On New Algebraic Systems

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Abstract:
The purpose of this paper is to give some results theorems , propositions and corollaries concerning new algebraic systems flower , garden and farm with accustomed algebraic systems groupoid , group and ring.
Keywords: Groupoid , Group , Ring ,Semiflower , Flower , Garden ,Farm , Lahhamian Group.

Introduction:

Many authors work for development and findout some mathematics systems .Al- Lahham [1] one of them , he was introduced new algebraic system flower , garden and farm ,he gave some results about them as a finite farm (S, $*$, $\Diamond$) such that $|S|$ >2, is not a field .Al-Lahham introduced type of groups is called a Lahhamian group, during his work depended on [2] and [3] . We can say he find some relations between his systems with another accustomed algebraic systems.

Preliminaries :

Throughout this paper S denote to a groupoid is an order pair (S, $*$)where S is a non empty set and $*$ is a binary operation on S [2] unless explicily stated .A binary operation $*$ on non empty set S is ATL if for all a,b,c $\in$ S ,a $*$ (b $*$ c)=c $*$ (b $*$ a).Asemiflower is an order pair (S, $*$) where S is a non empty set and $*$ is an ATL binary operation on S .Aflower is an order pair (S, $*$) where S is a non empty set and $*$ is a binary operation on S satisfying the following axioms :
(i)a $*$ (b $*$ c) = c $*$ (b $*$ a) for all a,b,c $\in$ S (ATL law).
(ii)There exists an element e in S such that a $*$ e =a for all a $\in$ S(e is a right identity of S).

(iii) a $*$ a=e for all a $\in$ S.Agroup (S, $*$) is called Lahhamian group if a $*$ a = e for all a $\in$ S , where e the identity of S.
Agarden is a triple (S, $*$, $\Diamond$) ,where S is non empty set has at least two elements , $*$ and $\Diamond$ are two binary operations on S, ($\Diamond$:S×S* $\to$ S)and S*=S-{0},(0is the right identity in (S, $*$)),satisfies the following axioms:
(i) (S, $*$)is a flower .(ii) $\Diamond$ is ATL (i.e.a $\Diamond$ (b $\Diamond$ c)= c $\Diamond$ (b $\Diamond$ a)for all a,b,c $\in$ S*).
(iii)The distributive law ,(a $*$ b) $\Diamond$ c= (a $\Diamond$ c) $*$ (b $\Diamond$ c) for all a,b $\in$ S, c $\in$ S*, holds in S. A farm (S, $*$, $\Diamond$)is a garden such that (S*, $\Diamond$)is aflower.A function p[λ]from a flower S to S is a right (left)translation of S if p(a)b= a(P(b)) [λ (ab)= λ (a)b]for all a,b $\in$ S .The operation o is composition of functions.
Now ,we will mention some results which we need to a chieve our work which apperared in[1]

Lemma 1 [Lemma 4]
Let (S, $*$, $\Diamond$)be a garden ,then for all a $\in$ S*, 0 $\Diamond$ a=0.

Lemma 2[Theorem6]
A flower (S, $*$) is a Lahhamian group if and only if it is commutative .

Lemma 3[Theorem4]

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Let S be a flower , then every right translation of S commutes with every left translation of S.

Proposition 5[Proposition 2]

If (S,.)is an abelian group ,then S with the new binary operation $*$ defined as follows : a$*$b=a.b^{-1} for all a,b$\in$S is a flower .

Lemma 4[Lemma2]

If (S, $*$) is a flower then for all a,b$\in$S,a$*$b=e$*$(b$*$a).

Proposition 6[Proposition1]

If (S, $*$) is a flower then , (b$*$c)$*$a=(b$*$a)$*$c for all a,b,c $\in$S.

Proposition 7[Proposition3]

If S is a Lahhamian group then S is a flower.

Proposition 8[Theorem 8]

Let (S,*) be a flower ,then the following conditions are equivalent:
(i) S is a Lahhamian group.
(ii) S has an identity.
(iii) S is commutative.
(iiii) S is associative.
It is our aim in this paper to give some new results concerning new algebraic systems.

The Main Results:
Theorem 1

Let (S, $*$) be a non–empty set such that a is idempotent element of S ,$*$ satisfy ATL law and left cancellation law hold in S,then (S, $*$) is flower .

Proof :We have a$*$a =a , a$\in$S ,$*$ satisfy ATL law .Then we must be prove
(i)a$*$ e=a ,a$\in$S. By replacing e by a, we obtain
a$*$e=a$*$a,a$\in$S.Then
=a , a$\in$S.Thus e is a right identity element of S.
(ii) a$*$a = e , a$\in$S. According to our relation ,we have
a*a=a ,a$\in$S. Since e is a right identity, then a$*$a=a$*$e, a$\in$S.Then a can be cancelled on the left our equation , we obtain a=e. Then a$*$a=e
Thus (S, $*$) is flower .

Remark 2
In Theorem 1 ,(S, $*$) is a semiflower .
Proposition 3
If (S, $*$) is a flower then the following are true for all a,b,c $\in$S.
(1) S is closed under $*$ (2) (e$*$a) $*$ (a$*$e)= e (e is a right identity element of S) .   (3)(b$*$a) $*$ a= b

Proof: (1)Since $*$ is binary operation ,then for all a,b$\in$S, we obtain , a$*$b$\in$S , replacing b by a ,we have a$*$a $\in$S ,by axaiom (iii) ,we get e$\in$S , e is a right identity element of S.
(2)We have (e$*$a) $*$ (a$*$e) =e .Let x=(e$*$a) and y = (a$*$e) ,then
x $*$ y =e.We set x =e$*$x and y=y$*$e ,then
(e$*$x) $*$ (y$*$e)
=(e$*$ (e$*$a)) $*$ ((a$*$e) $*$e) . By ATL law ,we get
=(a$*$ (e$*$e)) $*$ ((a$*$e) $*$e)
=a$*$ ((a$*$e) $*$e). Since e is a right identity element of S
=a$*$ (a$*$e) .By ATL law,we get
= e$*$ (a$*$a). By axiom (iii),we obtain.
=e$*$ e =e.

(3)We have (b$*$a) $*$ a =b,then
Replacing a by e$*$a ,we obtain
(b$*$ a) $*$ a=(b$*$ (e$*$a)) $*$ (e$*$a). By ATL law ,we get
=(a$*$ (e$*$b)) $*$ (e$*$a) .By Proposition 6,we obtain
=(a$*$ (e$*$a)) $*$ (e$*$b) .By ATL law ,we get
=b$*$ (e$*$ (a$*$ (e$*$a)) .By ATL law ,we obtain
=b$*$ ((e$*$a) $*$ (a$*$e)). By Part 2,then
=b$*$ e=b.

Theorem 4
Let (S, $*$) be a flower , if a$*$x=y$*$b then x=y for all a,b,x,y$\in$S.
Proof: We have a$*$x=y$*$b for all a,b,x,y$\in$S .Then

$a*(a*x)=a*(y*b)$ for all $a,b,x,y \in S$. By ATL law,we obtain

$x*(a*a)=b*(y*a)$ for all $a,b,x,y \in S$.By axiom (iii),we get

$x*e=b*(y*a)$ for all $a,b,x,y \in S$. By axiom (ii) ,we obtain

$x=b*(y*a)$ for all $a,b,x,y \in S$ .Then

$x*(e*e)=(b*(y*a))*(e*e)$ for all $a,b,x,y \in S$. By ATL law,we get

$e*(e*x)=e*(e*(b*(y*a)))$ for all $a,b,x,y \in S$ .By ATL law,we obtain $x*(e*e) = e*((y*a)*(b*e))$ for all $a,b,x,y \in S$.By ATL law,we get

$x*e =(b*e)*((y*a)*e)$ for all $a,b,x,y \in S$. Also by axiom(ii) ,we obtain $x=b*(y*a)$ for all $a,b,x,y \in S$. Since e is a right identity element of S ,then

$x=b*(y*(a*e))$ for all $a,b,x,y \in S$ .By ATL law,we obtain

$x=b*(e*(a*y))$ for all $a,b,x,y \in S$. By ATL law,we obtain

$x=(a*y)*(e*b)$ for all $a,b,x,y \in S$ .Let $b=e*b$,then

$x=(a*y)*(e*(e*b))$ for all $a,b,x,y \in S$.By ATL law ,we obtain

$x=(a*y)*(b*e)$ for all $a,b,x,y \in S$.Then

$x=(a*y)*b$ for all $a,b,x,y \in S$.By Proposition 6 ,we obtain

$x=(a*b)*y$ for all $a,b,x,y \in S$. Replacing a by b ,we obtain

$x=(b*b)*y$ for all $b,x,y \in S$ .By axiom (iii),we obtain $x=e*y$ for all $x,y \in S$. Replacing y by $e*y$,we get

$x=e*(e*y)$ for all $x,y \in S$ .By ATL law ,we get $x=y$ for all $x,y \in S$.

Theorem 5

If (S,.) is an abelian group ,then S with the new binary operation $*$ defined as follows: $a*b= a^{-1}.b$ for all $a,b \in S$ is a flower ,where e is identity element for $*$ on S.

Proof: (i) $c*(b*a) =c^{-1}.(b*a)$ for all $a,b,c \in S$.

Since S is belain then
$c^{-1}.(b^{-1}.a) = a.(b^{-1}.c^{-1})$ for all $a,b,c \in S$. Since e is identity element of S ,we obtain

$=(a.e).(b.c)^{-1}$

$=((b.c).(a.e)^{-1})^{-1}$for all $a,b,c \in S$.

$=((b.c).(a^{-1}.e^{-1}))^{-1}$ for all $a,b,c \in S$.

$=(a^{-1}.e^{-1})^{-1}.(b.c)^{-1}$ for all $a,b,c \in S$.Since e is identity element of S,we get $=((a^{-1}.e).e^{-1})^{-1}.(b^{-1}.e.c^{-1}.e)$ for all $a,b,c \in S$. Then

$=((a^{-1}.e)^{-1}.(e^{-1})^{-1}).(b^{-1}.(c^{-1}.e))$ for all $a,b,c \in S$.

$=((a^{-1}.e)^{-1}.e).(b*(c*e))$ for all $a,b,c \in S$.

$= ((a*e)^{-1}.e).(b*(c*e))$.Since e is identity elemnt of S,we get

$=(a*e)^{-1}.(b*(c*e))$ for all $a,b,c \in S$.

$= (a*e)*(b*(c*e))$ .Since e is identity for $*$ on S.

$= a*(b*c)$ for all $a,b,c \in S$.

Then $*$ satisfying ATL law.

(ii) Satisfying from our hypothesis (e is a right identity element for $*$ on S).

(iii) $a*a=a^{-1}.a =e$ for all $a \in S$.Then (S, $*$) is a flower.

Proposition 6

Let (S, $*$)be a flower such that $a*b =a.b^{-1}$ for all $a,b \in S$,then S is Lahhamian group ,where (S,.)is a group.

Proof: We have $a*b=a.b^{-1}$ for all $a,b \in S$.Then we must prove $a*b= b*a$ for all $a,b \in S$ .Then we suppose $a*b \neq b*a$ for some $a, b \in S$ .Then

$a.b^{-1} \neq b.a^{-1}$ for some $a,b \in S$. Then

$a.b^{-1}.a \neq b.a^{-1}.a$ for some $a,b \in S$. Then

$a.b^{-1}.a \neq b.e$ for some $a,b \in S$.Since e is a right identity element of S , we obtain

$a.b^{-1}.a \neq b$ for some $a,b \in S$ .Then replacing b by a,we get $a.a^{-1}.a \neq a$ for some $a \in S$ .Then

$a.e \neq a$ for some $a \in S$,then $a \neq a$ for some $a \in S$.

This lead to contradiction .

Then (S, $*$)is a commutative ,by Lemma 2,(S, $*$) is Lahhamian group.

Theorem 7

Every flower is commutative. .

Proof: Suppose (S, $*$) is a flower ,then we have $a*(b*c)=c*(b*a)$ for all $a,b,c \in S$. Then by ATL law ,we obtain

$c * (b * a) = a * (b * c)$ for all $a,b,c \in S$ .Then

$e * (c * (b * a)) = e * (a * (b * c))$ for all $a,b,c \in S$, e is a right identity element of S.By ATL law,we get

$(b * a) * (c * e) = (b * c) * (a * e)$ for all $a,b,c \in S$. Since e is a right identity, then

$(b * a) * c = (b * c) * a$ for all $a,b,c \in S$. Then

Replacing a by $(b * c)$ and c by $(b * a)$, we obtain

$(b * (b * c)) * (b * a) = (b * (b * a)) * (b * c)$ for all $a,b,c \in S$.

By ATL law ,we get

$(c * (b * b)) * (b * a) = (a * (b * b)) * (b * c)$ for all $a,b,c \in S$ .Then

$(c * e) * (b * a) = (a * e) * (b * c)$ for all $a,b,c \in S$.

Replacing a by b ,we obtain $(c * e) * (b * b) = (b * e) * (b * c)$ for all $b,c \in S$.According to axiom (iii),we get $(c * e) * e = e$.Since e is a right identity of S. Then c=e for all $c \in S$,then $c * b = e * b$ for all $c,b \in S$. By Theorem 4 ,We get b=e for all $b \in S$. Then

$c * b = b * e$ for all $c,b \in S$. Since c=e ,then $c * b = b * c$ for all $c,b \in S$.Then $(S, *)$is commutative.

Corollary 8:

Let (S,o) be a flower with left and right translation of S ,then (S,o)is Lahhamian group.

Proof: By Lemma 3 ,left and right translation are commutative i.e. (S,o)is commutative flower ,by Lemma 2 , we get (S,o)is Lahhamian group.

Theorem 9

The Boolean ring $(S, *,\Diamond)$is garden ,where $(S, *)$ is Lahhamian group.

Proof: (i)By Proposition 7 , $(S, *)$ is flower .

(ii) For all $a,b,c \in S^*$,we have $a \Diamond (b \Diamond c)$,replacing b by c ,we obtain

$a \Diamond (b \Diamond c) = a \Diamond (c \Diamond c)$. Since $c \Diamond c = c, c \in S^*$, S is Boolean ring, then

$= a \Diamond c$,By replacing c by a ,we obtain

$= a \Diamond a = a$.

Also , we have $c \Diamond (b \Diamond a)$,by same method ,we obtain $c \Diamond (b \Diamond a) = a$.

Thus ,we obtain $a \Diamond (b \Diamond c) = c \Diamond (b \Diamond a)$ for all $a,b,c \in S^*$.

  i.e. $\Diamond$ is ATL law.

Theorem 10

Let $(S, * , \Diamond)$ be a garden ,then

(1) $(a \Diamond b) * (c \Diamond d) = o$ for all $a,b,c,d \in S^*$.

(2)$c * (b \Diamond a) = 0$ for all $c \in S, a,b \in S^*$.

(3)$o \Diamond o = o$.

(4)$(a * b) \Diamond (c*d) = o$ for all $a,b,c,d \in S$.

Proof:(1) We have $(a \Diamond b) * (c \Diamond d)$,replacing a by c and b by d , we obtain

$(a \Diamond b) * (c \Diamond d) = (c \Diamond d) * (c \Diamond d) = o$ for all $c,d \in S^*$.(o is a right identity element of $(S,*)$).

(2) By replacing c by $(b \Diamond a)$,we get

$c * (b \Diamond a) = (b \Diamond a) * (b \Diamond a) = o$ for all $c \in S, a,b \in S^*$. (o is a right identity of $(S, *)$).

(3)Since o is a right identity of $(S, *)$, then we have

$o \Diamond o = (a * a) \Diamond (a * a)$ for all $a \in S$. Since o is a right identity of $(S,*)$,then for all $a \in S$.

$= ((a * a) * o) \Diamond (a * a)$. By distributive law,we obtain

$= ((a * a) \Diamond (a * a)) * (o \Diamond (a * a))$ for all $a \in S$.

Since o is a right identity of $(S,*)$,then

$= ((a * a) \Diamond (a * a)) * ((a * a) \Diamond (a * a)) = o$ for all $a \in S$.

(4)For all $a,b,c,d \in S$ we have $(a * b) \Diamond (c * d)$. By take a=b,we obtain

$(b * b) \Diamond (c * d) = o \Diamond (c * d)$.Since o is a right identity of $(S, *)$, then

$= ((c * d) * (c * d)) \Diamond (c * d)$ .By distributive law ,we get

$= ((c * d) \Diamond (c * d)) * ((c * d) \Diamond (c * d))$

$= o$ (o is a right identity of $(S, *)$).

Theorem 11

In a garden $(S, *, \diamond)$ for all $a,b \in S$, $c \in S^*$,then $c \diamond (a*b)=0$.

Proof: We have $c \diamond (a*b)$,then $(c*o) \diamond (a*b)$ for all $a,b \in S, c \in S^*$.By distributive law ,we get

$(c*o) \diamond (a*b)=(c \diamond (a*b)*(o \diamond (a*b))$ for all $a,b \in S, c \in S^*$.

Let $a=b$,we obtain

$= (c \diamond (a*a)) * (o \diamond (a*a))$ for all $a \in S, c \in S^*$.

Since $(S, *)$ is a flower then by axiom (iii),we get

$= (c \diamond o)*(o \diamond o)$.   By distributive law,we obtain

$=(c*o) \diamond o$ (o is right identity of $(S, *)$). By Lemma1,we get $= (c* (o \diamond y)) \diamond (o \diamond y)$ for all $c, y \in S^*$ .By distributive law ,we obtain $= (c \diamond (o \diamond y)) * ((o \diamond y) \diamond (o \diamond y))$ for all $c,y \in S^*$.Then

$=(c \diamond (o \diamond y))*o$ for all $c, y \in S^*$. Since o is a right identity of $(S,*)$,then

$=c \diamond (o \diamond y)$ for all $c,y \in S^*$. By Lemma 1 , we obtain

$=c \diamond o$ for all $c \in S^*$. By Theorem 10(3),we obtain

$=c \diamond (o \diamond o)$ for all $c \in S^*$. By ATL law,we get

$=o \diamond (o \diamond c)$ for all $c \in S^*$. By Lemma1,we obtain

$= o \diamond o$.By Theorem 10(3),we get

$= o$

As a corollary of Theorem 11, we can write

Corollary 12

Let $(S, *, \diamond)$ be a garden ,then for all $c \in S^*$,$c \diamond o=o$.

Proof: We have $c \diamond o$,for all $c \in S^*$,by Theorem 10(3),we obtain

$c \diamond o=c \diamond (o \diamond o)$ for all $c \in S^*$.By ATL law ,we get

$= o \diamond (o \diamond c)$ for all $c \in S^*$.By Lemma1,we obtain

$=o \diamond o$ .Again by Theorem 10(3),we get

$=o$

Proposition 13

In a farm $(S, *, \diamond)$ for all $a,b,c \in S$, then $c \diamond (a*b)=o$

Proof: We can write the proof of our proposition by same method in Theorem 11**.**

From precedence Theorem 10, Theorem 11and Corollary 12, we can write the following remark.

Remark 14

Let $(S, *, \lozenge)$ be a garden ,then

$(a \lozenge b) * (c \lozenge d)=(a*b) \lozenge (c*d) =c*(b \lozenge a)=c \lozenge (a*b)=c \diamond o=o \lozenge o=o$.

Theorem 15

If $(S, *, \lozenge)$ be a ring ,then S with the new operations $\lozenge$ and $*$ defined as follows :

$a*b=a.b^{-1}$ for all $a,b \in S$.

$a \lozenge b=a$ for all $a,b \in S^*$.Is a garden ,where $(S,.)$ is a belian group.

Proof:(i) By Proposition 5 ,$(S, *)$ is a flower .

(ii)We have $c \lozenge (b \lozenge a)$ according to defined of $\lozenge$,then $c \lozenge (b \lozenge a)=c \lozenge b$ for all $a,b,c \in S^*$.Since$(a \lozenge b=a)$,then

$= (c \lozenge a) \lozenge (b \lozenge a)$ for all $a,b,c \in S^*$.

Replacing a by c and c by a ,we obtain

$= (a \lozenge c) \lozenge (b \lozenge c)$ for all $a,b,c \in S^*$ .Since $(a \lozenge c=a)$ according to defined of $\lozenge$, we obtain

$=a \lozenge (b \lozenge c)$ for all $a,b,c \in S^*$

Thus $\lozenge$ is ATL law.

(iii) We have $(a*b) \lozenge c=a*b$ for all $a,b \in S$,$c \in S^*$. According to defined of $\lozenge$,we obtain

$= (a \lozenge c) * (b \lozenge c)$ for all $a,b \in S, c \in S^*$.

Thus the distributive law hold in S.

Then $(S, *, \lozenge)$ is a garden .

Theorem 16

If $(S, *, \lozenge)$ be a ring ,then S with new operation $\lozenge$ and $*$ defined as follows:

$a*b=a.b^{-1}$ for all $a,b \in S$ and $a \lozenge b=a$ for all $a,b \in S^*$. Is a farm, where $(S,.)$ is a belian group.

Proof: According to Theorem 15 ,we have (S, $*$) is a flower and the distributive law hold in S,therefore ,we must prove $(S^*, \Diamond)$is a flower ,then

(i) Form Theorem 15, $\Diamond$ is ATL law of S^*.

(ii) According to $a \Diamond b=a$ for all $a,b \in S^*$,there exists an element e in S^* such that $a \Diamond e=a$ for all $a \in S^*$, i.e. e is a right identity element of S^*.

(iii) According to $a \Diamond b=a$ for all $a,b \in S^*$,then $a \Diamond a=a$ for all $a \in S^*$,therefore,

$a \Diamond a=a \Diamond (a \Diamond e)$.By ATL law ,we get

$= e \Diamond (a \Diamond a)$ for all $a \in S^*$.According to defined $\Diamond$,we obtain

$=e \Diamond a$ for all $a \in S^*$.According to defined of $\Diamond$,we obtain

$=e$. Then $(S^*, \Diamond)$ is a flower

Thus ,we obtain $(S, *, \Diamond)$ is a farm.

As a corollary of Theorem 8 and Proposition 8 ,we write the following theorem.

Theorem 17

If $(S, *)$ is a flower .Then

(i) S is a Lahhamian group.

(ii)S is has an identity.

(iii) S is commutative.

(iv)S is a ssociative.

The following conjecture appeared in [1] with out proof ,we give the proof .

Al-Lahham Conjecture 18

If (S, $*$)is a groupoid with a right identity element e ,such that $x * x= e$ for all $x \in S$,then for all elements $a,b,c \in S$:

$a * (b * c)=c * (b * a)$ if and only if $(a * b) * c=(a * c) * b$.

Proof: For all $a,b,c \in S$, we have $a * (b * c)=c * (b * a)$ .Then we will deal with (S, $*$) as a flower. By Proposition 6 ,we get

$(a * c) * c=(a * c) * b$.

Also , we have for all $a,b,c \in S$,such that $(a * b) * c=(a * c) * b$.

Since $x * x =e$ for all $x \in S$.

Then (S, $*$)is Lahhamian group ,by Proposition 7 ,$(S, *)$ is flower,then $a * (b * c)= c * (b * a)$is satisfy.

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الخلاصة :

الغرض الرئيسي من هذا البحث اعطاء بعض النتائج مبرهنات وتمهيديات وملاحظات بخصوص هذه البنى الجبرية الحديثة ,الزهرة والحديقة والمزرعة مع بنى جبرية مألوفة ,نصف الزمرة والزمرة والحلقة.