined as:

$$w_j(x) = \frac{\partial y_N(x)}{\partial c_j} \qquad j = 0,1,\ldots,N$$

where $y_N(x)$ is the approximate solution of the problem, then equ.(4) becomes.

$$\int_D \frac{\partial y_N(x)}{\partial c_j} E(x)dx = 0 \qquad j = 0,1,\ldots,N$$

...(6)

And will provide (N+1) simultaneous equations for determination of c_0 , c_1 , …. c_N.

1.2 Least Square Method: [7]

In this method weighting function w_j is defined as:

$$w_j(x) = \frac{\partial E(x)}{\partial c_j} \qquad j = 0,1,\ldots,N$$

And due to equ.(5) we have

$$\int_D \frac{\partial E(x)}{\partial c_j} E(x)dx = 0 \qquad j = 0,1,\ldots.,N \qquad \ldots(7)$$

then the square of the error on the domain D is

$$J = \int_D [E(x)]^2 dx$$

Now, we compute the derivatives

$$\frac{\partial J}{\partial c_j} = 2\int_D E(x)\frac{\partial E(x)}{\partial c_j}dx \qquad j = 0,1,\ldots.,N \quad \ldots(8)$$

it implies from equ.(7),(8) that

$$\frac{\partial J}{\partial c_j} = 2\int_D E(x)\frac{\partial E(x)}{\partial c_j}dx = 0 \qquad j = 0,1,\ldots.,N$$

Therefore J is stationary and the square of residual E(x) attains its minimum

Solution of Linear FDE's by Weighted Residual Methods:

The general form nth order non-homogenous linear functional differential equation is

$$y^{(n)}(x) + \sum_{p=0}^{n} a_p y^{(p)}(x) + \sum_{m=0}^{(n-1)} b_m y^{(m)}(x-\tau) + \sum_{l=0}^{(n)} d_l y^{(l)}(x-\tau) = g(x)$$

... (9)

where τ is a prescribed constant and a_p , b_m and d_l are constants with initial condition

$$y(0) = y_0, \quad y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}, \quad y^{n}(0) = y_n$$

Using operator forms this equation can be written as

$$L[y] = g(x) \qquad \ldots(10)$$

Where the operator L is defined as

$$L[y]=y^{(n)}(x)+\sum_{p=0}^{n}a_p y^{(p)}(x)+\sum_{m=0}^{n-1}b_m y^{(m)}(x-\tau)+\sum_{l=0}^{n}d_l y^{(l)}(x-\tau)$$

The unknown function y(x) is approximated by the form

$$y_N(x)=\sum_{j=0}^{N}c_j\psi_j(x) \qquad \dots (11)$$

Substituting equ.(11) in equ.(10) we get

$$L[y_N]=g(x)+E_N(x)$$

Where

$$L[y_N]=y_N^{(n)}(x)+\sum_{p=0}^{n}a_p y_N^{(p)}(x)+\sum_{m=0}^{n-1}b_m y_N^{(m)}(x-\tau)+\sum_{l=0}^{n}d_l y_N^{(l)}(x-\tau)$$

Which we have the residue equation

$E_N(x) = L[y_N(x)] - g(x)$ … (12)

Substituting the equ.(11) in equ.(12), we get

$$E_N(x)=L\left[\sum_{j=0}^{N}c_j\psi_j\right]-g(x)=\sum_{j=0}^{N}c_jL[\psi_j(x)]-g(x)$$

...(13)

Evidently from equ.(5) that

$$\int_D w_jE_N(x)=0 \qquad \dots (14)$$

Inserting equ.(13) in equ.(14) yields

$$\int_D w_j\left(\sum_{j=0}^{N}c_jL[\psi_j(x)]-g(x)\right)=0$$

$$\sum_{j=0}^{N}c_j\int_D w_jL[\psi_j(x)]=\int_D w_jg(x) \qquad \dots (15)$$

Where

$$L(\psi_j(x))=\psi_j^{(n)}(x)+\sum_{p=0}^{n}a_p\psi_j^{(p)}(x)+\sum_{m=0}^{n-1}b_m\psi_j^{(m)}(x-\tau)+\sum_{l=0}^{n}d_l\psi_j^{(l)}(x-\tau)$$

for j=0,1,….,N ... (16)

Introducing matrix k and vector h as

$$k_{ji}=\int_D w_jL(\psi_i(x))dx \qquad i=0,1,\dots,N \qquad \dots (17)$$

$$h_j=\int_D w_jg(x)dx \qquad j=0,1,\dots,N$$

We can write equ.(15) in matrix form

K C = H ... (18)

Where

$$C=\begin{pmatrix}c_0\\c_1\\\vdots\\c_N\end{pmatrix}$$

Finally, we have N+1 linear equations (18) for determing N+1 coefficient c0, c_1 …, c_N.

1. Using Galerkin's Method:

It is a simple technique for obtaining a linear approximation y_N , the weight function w_j in equ.(5) is defined as [6]

$$w_j(x)=\frac{\partial y_N(x)}{\partial c_j} \qquad i=0,1,\dots,N$$

now since

$$y_N(x)=\sum_{i=0}^{N}c_i\psi_i$$

Therefore $w_j(x)=\frac{\partial}{\partial c_j}\sum_{i=0}^{N}c_i\psi_i=\psi_j \quad , \quad j=0,1,\dots,N$

… (19)

Substituting equ.(19) in equ.(15) we get

$$\sum_{i=0}^{N}c_j\psi_jL(\psi_j)=\int_D\psi_jg(x), \quad j=0,1,\dots,N$$

The linear system (18) becomes

$$\begin{bmatrix}\int_D\psi_0L(\psi_0)dx & \int_D\psi_0L(\psi_1)dx & \dots & \int_D\psi_0L(\psi_N)dx\\ \int_D\psi_1L(\psi_0)dx & \int_D\psi_1L(\psi_1)dx & \dots & \int_D\psi_1L(\psi_N)dx\\ \vdots & & & \\ \int_D\psi_NL(\psi_0)dx & \int\psi_NL(\psi_1)dx & \dots & \int_D\psi_NL(\psi_N)dx\end{bmatrix}\begin{bmatrix}c_0\\c_1\\\vdots\\c_N\end{bmatrix}=\begin{bmatrix}\int_D\psi_0gdx\\\int_D\psi_1gdx\\\vdots\\\int_D\psi_Ngdx\end{bmatrix}$$

... (20)

Where

$$K=\begin{bmatrix}\int_D\psi_0L(\psi_0)dx & \int_D\psi_0L(\psi_1)dx & \dots & \int_D\psi_0L(\psi_N)dx\\ \int_D\psi_1L(\psi_0)dx & \int_D\psi_1L(\psi_1)dx & \dots & \int_D\psi_1L(\psi_N)dx\\ \vdots & & & \\ \int_D\psi_NL(\psi_0)dx & \int\psi_NL(\psi_1)dx & \dots & \int_D\psi_NL(\psi_N)dx\end{bmatrix},$$

$$C=\begin{bmatrix}c_0\\c_1\\\vdots\\c_N\end{bmatrix} \quad \& \quad H=\begin{bmatrix}\int_D\psi_0gdx\\\int_D\psi_1gdx\\\vdots\\\int_D\psi_Ngdx\end{bmatrix}$$

... (21)

Hence solving the system KC = H for coefficient c_j's. Substitute the coefficient in equ.(10) to obtain approximate solution.

Algorithm (orthogonal chebyshev Galerkin's OCG)

This method can be summarized in the following steps

Step 1: Select ψ_j for j = 0, 1, …., N.

Step 2: Compute $L(\psi_j)$ by using equ.(16) for j = 0, 1, …., N.

Step 3: Compute the matrices K and H by using equ.(21).

Step 4: Solve the system (20) for coefficients c_j's .

Step 5:Substitute c_j's in transforming form to obtain the approximate solution of y(x).

2. Using Least Square Method:

The standard least squares approach is to attempt to minimize the sum of the squares of the residual at each point in the domain D.

Here this minimization of [6]

$$I(\bar{c}) = \frac{1}{2}\int_D [E_N(x)]^2 dx$$

Where $\bar{c} = (c_0, c_1, \ldots, c_N)$

This technique is to solve the (N+1) normal equation given by

$$\frac{\partial I}{\partial c_i} = \int_D E_n(x)\frac{\partial E_n(x)}{\partial c_i}dx = 0 \quad , \quad i = 0,1,\ldots,N$$

Substituting eq.(15), yields

$$\int_D \sum_{j=0}^{N} c_j L(\psi_i(x))L(\psi_j(x))dx = \int_D L(\psi_i(x))g(x)dx$$

$$i = 0,1,\ldots.N$$

The linear system (18) becomes

$$\begin{bmatrix} \int_D L(\psi_0)L(\psi_0)dx & \int_D L(\psi_0)L(\psi_1)dx & \ldots. & \int_D L(\psi_0)L(\psi_n)dx \\ \int_D L(\psi_1)L(\psi_0)dx & \int_D L(\psi_1)L(\psi_1)dx & \ldots. & \int_D L(\psi_1)L(\psi_n)dx \\ \vdots & & & \\ \int_D L(\psi_n)L(\psi_0)dx & \int_D L(\psi_n)L(\psi_1)dx & \ldots. & \int_D L(\psi_n)L(\psi_n)dx \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} \int_D L(\psi_0)gdx \\ \int_D L(\psi_1)gdx \\ \vdots \\ \int_D L(\psi_n)gdx \end{bmatrix}$$

... (22)

Where

$$K = \begin{bmatrix} \int_D L(\psi_0)L(\psi_0)dx & \int_D L(\psi_0)L(\psi_1)dx & \ldots. & \int_D L(\psi_0)L(\psi_n)dx \\ \int_D L(\psi_1)L(\psi_0)dx & \int_D L(\psi_1)L(\psi_1)dx & \ldots. & \int_D L(\psi_1)L(\psi_n)dx \\ \vdots & & & \\ \int_D L(\psi_n)L(\psi_0)dx & \int_D L(\psi_n)L(\psi_1)dx & \ldots. & \int_D L(\psi_n)L(\psi_n)dx \end{bmatrix}$$

$$C = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} \quad \& \quad H = \begin{bmatrix} \int_D L(\psi_0)gdx \\ \int_D L(\psi_1)gdx \\ \vdots \\ \int_D L(\psi_n)gdx \end{bmatrix}$$

… (23)

Hence the system of N+1 linear equations with N+1 unknown N+1 coefficients c_0 , c_1 , …. , c_N will construct.

By solving this system, the values of c_j's will be found and then substituted in equ.(10) to obtain the approximate solution.

Algorithm (orthogonal chebyshev least square OCL)

This method can be summarized in the following steps

Step 1: Select ψ_j for j = 0, 1, …., N.

Step 2: Compute $L(\psi_j)$ & $L(\psi_i)$ by using equ.(16) for i = 0,1,….,N,j = 0, 1, …., N.

Step 3: Compute the matrices K and H by using equ.(23).

Step 4: Solve the system (22) for coefficients c_j's .

Step 5: Substitute c_j's in equ.(10) to obtain the approximate solution.

Transforming the Interval:

Sometimes it is necessary to take a problem state on interval [a, b], then we can convert the variable so that the problem is reformulated on [-1, 1], as follows $x = \dfrac{(b-a)t + b + a}{2}$,

So that $dx = \dfrac{(b-a)}{2} dt$

Then $\int_a^b g(x)dx = \dfrac{b-a}{2} \int_{-1}^{1} g\left(\dfrac{(b-a)t + b + a}{2}\right) dt$

Conversely, transform the limits of integration from [-1, 1] to [a, b] using

$t = 2\left(\dfrac{x-a}{b-a}\right) - 1$,

so that $dt = \dfrac{2}{b-a} dx$

Then $\int_{-1}^{1} g(t)dt = \dfrac{2}{b-a} \int_a^b g(2\left(\dfrac{x-a}{b-a}\right) - 1) dx$

Test Example:

Example (1):

Consider the following first order linear functional differential equation of retarded type:

$$y'(t) + y(t) - y(t-1) = t \qquad t \geq 0$$

With initial function

$$y(t) = t \qquad -1 \leq t \leq 0$$

Which has exact solution

$$y(t) = \frac{t}{8} + \frac{t^2}{4} \qquad 0 \leq t \leq 1$$

Assume the approximate solution in the form

$$y_N(t) = \sum_{j=1}^{N} a_j \psi_j (2x - 1)$$

Table (1) lists the least square errors obtained by running the programs for algorithms (OCG and OCL) with different values to find approximate solution of above equation.

Table (1) the approximated values of y (t) against the exact value for N=2 and Δ t=10

t	Exact	Chybeshev polynomial	
		OCG	OCL
0.0	0	0	0
0.1	0.0150	0.0150	0.0150
0.2	0.0350	0.0350	0.0350
0.3	0.0600	0.0600	0.0600
0.4	0.0900	0.0900	0.0900
0.5	0.1250	0.1250	0.1250
0.6	0.1650	0.1650	0.1650
0.7	0.2100	0.2100	0.2100
0.8	0.2600	0.2600	0.2600
0.9	0.3150	0.3150	0.3150
1.0	0.3750	0.3750	0.3750
L.S.E.		0	0

Example (2):

Consider the following first order delay differential equation of neutral type:

$$y'(t) = 1 - y'(t - \frac{y(t)^2}{4})$$

With initial function

$$y(t) = 1 + t \qquad 0 \leq t \leq 1$$

Which has exact solution

$$y(t) = 1 + \frac{t}{2} + \frac{t^2}{4} \qquad 0 \leq t \leq 1$$

Assume the approximate solution as in the form

$$y_N(t) = \sum_{j=1}^{N} a_j \psi_j (2x - 1)$$

Table (2) lists the least square errors obtained by running the programs for algorithms (OCG, and OCL) with different values to find approximate solution of above equation.

Table (2)The approximated values of y (t) against the exact value for N=5 and Δ t=10

t	Exact	Chybeshev polynomial	
		OCG	OCL
0.0	1	1	1
0.1	1.0525	1.0525	1.0526
0.2	1.1100	1.1100	1.1101
0.3	1.1725	1.1725	1.1726
0.4	1.2400	1.2400	1.2401
0.5	1.3125	1.3125	1.3126
0.6	1.3900	1.3900	1.3901
0.7	1.4725	1.4725	1.4726
0.8	1.5600	1.5600	1.5601
0.9	1.6525	1.6525	1.6526
1.0	1.7500	1.7500	1.7501
L.S.E.		1.0000e-011	1.0000e-007

Conclusions:

The approximate solution using the basis functions orthogonal functions using chebyshev polynomial with the aid of weighted residual methods has been obtained for two examples. Good results were obtained and the following conclusion points are drawn.

In terms of the results, Galerkin's method gives more accurate solution than collocation method and least square method, see tables

The good approximation depends on:

the number of the orthogonal polynomials

The number of the root's x_i's of the orthogonal polynomials, i.e. as it increases the L.S.E. approaches to zero.

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الدوال المتعامدة لحل معادلات الدوال التفاضلية الخطية باستخدام متعددات الحدود جبشيف

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الخلاصة:

وصفت طريقة تقريبية لحساب معادلات الدوال التفاضلية الخطية حيث كتبت الدالة على شكل تركيب خطي من دوال متعامدة هي دوال جبشيف .المعاملات للتركيب الناتج تم حسابها بطريقتين المربعات الصغرى وكالكن .خاصية متعددات الحدود جبشيف أدت الى نتائج جيدة تم توضيحها بأمثلة.