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Recurrency on the Space of Hilbert-Schmidt Operators

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Abstract

In this paper, it is proved that if a C0-semigroup is chaotic, hypermixing or supermixing, then the related left multiplication C0-semigroup on the space of Hilbert-Schmidt operators is recurrent if and only if it is hypercyclic. Also, it is stated that under some conditions recurrence of a C0-semigroup and the recurrency of the left multiplication C0-semigroup that is related to it, on the space of Hilbert-Schmidt operators are equivalent. Moreover, some sufficient conditions for recurrency and hypercyclicity of the left multiplication C0-semigroup are presented that are based on dense subsets.

Keywords: Hilbert-Schmidt Operators, Hypercyclic Operators, Left Multiplication, Recurrent Operators, Recurrent Semigroup.

Introduction

Let $B(H)$ be a set of bounded and linear operators on a Hilbert space H. An operator $T \in B(H)$ is called hypercyclic if $h \in H$ exists such that $orb(T,h)$ be dense in H, that is $\overline{\{h, Th, \ldots, T^n h, \ldots\}} = H$ [1]. Hypercyclicity of an operator implies for any nonempty open sets $U, V \subseteq H$, there is $n \in \mathbb{N}$ such that $T^{-n}(U) \cap V \neq \emptyset$ [1]. If for any nonempty open set $U \subseteq H$, there is $n \in \mathbb{N}$ such that $T^{-n}(U) \cap U \neq \emptyset$, T is called a recurrent operator [2]. Hence, hypercyclicity implies recurrence. These concepts and related topics are investigated by many mathematicians. To see a history of these concepts, one can refer to [3] and [4].

Another interesting structure that concepts like hypercyclicity and recurrency investigated on it is a C_0-semigroup. A C_0-semigroup on a Hilbert space H is a family $(T_t)_{t\geq 0}$ of operators on H with these properties that $T_0 = I$, $T_t T_s = T_{s+t}$ for any $s, t \geq 0$, and for any $h \in H$, $\lim_{s\to t} T_s h = T_t h$ for any $t \geq 0$ [1].

A C_0-semigroup $(T_t)_{t\geq 0}$ is said a hypercyclic C_0-semigroup if $orb((T_t)_{t\geq 0}, h)$ is dense in H for some $h \in H$. It is well known that the hypercyclicity of a C_0-semigroup implies that for any nonempty open sets $U, V \subseteq H$, $t > 0$ can be found such that $T_t^{-1}(U) \cap V \neq \emptyset$ [1]. One can find more about the hypercyclicity of a C_0-semigroup in [5]. It is proved in Theorem 2.4 of [1] that hypercyclic C_0-semigroups can be found in every infinite-dimensional separable Hilbert space. Also, one can see [6-9] for more information.

There are some criteria for C_0-semigroups. Recall hypercyclicity criterion (HCC) and recurrent hypercyclicity criterion (RHCC) given from [10] as follows. In the following, H always indicates a separable Hilbert space.

Definition 1: (see [10]) (HCC) A C_0-semigroup $(T_t)_{t\geq 0}$ on H fulfills the hypercyclicity criterion if and only if $t > 0$ can be found with these properties

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that $T_t(U) \cap W \neq \emptyset$, and $T_t(W) \cap V \neq \emptyset$, where W is a neighborhood of zero in H and $U, V \subseteq H$ are nonempty open sets.

Definition 2: (see [10]) (RHCC) A C_0-semigroup $(T_t)_{t\geq 0}$ on H fulfills recurrent hypercyclicity criterion if and only if for any nonempty open sets $U, V \subseteq H$ and any neighborhood W of zero in H, L_1 and L_2 can be chosen with this property that, $q_1 \in [t, t+L_1]$ and $q_2 \in [t, t+L_2]$ can be found such that for any $t > 0$

$$T_{q_1}(U) \cap W \neq \emptyset \;, \quad \text{and} \quad T_{q_2}(W) \cap V \neq \emptyset.$$

One can see some relations between HCC and RHCC in [10].

A C_0-semigroup $(T_t)_{t\geq 0}$ on H is called recurrent if for any nonempty open set $U \subseteq H$, there is $t > 0$ such that $T_t^{-1}(U) \cap U \neq \emptyset$ [2]. By Definition 1, and Definition 2, it is deduced that for C_0-semigroups, hypercyclicity implies recurrency. It is proved in Theorem 5 of [2] that recurrent C_0-semigroups can be found in finite-dimensional spaces, too. If for a vector $h \in H$, an increasing sequence (t_n) exists such that $T_{t_n}h \to h$, then h is called a recurrent vector for $(T_t)_{t\geq 0}$ [2]. The set of recurrent vectors for $(T_t)_{t\geq 0}$ is denoted by $Rec(T_t)_{t\geq 0}$. It is proved that the recurrence of $(T_t)_{t\geq 0}$ is equivalent to this condition that $(T_t)_{t\geq 0}$ has a dense set of recurrent vectors [2].

It is interesting to investigate recurrency on $B_2(H)$, where $B_2(H)$ is the algebra of Hilbert-Schmidt operators. Remind that if a separable Hilbert space H has the basis $\{e_i\}$, then

$$||T||_2 = (\sum_{i=1}^{\infty} ||Te_i||^2)^{\frac{1}{2}}.$$

If $||T||_2 < \infty$, T is said a Hilbert-Schmidt operator. Also, it is interesting to investigate recurrency on the operator algebra $B(H)$, when H is a Hilbert space on $\mathbb{C}$, the field of complex numbers, where H is separable and infinite-dimensional. The norm topology of $B(H)$ is not separable [11]. Also, $B(H)$ is separable with strong operator topology or briefly SOT-topology [11].

A considerable operator on $B(H)$ and $B_2(H)$ is left multiplication. Recall that the left multiplication $L_T: B(H) \to B(H)$ is defined with $L_T S = TS$ for any $S \in B(H)$. The operator L_T defines similarly on $B_2(H)$. Yousefi and Rezaei proved that the hypercyclicity of L_T on $B(H)$ is equivalent to the hypercyclicity of L_T on $B_2(H)$, and equivalent to this condition that T satisfies in hypercyclicity criterion for operators on H [12]. One can see [13] and [14] for some related matter to operators on Hilbert-Schmidt operators.

For a C_0-semigroup $(T_t)_{t\geq 0}$ on H and any operator $S \in B(H)$, if $(L_{T_t})_{t\geq 0}$ is defined on $B(H)$, with $(L_{T_t}S)_{t\geq 0} = (T_t S)_{t\geq 0}$, then $(L_{T_t})_{t\geq 0}$ is a C_0-semigroup, since for any $t, q \geq 0$ and any $S \in B(H)$,

$$L_{T_0}S = L_I S = IS = I,$$
$$L_{T_t}L_{T_q}S = L_{T_t}(L_{T_q}S) = L_{T_t}(T_q S) = T_t(T_q S) = T_{t+q}S,$$
$$\lim_{t\to q} L_{T_t}S = \lim_{t\to q} T_t S = T_q S.$$

$(L_{T_t})_{t\geq 0}$ Is called a left multiplication C_0-semigroup related to $(T_t)_{t\geq 0}$ or simplify a left multiplication C_0-semigroup.

It is proved in this paper that the recurrency of $(L_{T_t})_{t\geq 0}$ on $B_2(H)$ with $||.||_2$-topology and on $B(H)$ with SOT-topology is equivalent. It is established that if $(T_t)_{t\geq 0}$ satisfies the RHCC or HCC, then $(L_{T_t})_{t\geq 0}$ is recurrent. It is proved that if a C_0-semigroup is chaotic, hypermixing or supermixing, then its related left multiplicationC_0-semigroup on the space of Hilbert-Schmidt operators is recurrent if and only if it is hypercyclic. Moreover, some sufficient conditions for hypercyclicity and recurrency of $(L_{T_t})_{t\geq 0}$ that are based on dense subsets of $B(H)$ are stated.

Results and Discussion

Main Results

In the beginning, it is proved that the recurrency of left multiplication semigroup on operator algebra and its recurrency on the space of Hilbert-Schmidt operators are equivalent.

In the following, consider $S(H)$ as the set of finite rank operators. That means for any $T \in S(H)$, a

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natural number m_T can be found such that $Te_i = 0$, when $i \geq m_T$.

Theorem 1: For a C_0-semigroup $(T_t)_{t\geq 0}$ on H, the following are equivalent:
(a) $(L_{T_t})_{t\geq 0}$ is recurrent on $B_2(H)$ with $||.||_2$-topology,
(b) $(L_{T_t})_{t\geq 0}$ is recurrent on $B(H)$ with strong operator topology.
Proof: (a)→(b) Suppose $(L_{T_t})_{t\geq 0}$ is recurrent on $B_2(H)$ with $||.||_2$-topology. Hence $(L_{T_t})_{t\geq 0}$ has a dense set of recurrent vectors on $B_2(H)$ by Theorem 3 in [2]. That means $Rec((L_{T_t})_{t\geq 0})$ is dense in $B_2(H)$ with $||.||_2$-topology. Consider $x \in Rec((L_{T_t})_{t\geq 0})$. Hence, $L_{T_{t_n}}x \to x$ with $||.||_2$-topology for some increasing sequence (t_n). Therefore, $L_{T_{t_n}}x \to x$ with SOT-topology. So x is a recurrent vector for $(L_{T_t})_{t\geq 0}$ in the SOT-topology on $B_2(H)$. Hence, $Rec((L_{T_t})_{t\geq 0})$ is dense in $B_2(H)$ with SOT-topology. As it is known, $B_2(H)$ is dense in $B(H)$ with SOT-topology [12]. Hence, $(L_{T_t})_{t\geq 0}$ has a dense set of recurrent vectors on $B(H)$ with SOT-topology. So $(L_{T_t})_{t\geq 0}$ is recurrent on $B(H)$ with strong operator topology by Theorem 3 in [2].
(b)→(a) Suppose that $(L_{T_t})_{t\geq 0}$ is recurrent on $B(H)$ with SOT-topology. Let U be a norm open and nonempty set. Let $A \in U \cap S(H)$. Suppose that $Ae_i = 0$ for any $i > N$. Let D=$\sum_{i=1}^{N} e_i \otimes e_i$. Hence,

$$AD = \sum_{i=1}^{N} A(e_i \otimes e_i) = A.$$

Now, let

$$U_k = \left\{V \in B(H): \left||Ve_i - Ae_i\right|| < \frac{1}{k} :\ 1 \leq i \leq N\right\}.$$

U_k is a SOT-open set in $B(H)$. By assumption, there is $t_0 > 0$ with this property that $L_{T_{t_0}}^{-1}(U_k) \cap U_k \neq \emptyset$. Hence, there is $S_k \in B(H)$ such that $S_k \in U_k$ and $T_{t_0}S_k \in U_k$. That means, for any i with $1 \leq i \leq N$,

$$\left||S_k e_i - Ae_i\right|| < \frac{1}{k}, \text{ and } \left||T_{t_0}S_k e_i - Ae_i\right|| < \frac{1}{k}.$$

Now,

$$\left||S_k D - AD\right||_2^2 = \sum_{i=1}^{N} \left||(S_k - A)e_i\right||^2 \leq \frac{N}{k^2},$$

and

$$\left|\left|L_{T_{t_0}}(S_k D) - AD\right|\right|_2^2 = \sum_{i=1}^{N} \left||(T_0 S_k - A)e_i\right||^2 \leq \frac{N}{k^2}.$$

Therefore, $S_k D \to AD$ with $||.||_2$ and $L_{T_{t_0}}(S_k D) \to AD$ with $||.||_2$, where $k \to \infty$. So $S_k D \in U$, and $L_{T_{t_0}}(S_k D) \in U$. Moreover, $S_k D$ and $L_{T_{t_0}}(S_k D)$ are finite rank operators and hence, they are Hilbert-Schmidt operators. Therefore, $L_{T_{t_0}}^{-1}(U) \cap U \neq \emptyset$.

The following theorem indicates that satisfying HCC and RHCC are sufficient conditions for the recurrence of left multiplication C_0-semigroup.

Theorem 2: If $(T_t)_{t\geq 0}$ satisfies HCC or RHCC on $B_2(H)$, then $(L_{T_t})_{t\geq 0}$ is recurrent.
Proof: If $(T_t)_{t\geq 0}$ satisfies RHCC, then it satisfies HCC by Proposition 2.2 of [10]. So T_1 satisfies HCC [1]. By [12], L_{T_1} is hyper on $B_2(H)$. Hence, L_{T_1} is recurrent on $B_2(H)$. Therefore, $(L_{T_t})_{t\geq 0}$ is recurrent by Theorem1 in [2].

Now, this question arises is the recurrency of $(L_{T_t})_{t\geq 0}$ implying that $(T_t)_{t\geq 0}$ is recurrent? In the next theorem, it is shown that if $(L_{T_t})_{t\geq 0}$ satisfies HCC or RHCC, then the answer to this question is positive.

Theorem 3: Let $(T_t)_{t\geq 0}$ be a C_0-semigroups on H. If $(L_{T_t})_{t\geq 0}$ satisfies RHCC or HCC on $B_2(H)$, then $(T_t)_{t\geq 0}$ is recurrent on H.
Proof: Let $(L_{T_t})_{t\geq 0}$ satisfies RHCC. Hence $(L_{T_t})_{t\geq 0}$ satisfies HCC [10]. So $(L_{T_t} \oplus L_{T_t})_{t\geq 0}$ is hypercyclic by Theorem 2.3 in [15]. Therefore, $L_{T_t} \oplus L_{T_t}$ is hypercyclic for any $t > 0$ by Theorem 2.3 in [16]. Consider $t_0 > 0$. By Theorem 2.3 in [15], $L_{T_{t_0}}$satisfies in HCC. By Proposition 2.3 in [12], $\bigoplus_{n=1}^{\infty} T_{t_0}$ satisfies in HCC. Therefore, T_{t_0} is hypercyclic. Thus T_{t_0} and so $(T_t)_{t\geq 0}$ is recurrent.

By Theorem 2, and Theorem 3, the following corollary is concluded.

Corollary 1: Let $(T_t)_{t\geq 0}$ on H and $(L_{T_t})_{t\geq 0}$ on $B_2(H)$ both satisfy one of RHCC or HCC. Then the following conditions are equivalent:
(a) $(L_{T_t})_{t\geq 0}$ is recurrent on $B_2(H)$ with $||.||_2$-topology.
(b) $(L_{T_t})_{t\geq 0}$ is recurrent on $B(H)$ with SOT-topology.
(c) $(T_t)_{t\geq 0}$ is recurrent on H.

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A C_0-semigroup $(T_t)_{t\geq 0}$ on H is chaotic if it is hypercyclic and has a dense set of periodic points in H. That means, there is a dense set of points like $h \in H$ such that $T_{t_0}h = h$ for some $t_0 > 0$ [1]. Therefore, a chaotic C_0-semigroup is recurrent.

Example 1: Consider $H := l^p$, $1 \leq p < \infty$. Let $(T_t)_{t\geq 0}$ be a C_0-semigroups on l^p such that $T_{t_0} = \lambda_0 B$ for some $t_0 > 0$, where B is the backward shift on l^p and $|\lambda_0| > 1$. By Example 2.32 [1], $T_{t_0} = \lambda_0 B$ is chaotic. Hence, $(T_t)_{t\geq 0}$ is a chaotic C_0-semigroups on l^p since any periodic point of $T_{t_0} = \lambda_0 B$ is a periodic point of $(T_t)_{t\geq 0}$. Hence, $(T_t)_{t\geq 0}$ is recurrent on l^p. So, $(L_{T_t})_{t\geq 0}$ is recurrent on $B_2(H)$ with $||.||_2$-topology, and $(L_{T_t})_{t\geq 0}$ is recurrent on $B(H)$ with SOT-topology by Corollary 1.

By Theorem 3, the following corollary about chaotic semigroups is concluded.

Corollary 2: If $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are chaotic C_0-semigroups on H, then $(T_t \oplus S_t)_{t\geq 0}$ is recurrent on $H \oplus H$.
Proof: By Corollary 6.2 in [10], a chaotic semigroup satisfies RHCC. By Corollary 5.6 in [10], $(T_t \oplus S_t)_{t\geq 0}$ satisfies in RHCC. So $(T_t \oplus S_t)_{t\geq 0}$ is recurrent.

The following corollary is a direct result of Corollary 1, and Corollary 2.

Corollary 3: If $(T_t)_{t\geq 0}$ is a chaotic C_0-semigroups on H, then $(L_{T_t})_{t\geq 0}$ is recurrent on $B_2(H)$ and $B(H)$. Moreover, $(L_{T_t \oplus T_t})_{t\geq 0}$ is recurrent.

Some Sufficient Conditions

This section is started with a sufficient condition for hypercyclicity and hence, recurrency for a C_0-semigroup.

Theorem 4: Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a Hilbert space H. Suppose $A \subseteq H$ exists such that $\bar{A} = H$ and suppose that $(S_t)_{t\geq 0}$ exists on H with these properties:

(a) $T_t S_t = I$ on A,

(b) $||T_t(a)|| \to 0$, when $t \to \infty$ for any $a \in A$,

(c) $||S_t(a)|| \to 0$, when $t \to \infty$ for any $a \in A$.

Then $(T_t)_{t\geq 0}$ is hypercyclic. Especially, $(T_t)_{t\geq 0}$ is recurrent.

Proof: It is sufficient to consider Theorem 7.29 in [1].

The idea of the following theorem is given from Theorem 2.1 in [1].

Theorem 5: Let $(T_t)_{t\geq 0}$ be a C_0-semigroups on $B(H)$. Suppose that there is a C_0-semigroup $(S_t)_{t\geq 0}$ on $B(H)$ such that $T_t S_t = I$ for any $t > 0$. If there is a SOT-dense set $D \subseteq B(H)$ such that for any $g \in D$,

$$||T_t(g)|| \to 0, \text{ and } ||S_t(g)|| \to 0,$$

when $t \to \infty$, then $(T_t)_{t\geq 0}$ is hypercyclic. Especially, $(T_t)_{t\geq 0}$ is recurrent.

Proof: Let $D' = \{f_k : k \geq 1\}$ be a countable SOT-dense subset of D. By hypothesis, t_1 can be found such that

$$\left|\left|T_{t_1}(f_1)\right|\right| < \tfrac{1}{2}.$$

Also, t_2 can be found such that

$$\left|\left|T_{t_2}(f_2)\right|\right| < \tfrac{1}{2^{1+2}}, \text{ and } \left|\left|S_{t_2}(f_1)\right|\right| < \tfrac{1}{2^2}.$$

Also, one can find t_3 such that

$$\left|\left|T_{t_1+t_2+t_3}(f_3)\right|\right| < \tfrac{1}{2^{1+2+3}}, \left|\left|T_{t_2+t_3}(f_3)\right|\right| < \tfrac{1}{2^{2+3}}, \left|\left|T_{t_3}(f_3)\right|\right| < \tfrac{1}{2^3},$$

$$\left|\left|S_{t_2+t_3}(f_1)\right|\right| < \tfrac{1}{2^{2+3}}, \text{ and } \left|\left|S_{t_3}(f_2)\right|\right| < \tfrac{1}{2^3}.$$

In such a way, one can find t_k such that for $m = 1,2,\dots,k$ and for $i = 2,3,\dots,k$,

$$\left|\left|T_{t_m+t_{m+1}+\cdots+t_k}(f_k)\right|\right| < \tfrac{1}{2^{m+(m+1)+\cdots+k}}, \quad (1)$$

and

$$\left|\left|S_{t_i+t_{i+1}+\cdots+t_k}(f_{i-1})\right|\right| < \tfrac{1}{2^{i+(i+1)+\cdots+k}}. \quad (2)$$

Let

$$f = \sum_{k=1}^{\infty} S_{t_1+t_2+\cdots+t_k}(f_k),$$

The above definition is meaningful. Since the series is absolutely convergent by Eq 2.

Consider $m \geq 2$. Then by the boundedness of $T_{t_1+t_2+\cdots+t_m}$,

$$T_{t_1+t_2+\cdots+t_m}(f) = \sum_{k=1}^{\infty} T_{t_1+t_2+\cdots+t_m} S_{t_1+t_2+\cdots+t_k}(f_k) = \sum_{k=1}^{m-1} T_{t_1+t_2+\cdots+t_m}(f_k) + f_m + \sum_{k=m+1}^{\infty} S_{t_{m+1}+\cdots+t_k}(f_k). \quad 3$$

By Eq 1 and Eq 2, it is concluded from Eq 3 that

$$\lim_{m\to\infty} || T_{t_1+t_2+\cdots+t_m}(f) - f_m|| = 0. \quad 4$$

f is a hypercyclic vector for $(T_t)_{t\geq 0}$. For this, let U be a nonempty open set in SOT-topology. So there are $h_1, h_2, \dots, h_n \in H$ and $f_0 \in B(H)$ such that

$$U = U(f_0, \varepsilon; h_1, h_2, \dots, h_n) = \{g \in B(H): ||(g - f_0)h_i|| < \varepsilon,\ i = 1,2,\dots,N\}.$$

If $h_1 = h_2 = \dots = h_n = 0$, then $U = B(H)$ and hence $U \cap orb((T_t)_{t\geq 0}, f) \neq \emptyset$.

If one of the h_i's is non-zero, consider

$$\alpha = \max\{1, ||h_i|| : 1 \leq i \leq N\}.$$

By Eq 4, a positive integer M exists with this property that for any $m \geq M$,

$$||T_{t_1+t_2+\cdots+t_m}(f) - f_m|| < \frac{\varepsilon}{2\alpha}.$$

Hence, for any $1 \leq i \leq N$ and for any $m \geq M$,

$$||T_{t_1+t_2+\cdots+t_m} f(h_i) - f_m(h_i)|| < \frac{\varepsilon}{2\alpha} ||h_i|| < \frac{\varepsilon}{2\alpha}.$$

On the other hand, SOT-density of $\{f_k : k \geq 1\}$ in $B(H)$ implies that $\{f_k : k \geq m\}$ is SOT-dense in $B(H)$, too. Hence, there is $k_0 \geq M$ such that $f_{k_0} \in U$. So for any $k_0 \geq M$,

$$\left||f_{k_0}(h_i) - f_0(h_i)|\right| < \frac{\varepsilon}{2}. \quad 5$$

Therefore, by Eq 5 for any $1 \leq i \leq N$

$$||T_{t_1+t_2+\cdots+t_{k_0}} f(h_i) - f_0(h_i)|| \leq ||T_{t_1+t_2+\cdots+t_{k_0}} f(h_i) - f_{k_0}(h_i)|| + \left||f_{k_0}(h_i) - f_0(h_i)|\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad 6$$

Now, Eq 6 implies that $T_{t_1+t_2+\cdots+t_{k_0}}(f) \in U$. That means in this case, $U \cap orb((T_t)_{t\geq 0}, f) \neq \emptyset$. So f is a hypercyclic vector for $(T_t)_{t\geq 0}$. This completes the proof.

Theorem 6: Let $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ be two C_0-semigroups on $B(H)$. Suppose that $A \subseteq H$ exists such that $\bar{A} = H$, and for any $x \in A$,

$$\lim_{t\to\infty} \left||T_t x|\right| = 0, \text{ and } \lim_{t\to\infty} \left||S_t x|\right| = 0.$$

Then there is $D \subseteq B(H)$ such that D is dense in $B(H)$ with SOT-topology with this property that for any $W \in D$,

$$\lim_{t\to\infty} \left||L_{T_t}(W)|\right| = 0, \text{ and } \lim_{t\to\infty} \left||L_{S_t}(W)|\right| = 0.$$

Proof: A countable set $A' \subseteq A$ can be found such that $\overline{A'} = H$. Let $\{e_k : k \geq 1\}$ be an orthonormal basis for H. Consider

$$D := \{W \in B(H): \exists n_W;\ We_k = 0 \ when\ k > n_W \ and\ We_k \in A' \ when\ k \leq n_W \}. \quad 7$$

The set D is dense in $B(H)$ [4, Lemma 3.1]. Also, if $f = \sum_{k=1}^{\infty} a_k e_k$ with $\sum_{k=1}^{\infty} |a_k|^2 < \infty$, then by Eq 7 for any $W \in D$,

$$||L_{T_t}(W)f||^2 = ||T_t(Wf)||^2 = ||T_t\left(\sum_{k=1}^{n_W} a_k W(e_k)\right)||^2 = ||\sum_{k=1}^{n_W} a_k T_t W(e_k)||^2 \leq \sum_{k=1}^{n_W} (|a_k|\, ||T_t(We_k)||)^2 \leq (\sum_{k=1}^{n_W} |a_k|^2)(\sum_{k=1}^{n_W} ||T_t(We_k)||^2) \leq ||f||^2 \ \sum_{k=1}^{n_W} ||T_t(We_k)||^2. \quad 8$$

For each $W \in D$, $We_k \in A'$ for any $k \leq n_W$. So $||T_t(We_k)|| \to 0$, when $t \to \infty$. Hence, it is concluded from Eq 8 that $||L_{T_t}(W)|| \to 0$, when $t \to \infty$. So, D is the desired set.

Theorem 6 and Theorem 4 lead us to the next corollary. Note that if $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are two C_0-semigroups on H with this property that $T_tS_t = I$ on H, then $L_{T_t}L_{S_t} = I$ on $B(H)$. Because for any $W \in B(H)$,

$$L_{T_t}L_{S_t}W = L_{T_t}(L_{S_t}W) = L_{T_t}(S_tW) = T_t\, S_tW = W. \qquad 9$$

Corollary 4: Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on H. If $(T_t)_{t\geq 0}$ satisfies the conditions of Theorem 4, then $(L_{T_t})_{t\geq 0}$ is hypercyclic on $B_2(H)$ with $||.||_2$-topology. Especially, $(T_t)_{t\geq 0}$ is recurrent on $B(H)$ with SOT-topology and on $B_2(H)$ with $||.||_2$-topology.

Proof: If $(T_t)_{t\geq 0}$ satisfies the conditions of Theorem 4, then $(T_t)_{t\geq 0}$ satisfies the conditions of Theorem 6. Also, because $(S_t)_{t\geq 0}$ is the right inverse of $(T_t)_{t\geq 0}$, then $(L_{W_t})_{t\geq 0}$ is the right inverse of $(L_{T_t})_{t\geq 0}$ as it is shown in Eq 9.

Supermixing, Hypermixing, and Recurrency of Left Multiplication C_0-semigroup

A C_0-semigroup $(T_t)_{t\geq 0}$ on H is named supermixing if $\bigcup_{i=0}^{\infty}\bigcap_{t\geq i} T_t(U)$ is dense in H for any nonempty open subset U of H and if $H \setminus \{0\} \subseteq \bigcup_{i=0}^{\infty}\bigcap_{t\geq i} T_t(U)$, then $(T_t)_{t\geq 0}$ is called hypermixing [17]. The set of hypermixing and supermixing C_0-semigroups are proper subsets of the set of hypercyclic C_0-semigroups [17]. It is shown in the next theorem that hypermixing (supermixing) of a C_0-semigroup indicates hypercyclicity and recurrency of the related left multiplication.

Theorem 7: Let $(T_t)_{t\geq 0}$ be a hypermixing (supermixing) C_0-semigroup on H. Then

(a) $(L_{T_t})_{t\geq 0}$ is hypercyclic on $B_2(H)$ with $||.||_2$-topology,

(b) $(L_{T_t})_{t\geq 0}$ is recurrent on $B_2(H)$ with $||.||_2$-topology,

(c) $(L_{T_t})_{t\geq 0}$ is recurrent on $B(H)$ with SOT-topology.

Proof: First note that $(T_t)_{t\geq 0}$ satisfies HCC because $(T_t)_{t\geq 0}$ is hypermixing by Theorem 3.6 in [17]. Similar to the proof of Theorem 2, L_{T_1} is hypercyclic on $B_2(H)$. So $(L_{T_t})_{t\geq 0}$ is hypercyclic on $B_2(H)$.

Part (a) implies part (b) because recurrency is concluded from hypercyclicity.

Finally, Theorem 1 asserts part (c), since by Theorem 1, the recurrency of $(L_{T_t})_{t\geq 0}$ on $B_2(H)$, and recurrency of $(L_{T_t})_{t\geq 0}$ on $B(H)$ with SOT-topology are equivalent.

The proof is similar when $(T_t)_{t\geq 0}$ is supermixing.

The following theorem shows that hypermixing (supermixing) of left multiplication C_0-semigroup $(L_{T_t})_{t\geq 0}$, implies the recurrency of $(T_t)_{t\geq 0}$.

Theorem 8: Let $(L_{T_t})_{t\geq 0}$ is a hypermixing(supermixing) C_0-semigroup on $B_2(H)$ with $||.||_2$-topology. Then $(T_t)_{t\geq 0}$ is hypercyclic on H. Especially, $(T_t)_{t\geq 0}$ is recurrent.

Proof: By hypothesis, $(L_{T_t})_{t\geq 0}$ is a hypermixing. So by Theorem 3.6 in [17], $(L_{T_t})_{t\geq 0}$ satisfies HCC. Similar to the proof of Theorem 1, the operator T_{t_1} is hypercyclic. Hence, $(T_t)_{t\geq 0}$ is hypercyclic on H and hence, it is recurrent.

Similarly, the supermixing of $(L_{T_t})_{t\geq 0}$ indicates that $(T_t)_{t\geq 0}$ is recurrent.

Since hypermixing (supermixing) C_0-semigroups are hypercyclic they do not exist on $B(H)$ with SOT-topology. Also, the following corollary about the left multiplication operator is given.

Corollary 5: If $(L_{T_t})_{t\geq 0}$ is a hypermixing (supermixing) C_0-semigroup $B_2(H)$ with $||.||_2$-topology, then L_{T_t} and T_t are recurrent, respectively on $B_2(H)$ and H for any $t > 0$.

Proof: It is deduced from Theorem 3.6 in [17] that $(L_{T_t})_{t\geq 0}$ satisfies HCC. So $(L_{T_t})_{t\geq 0}$ is hypercyclic on $B_2(H)$. Hence, L_{T_t} is hypercyclic on $B_2(H)$ for any $t > 0$ by Theorem 2.3 in [16]. Hence, L_{T_t} is hypercyclic on $B_2(H)$ for any $t > 0$. Moreover, the hypercyclicity of L_{T_t} implies that T_t satisfies HCC on H by Theorem 2.2 in[12]. Therefore T_t is hypercyclic and so recurrent for any $t > 0$.

Conclusion

A C_0-semigroup is an important structure for mathematicians. In this paper, the recurrency of the C_0-semigroups on the space of Hilbert-Schmidt operators are investigated which is an exciting matter in dynamical systems. In this paper, it is proved that the recurrence of a C_0-semigroup $(T_t)_{t\geq 0}$ on H, and the recurrence of its related left multiplication C_0-semigroup on $B_2(H)$ are equivalent. It is interesting to know if this issue can be stated for the related right multiplication C_0-semigroup as well? Recall that $(R_{T_t})_{t\geq 0}$ is the related right multiplication C_0-semigroup such that $R_{T_t}: B(H) \to B(H)$ is defined with $R_{T_t}S = ST_t$ for any $S \in B(H)$. In Theorem 4, Theorem 5, and Theorem 6, some sufficient conditions for a C_0-semigroup to be recurrent are stated that are based on dense sets. In Theorem 7, it is shown that if a C_0-semigroup $(T_t)_{t\geq 0}$ on H is hypermixing (supermixing), then hypercyclicity, and recurrency of its related left multiplication C_0-semigroup on $B_2(H)$ are equivalent. This question arises can one state this equivalence for related right multiplication a C_0-semigroup or not?

Author's Declaration

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Furthermore, any Figures and images, that are not mine, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Farhangian University, Tehran, Iran.

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التكرار في فضاء مؤثرات هلبرت-شميدت

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الخلاصة

في هذا البحث, تم برهان بانه اذا كانت شبه الزمرة-C0 فوضوية وزائدة او فائقة الاختلاط فان الضرب من جهة اليسار المرتبط بشبه الزمرة-C0 في فضاء مؤثرات هلبرت-شميدت متكرر اذا وفقط اذا كان مفرط الدوران. كذلك, تم بيان انه في ظل بعض الشروط يكون تكرار شبه الزمرة-C0 وتكرار الضرب لشبه الزمرة-C0 من جهة اليسار المرتبط بها في فضاء مؤثرات هلبرت-شميدت متكافئين. علاوة على ذلك, بعض الشروط الكافية للتكرار وزيادة الدوران للضرب من جهة اليسار لشبه الزمرة -C0 تم عرضها بالاعتماد على المجموعات الجزئية المتشعبة.

الكلمات المفتاحية: مؤثرات هلبرت- شمدت، مؤثرمفرط الدوران، الضرب من جهة اليسار، مؤثرالتكرار، شبه الزمرة المكررة .