2024, 21(7): 2378-2394
https://doi.org/10.21123/bsj.2023.8945
P-ISSN: 2078-8665 - E-ISSN: 2411-7986

Baghdad Science Journal

Hopf Bifurcation of Three-Dimensional Quadratic Jerk System

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Received 16/04/2023, Revised 15/07/2023, Accepted 17/07/2023, Published Online First 25/12/2023, Published 1/7/2024

Abstract

This paper is devoted to investigating the Hopf bifurcation of a three-dimensional quadratic jerk system. The stability of the singular points, the appearance of the Hopf bifurcation and the limit cycles of the system are studied. Additionally, the Liapunov quantities technique is used to study the cyclicity of the system and find how many limit cycles can be bifurcated from the Hopf points. Due to the computational load required for computing Liapunov quantities, some parameters are fixed. Currently, the analysis shows that three limit cycles can be bifurcated from the Hopf points. The results presented in this study are verified using MAPLE program.

Keywords: Hopf bifurcation, Jerk system, Limit cycle, Liapunov quantities, Stability.

Introduction

Consider the following differential systems

$$\dot{X} = Ax + F(X,\mu) \qquad 1$$

where $X \in \mathbb{R}^3$ and $\mu \in \mathbb{R}^3$ represent the phase variables and the parameters, respectively, and F is an analytic function. Let's assume that the system above has an isolated singular point at the origin and the Jacobian matrix of the system (Eq 1) at that point has a simple pair of pure imaginary eigenvalues. When the third eigenvalue has a non-zero real part, the origin is called the Hopf point. However, it is called a Zero Hopf point when the third eigenvalue is zero.

In classical mechanics, a jerk equation is a differential equation of the form

$$\dddot{x} = f(x, \dot{x}, \ddot{x}) \qquad 2$$

where x , $\dot{x}$, $\ddot{x}$ and $\dddot{x}$ represent the displacement, velocity, acceleration, and jerk, respectively. It is appropriate to express Eq 2 in system notation by introducing two additional phase variables as

$$y = \dot{x} \ \ and \ \ z = \dot{y} = \ddot{x} \qquad 3$$

Therefore, the status variables x, y, z provide a mechanical meaning of displacement, velocity, and acceleration, respectively[1]. Therefore, Eq 2 can be transformed into the following jerk system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = f(x,y,z). \end{cases}$$

Many chaotic jerk systems can be found with various forms of $f(x,y,z)$.

Let us consider the general quadratic jerk system

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$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 xy + a_6 xz + a_7 y^2 + a_8 yz + a_9 z^2 \end{cases} \quad 4$$

where a_i for $i = 0, \cdots, 9$ are real parameters. Molaie et al.[2] found twenty-three simple chaotic flows of Eq. 4 and they showed that a single stable singular point exists when $a_4 = 0$. Wei et al. [3] showed that Eq. 4 when $\Delta = a_1^2 - 4a_4 a_0 = 0$; $a_4 \neq 0$ does not have classical Hopf bifurcation and has found coexisting attractors with a single non-hyperbolic singular point. However, they proved that an unstable periodic orbit can be bifurcated from the zero-Hopf singular point. It was proved that the solution and its derivatives are bounded for the special case when $a_4 = a_5 = a_6 = 0$ [4]. Sang and Huang [5] studied some types of bifurcations for Eq. 4 such as transcritical, and saddle-note, and also showed that at most two periodic solutions bifurcate from a zero-Hopf singular point [5]. Some examples of zero-Hopf bifurcation analysis can be found in[6-10]. Salih et al.[11] studied bifurcated periodic orbits from the centre for the general quadratic jerk system (Eq 4) when $a_0 = 0$. The inverse Jacobi multiplier was used to obtain sufficient conditions for the existence of a centre. They showed that three limit cycles can be bifurcated from the origin under two sets of conditions and four limit cycles can bifurcate under the other sets of conditions.

Sang[12] presented a chaotic jerk system for the Genesio-Tesi system and they carried out a substantial simplification for the classical Hopf bifurcation formula. The results of that it can be extended to multi-dimensional quadratic systems. The dynamics of the system are investigated by using bifurcation diagrams, Liapunov exponents, and Poincare maps. They exhibited that the system can generate chaos using a Hopf bifurcation and period doubling cascade as the control parameter change. Liu et al.[13] investigated the general quadratic jerk system when $a_0 = a_4 = a_6 = a_7 = a_8 = 0$. In their work, the chaotic attractors were studied by using Hopf bifurcation analysis, bifurcation diagrams, Liapunov exponents, and cross sections. They found self-exited and hidden chaotic attractors by using Hopf bifurcation and period-doubling cascades. The authors in [14] investigated an ecological model that includes the Allee effect and intra-specific predator competition. They studied the equilibrium points and their stability with the help of the prey and predator nullcline. In their work, they established the conditions for encouraging Hopf bifurcation around the equilibrium points. Furthermore, they studied the nature of Hopf bifurcation around the equilibrium point. Finally, they performed a thorough numerical simulation to confirm their analytical results.

This paper is organized as follows: Firstly, the singular points of the system described in Eq 4 are identified, and their stability is also studied. Then, conditions for the occurrence of Hopf bifurcations are examined. Additionally, to investigate the cyclicity of the system, the Liapunov quantities technique is applied to obtain the number of limit cycles that can be bifurcated from the Hopf points. A numerical example is used to confirm the analytical results. Finally, the conclusions are presented at the end of the paper.

Singular Points with Their Stability

In this section, the singular points with their stability are studied.

Proposition 1: Consider the three-dimensional quadratic Jerk system described in Eq 4.

Let $\Delta = a_1^2 - 4\, a_4\, a_0$.

1. If $\Delta < 0$, then the system has no any singular point.

2. When $\Delta > 0$ and

(a) $a_4 \neq 0$, then the system has two single singular points, $E_{\pm} = \left(\frac{-a \pm \sqrt{\Delta}}{2\, a_4}, 0, 0\right)$.

(b) $a_4 = 0$ (in this case $a_1 \neq 0$), then the system has a single singular point $\left(-\frac{a_0}{a_1}, 0, 0\right)$.

3. When $\Delta = 0$ and

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(a) $a_4 \neq 0$, then the system has a single singular point.

(b) $a_4 = 0$ (in this case $a_1 = 0$), provided that $a_0 \neq 0$, then the system has no any singular point.

(c) $a_4 = 0$ (in this case $a_1 = 0$) and $a_0 = 0$, then the system has a line of singularity which is located on the x- axis.

Proof: In order to find the singular points of Eq 4, the right-hand sides of these equations are set to zeros:

$$y = 0,$$
$$z = 0,$$
$$a_0 + a_1 x + a_2 y + a_3 y + a_4 x^2 + a_5 xy + a_6 xz + a_7 y^2 + a_8 yz + a_9 z^2 = 0. \qquad 5$$

By substituting the values of $y = 0, z = 0$ into Eq.5, it becomes

$$a_4 x^2 + a_1 x + a_0 = 0, \qquad 6$$

Its roots are $x = \frac{-a \pm \sqrt{\Delta}}{2\,a_4}$, where $\Delta = a_1^2 - 4a_4\, a_0$. Let's assume that the real root $x = x_0$ of Eq. 6 is the singular point $(x_0 , 0, 0)$ of the system, which is located on $x -$ axis.

1. When $\Delta < 0$, Eq 6 has no any real root. Thus, the system does not have any real roots, this indicates that the system has no any singular point.

2.a) When $\Delta > 0$ and $a_4 \neq 0$, the system will have two singular points at $\left(x = \frac{-a \pm \sqrt{\Delta}}{2\,a_4}, y = 0, z = 0\right)$.

(b) If $\Delta > 0, a_4 = 0$ and $a_1 \neq 0$, the system will have a single singular point at $\left(x = -\frac{a_0}{a_1}\right.$, $\left. y = 0,\ z = 0\right), a_1 \neq 0.$

3. When $\Delta = 0$

a) If $a_4 \neq 0$, the system will have a single singular point $\left(x = -\frac{a_1}{2a_4}, y = 0, z = 0\right)$.

b) If $a_4 = a_1 = 0$ and $a_0 \neq 0$, the system has no any singular point.

c) If $a_4 = a_1 = a_0 = 0$, the system will have a line of singularity at $(x = x, y = 0, z = 0)$. ■

The stability of singular points of Eq 4 are determined by the following proposition.

Proposition 2: For the quadratic Jerk system described in Eq 4 with $\Delta = a_1^2 - 4a_4\, a_0$.

1. If $\Delta > 0$ and $a_4 \neq 0$, then the singular point E_+ is always unstable and E_- is asymptotically stable if and only if the following conditions hold $\frac{-\sqrt{\Delta}\, a_6 - a_6\, a_1 + 2\, a_3\, a_4}{2\, a_4} < 0$ and $(-\sqrt{\Delta}\, a_6 - a_6\, a_1 + 2\, a_3\, a_4)(-\sqrt{\Delta}\, a_5 - a_5\, a_1 + 2\, a_2\, a_4) > 4\, a_4^2 \sqrt{\Delta}$.

2. If $\Delta > 0$ and $a_4 = 0$, then the singular point $\left(-\frac{a_0}{a_1}, 0, 0\right)$, $a_1 \neq 0$ is unstable when $a_1 > 0$ and it is asymptotically stable if and only if $a_1 < 0, a_1 a_3 > a_0 a_6$ and $(a_0\, a_5 - a_2\, a_1)(a_0\, a_6 - a_3\, a_1) > -a_1^3$.

3. If $\Delta = 0$ $\left(a_0 = \frac{a_1^2}{4a_4}\right)$ and $a_4 \neq 0$, then the singular point $\left(-\frac{a_1}{2a_4}, 0, 0\right)$ is non-hyperbolic and is unstable if one of the following conditions is satisfied:
   I. $\left(a_2 > \frac{a_5 a_1}{2a_4}\right)$, or $\left(a_2 < \frac{a_5\, a_1}{2a_4}\right.$ and $\left. a_3 > \frac{a_6 a_1}{2a_4}\right)$.
   II. $a_2 < \frac{a_5 a_1}{2a_4}$ and $a_3 < \frac{a_6 a_1}{2a_4}$.

Proof: The Jacobian matrix of the system at $(x^*, 0, 0)$ is given by

$$J(x^*) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2a_4\, x^* + a_1 & a_5\, x^* + a_2 & a_6\, x^* + a_3 \end{bmatrix},$$

and its characteristic equation is

$$\lambda^3 - (a_6\, x^* + a_3)\, \lambda^2 - (a_5\, x^* + a_2)\, \lambda - (2a_4\, x^* + a_1) = 0, \qquad 7$$

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1. When $x^* = \frac{-a_1 \pm \sqrt{\Delta}}{2a_4}$, the characteristic Eq 7 is

$$\lambda^3 - \frac{\pm\sqrt{\Delta}\ a_6 - a_6 a_1 + 2a_3\ a_4}{2a_4}\lambda^2 - \frac{\pm\sqrt{\Delta}\ a_5 - a_5 a_1 + 2a_2\ a_4}{2a_4}\lambda - (\pm\sqrt{\Delta}) = 0, \qquad 8$$

At E_+, the determinant of the Jacobian matrix is $\sqrt{\Delta} > 0$, then at least one of the eigenvalues is positive. Then, the singular point E_+ is unstable. At E_- , from Eq 8, $T = \frac{-\sqrt{\Delta}\ a_6 - a_6 a_1 + 2a_3\ a_4}{2a_4}$, $K = \frac{-\sqrt{\Delta}\ a_5 - a_5 a_1 + 2a_2\ a_4}{2a_4}$ and $D = -\sqrt{\Delta}$. By the Routh -Hurwitz stability criterion, the singular point E_- asymptotically stable if and only if $\frac{-\sqrt{\Delta}\ a_6 - a_6 a_1 + 2a_3\ a_4}{2a_4} < 0$ and $\left(-\sqrt{\Delta}\ a_6 - a_6 a_1 + 2a_3\ a_4\right)\left(-\sqrt{\Delta}\ a_5 - a_5 a_1 + 2a_2\ a_4\right) > 4a_4^2\sqrt{\Delta}$.

2. When $a_4 = 0$ and $x^* = -\frac{a_0}{a_1}$, the characteristic Eq 7 is

$$\lambda^3 + \frac{a_6 a_0 - a_3 a_1}{a_1}\lambda^2 + \frac{a_5 a_0 - a_2 a_1}{a_1}\lambda - a_1 = 0. \qquad 9$$

From Eq 9 the determinant of the Jacobian matrix is a_1. When $a_1 > 0$, then at least one of the eigenvalues is positive. Therefore, the singular point $(-\frac{a_0}{a_1}, 0, 0)$ is unstable. However, by the Routh-Hurwitz stability criterion, the singular point is asymptotically stable if and only if $a_1 < 0$, $a_1 a_3 > a_0 a_6$ and $(a_0 a_5 - a_2 a_1)(a_0 a_6 - a_3 a_1) > -a_1^3$.

3. When $a_0 = \frac{a_1^2}{4a_4}$ and $x^* = -\frac{a_1}{2a_4}$, $a_4 \neq 0$, the characteristic Eq 7 can be written in the form:

$$\lambda\left(\lambda^2 - \frac{2a_3 a_4 - a_6\ a_1}{2a_4}\lambda + \frac{a_5 a_1 - 2a_2\ a_4}{2a_4}\right) = 0. \qquad 10$$

From Eq 10,

I. If $\left(a_2 > \frac{a_5 a_1}{2a_4}\right)$ or $\left(a_2 < \frac{a_5\ a_1}{2a_4}\right.$ and $\left.a_3 > \frac{a_6 a_1}{2a_4}\right)$, then at least one of the roots is positive or has a positive real part. Thus, the singular point is unstable.

II. II. If $a_2 < \frac{a_5 a_1}{2a_4}$ and $a_3 < \frac{a_6 a_1}{2a_4}$, then the two non-zero roots are negative or have negative real parts. Therefore, the complete stability of the singular point depends on the behavior of the dynamics on the center manifold.

By scaling $x \longrightarrow x - \frac{a_1}{2a_4}$ and the following transformation,

$$\begin{cases} x = y_1 + y_2 + y_3\,, \\ y = \dfrac{-a_1 a_6 + 2a_3 a_4 + \sqrt{\Delta_1}}{4a_4} y_2 - \dfrac{a_1 a_6 - 2a_3 a_4 + \sqrt{\Delta_1}}{4a_4} y_3, \\ z = \dfrac{\Delta_1 - (a_1 a_6 - 2a_3 a_4)\sqrt{\Delta_1} + 4a_1 a_4 a_5 - 8a_2 a_4^2}{8a_4^2} y_2 + \dfrac{\Delta_1 + (a_1 a_6 - 2a_3 a_4)\sqrt{\Delta_1} + 4a_1 a_4 a_5 - 8a_2 a_4^2}{8a_4^2} y_3, \end{cases}$$

where $\Delta_1 = a_1^2 a_6^2 - 4\,a_1 a_3 a_4 a_6 + 4a_3^2\,a_4^2 - 8\,a_1 a_4\,a_5 + 16\,a_2 a_4^2$, Eq 4 is transformed to the following canonical form

$$\begin{aligned} \frac{dy_1}{dt} &= O(2), \\ \frac{dy_2}{dt} &= \frac{-a_1 a_6 + 2a_3 a_4 + \sqrt{\Delta_1}}{4a_4} y_2 + O(2), \\ \frac{dy_3}{dt} &= \frac{-a_1 a_6 + 2a_3 a_4 - \sqrt{\Delta_1}}{4a_4} y_3 + O(2). \end{aligned} \qquad 11$$

The one-dimensional local center manifold of that system near the origin can be expressed as the following set:

$W^c = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_2 = h_1(y_1), y_3 = h_2(y_1), |y_1| < 1$, with $h_1(0) = Dh_1(0) = h_2(0) = Dh_2(0) = 0\}$. Then $h_1(y_1) = \alpha_1\,y_1^2 + O(3)$ and $h_2(y_2) = \alpha_2\,y_1^2 + O(3)$ where

$$\alpha_1 = \frac{4a_4^3\left(a_6a_1-2a_3\,a_4+\sqrt{\Delta_1}\right)}{\left((-a_6a_1+2a_3\,a_4)\sqrt{\Delta_1}+(4a_3^2+16a_2)a_4^2-4a_1(a_3a_6+2a_5)a_4+a_1^2a_6^2\right)(a_5a_1-2a_2a_4)},$$

$$\alpha_2 = \frac{4a_4^3\left(a_6a_1-2a_3\,a_4-\sqrt{\Delta_1}\right)}{\left((a_6a_1-2a_3\,a_4)\sqrt{\Delta_1}+(4a_3^2+16a_2)a_4^2-4a_1(a_3a_6+2a_5)a_4+a_1^2a_6^2\right)(a_5a_1-2a_2a_4)}.$$

The vector field is restricted to the center manifold which is given by the following equation

$$\frac{dy_1}{dt} = \frac{2a_4^2}{a_5a_1-2a_2\,a_4}\;y_1^2 + O(3). \qquad 12$$

Since $\frac{2a_4^2}{a_5a_1-2a_2\,a_4} > 0$ and $y_1 = 0$ is unstable for Eq.12. Therefore, the singular point $\left(-\frac{a_1}{2a_4},0,0\right)$ is unstable. ■

Hopf Bifurcation

It is well known that Hopf bifurcation is a common phenomenon associated with the appearance or disappearance of a limit cycle around the singular point. It can be investigated by using some techniques: such as bifurcation formulas[15], Liapunov quantities[16], and focus quantities[17, 18]. The appearance or disappearance of Hopf bifurcation for the Prey-Predator Model and SIR Epidemic Model has been studied [19, 20].

In order to analyze Hopf bifurcation of Eq 1, some conditions are needed. The Hopf bifurcation occurs when the Jacobian matrix of the system at the origin has a simple pair of pure imaginary eigenvalues, $\lambda_{1,2} = \pm\omega$, with a non-zero real eigenvalue. For Eq 1, a sufficient condition for occurring a Hopf bifurcation is explained. Assume that

$$P(\lambda) = \lambda^3 - T\lambda^2 - K\lambda - D \qquad 13$$

be the characteristic polynomial at the origin, where $T, K,$ and D are the trace, sum of diagonal minors, and determinant of the Jacobian matrix of the system decribed in Eq 1 at the origin, respectively. Then, the Hopf bifurcation occurs at the origin if and only if

$$TK + D = 0, K < 0 \text{ and } T \neq 0. \qquad 14$$

Here, the Liapunov quantity technique is used to evaluate cyclicity in the three-dimensional system as follows. Let us introduce a function of the form

$$F(x,y,z) = x^2 + y^2 + \sum_{k=3}^{\infty} F_k(x,y,z;\mu), \qquad 15$$

where F_k is a polynomial in x, y and z of degree k and the coefficients of F_k satisfy

$$\chi(F) = \eta_2 r^2 + \eta_4 r^4 + \eta_6 r^6 + \cdots + \eta_{2i} r^{2i} + \ldots, \qquad 16$$

where $r^2 = x^2 + y^2$ or x^2 or y^2 or $(x^2 + y^2)^2$ or other suitable forms and χ is the vector field of the canonical form of the system. The $\eta_{2i}, i = 1, 2, \cdots$ are polynomials in the parameter μ of the system and are called i^{th} the Liapunov quantity or focal values. The origin is said to be a fine focus of order k if $\eta_2 = \eta_4 = \cdots = \eta_{2k} = 0$ but $\eta_{2k+1} \neq 0$. In this case, at most k limit cycles can be bifurcated from the origin under perturbation and its exact number is obtained by the independence of the Liapunov quantities. For more detail about this technique[21-25].

Proposition 3: The characteristic equation corresponding to the linearization of Eq 4 at the singular point $E_+ = \left(\frac{-a_1+\sqrt{\Delta}}{2a_4}, 0, 0\right)$ (for $E_- = \left(\frac{-a_1-\sqrt{\Delta}}{2a_4}, 0, 0\right)$ is similar by changing the sign of $\sqrt{\Delta}$) where $\Delta = a_1^2 - 4a_4a_0 > 0$ and $a_4 \neq 0$ has two pure imaginary with non-zero solutions if and only if the following conditions are satisfied, in this case, the solutions are $\lambda_1 = \frac{\sqrt{\Delta}}{\omega^2}$, $\lambda_{1,2} = \pm i\omega$ where $\omega = \sqrt{\frac{-a_5\sqrt{\Delta}+a_1a_5-2a_2a_4}{2a_4}}$:

i. $a_6\left(a_1 - \sqrt{\Delta}\right) \neq 2a_3a_4$,

ii. $\frac{a_5(\sqrt{\Delta}-a_1)+2a_2a_4}{2a_4} < 0$,

iii. $(4a_4^2 + (2a_2\, a_6 + 2a_5\, a_3)a_4 - 2a_1\, a_5 a_6)\sqrt{\Delta} + 4a_2\, a_3\, a_4^2 - 2\, a_1(a_2\, a_6 + a_5 a_3)\, a_4 + a_5\, a_6\, (a_1^2 + \Delta) = 0.$

Proof: By scaling $x \longrightarrow x + \frac{-a_1+\sqrt{\Delta}}{2a_4}$, the singular point E_+ is moved to the origin. The characteristic equation of the Jacobian matrix of system (Eq 4) at the origin is given by

$$\lambda^3 - \frac{(a_6\sqrt{\Delta} - a_6 a_1 + 2a_3\, a_4)}{2a_4}\lambda^2 - \frac{(a_5\sqrt{\Delta} - a_5 a_1 + 2a_2\, a_4)}{2a_4} - \sqrt{\Delta} = 0. \qquad 17$$

By comparing the above equation with Eq 13, the following values of T, K, and D are found:

$$T = \frac{(a_6\sqrt{\Delta} - a_6 a_1 + 2a_3\, a_4)}{2a_4},$$

$$K = \frac{(a_5\sqrt{\Delta} - a_5 a_1 + 2a_2\, a_4)}{2a_4}, \qquad D = \sqrt{\Delta}.$$

Since the polynomial defined in Eq 17 has two pure imagines with non-zero solutions (which is called a Hopf point) if and only if its coefficients satisfy conditions (14). Then, when

i. $T \neq 0$ implies that $a_6\left(a_1 - \sqrt{\Delta}\right) \neq 2a_3 a_4$,
ii. $K < 0$ implies that $\frac{a_5(\sqrt{\Delta}-a_1)+2a_2 a_4}{2a_4} < 0$,
iii. $TK + D = 0$ implies that $(4a_4^2 + (2a_2\, a_6 + 2a_5\, a_3)a_4 - 2a_1\, a_5 a_6)\sqrt{\Delta} + 4a_2\, a_3\, a_4^2 - 2\, a_1(a_2\, a_6 + a_5 a_3)\, a_4 + a_5\, a_6\, (a_1^2 + \Delta) = 0.$

This completes the proof. ■

Proposition 4: Assume that $\Delta > 0$ and $a_4 = 0$, the characteristic equation corresponding to the linearization of the system (Eq 4) at the singular point $\left(-\frac{a_0}{a_1}, 0, 0\right)$; $a_1 \neq 0$ has two pure imaginaries with non-zero solutions if and only if the following conditions are satisfied, in this case, the solutions are $\lambda_1 = \frac{a_1}{\omega^2}$, $\lambda_{2,3} = \pm i\,\omega$ where $\omega = \sqrt{\frac{a_5 a_0 - a_1 a_2}{a_1}}$:

i. $a_6\, a_0 \neq a_1\, a_3$,
ii. $\frac{a_5 a_0 - a_1 a_2}{a_1} > 0$,
iii. $a_0^2\, a_5\, a_6 - a_0 a_1\, a_6 a_2 - a_0 a_1 a_3\, a_5 + a_1^2\, a_2\, a_3 + a_1^3 = 0.$

Proof: By applying the translation $x \longrightarrow x - \frac{a_0}{a_1}$, the singular point is moved to the origin. The characteristic equation of the Jacobian matrix of system (Eq 4) at the origin is given by

$$\lambda^3 + \frac{a_6\, a_0 - a_3\, a_1}{a_1}\lambda^2 + \frac{a_5\, a_0 - a_2 a_1}{a_1}\lambda - a_1 = 0. \qquad 18$$

By comparing Eq 18 with Eq 13, the following values of T, K, and D are obtained:

$$T = \frac{-a_6\, a_0 + a_3\, a_1}{a_1},\ K = -\frac{a_5\, a_0 - a_2 a_1}{a_1},\ D = a_1. \qquad 19$$

Since the characteristic Eq 18 has two pure imagines with non-zero solutions (which is called a Hopf point) if and only if its coefficients satisfy conditions (14). Then, when

i. $T \neq 0$ implies that $a_6\, a_0 \neq a_1\, a_3$,
ii. $K < 0$ implies that $\frac{a_5 a_0 - a_1 a_2}{a_1} < 0$,
iii. $TK + D = 0$ implies that $a_0^2\, a_5\, a_6 - a_0 a_1\, a_6 a_2 - a_0 a_1 a_3\, a_5 + a_1^2\, a_2\, a_3 + a_1^3 = 0.$

This completes the proof. ■

In general, the cyclicity bifurcated from the above Hopf points for the three-dimensional quadratic jerk system (Eq 4) cannot be investigated due to the limited capacity of computers. Therefore, in the next theorems, some special cases are given.

Theorem 1: For system (Eq 4) when $\Delta > 0$ and $a_4 \neq 0$, three limit cycles can be bifurcated from the singular point at $(\frac{-a_1+\sqrt{\Delta}}{2a_4}, 0, 0)$ when the parameters satisfy the conditions of Proposition 3 with

i. $\Omega \neq 0$, which is defined in Eq 27.
ii. $\eta_4\,|_{\{b_2=b_2^*, b_6=b_6^*\}} \neq 0$.

Proof: Under the linear transformation $x \longrightarrow x + \frac{-a_1+\sqrt{\Delta}}{2a_4}$, the singular point is moved to the origin, and system (Eq 4) becomes

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https://doi.org/10.21123/bsj.2023.8945
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = a_0 + a_1\left(x + \frac{-a_1+\sqrt{\Delta}}{2a_4}\right) + a_2\, y + a_3\, z + a_4\left(x + \frac{-a_1+\sqrt{\Delta}}{2a_4}\right)^2 + a_5\left(x + \frac{-a_1+\sqrt{\Delta}}{2a_4}\right)y \\ \qquad + a_6\left(x + \frac{-a_1+\sqrt{\Delta}}{2a_4}\right)z + a_7\, y^2 + a_8 yz + a_9\, z^2. \end{cases} \quad 20$$

The Jacobian matrix of system (Eq 20) at the origin is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sqrt{\Delta} & \frac{a_5(-a_1+\sqrt{\Delta})}{2a_4} + a_2 & \frac{a_6(-a_1+\sqrt{\Delta})}{2a_4} + a_3 \end{bmatrix},$$

Its characteristic equation at the origin is given by

$$f(\lambda, a_3) = \lambda^3 - \frac{(a_6\sqrt{\Delta} - a_6 a_1 + 2a_3\, a_4)}{2a_4}\lambda^2 - \frac{(a_5\sqrt{\Delta} - a_5 a_1 + 2a_2\, a_4)}{2a_4} - \sqrt{\Delta} = 0. \quad 21$$

The conditions of Proposition 3 leads to

$$a_3 = a_3^* = -\frac{\Delta\, a_5\, a_6 - 2\sqrt{\Delta}\, a_1 a_5\, a_6 + 2\sqrt{\Delta}\, a_2 a_4\, a_6 + a_1^2\, a_5 a_6 - 2a_1 a_2\, a_4 a_6 + 4\sqrt{\Delta}\, a_4^2}{2a_4\left(a_5\sqrt{\Delta} - a_5\, a_1 + 2a_2 a_4\right)}$$

and $a_2 = a_2^* = -\frac{2\omega^2\, a_4 + a_5\sqrt{\Delta} - a_5 a_1}{2a_4}$ where $\omega = \sqrt{\frac{-a_5\sqrt{\Delta} + a_5\, a_1 - 2a_2\, a_4}{2a_4}}$

and the characteristic Eq 21 has a non-zero eigenvalue $\frac{\sqrt{\Delta}}{\omega^2}$ with a pair of purely imaginary eigenvalues $\pm i\omega$.

In order to verify the transversality condition, by using the implicit function theorem, the derivative of the complex eigenvalue $\lambda(a_3)$ with respect to a_3 can be found for the singular point as follows:

$$\frac{d\lambda}{da_3} = \frac{-\partial f/\partial a_3}{\partial f/\partial \lambda} = \frac{\lambda^2}{3\lambda^2 - \frac{(a_6\sqrt{\Delta} - a_6 a_1 + 2a_3\, a_4)\lambda}{a_4} - \frac{(a_5\sqrt{\Delta} - a_5 a_1 + 2a_2\, a_4)}{2a_4}}. \quad 22$$

Substituting $a_3 = a_3^*$ and $\lambda = i\omega$ into Eq 22, to obtain

$$\frac{dRe(\lambda_{1,2})}{da_3}\Big|_{\{a_3 = a_3^*, a_2 = a_2^*, \lambda = i\omega\}} = \frac{\omega^6}{2(\omega^6 + \Delta)} > 0\,. \quad 23$$

which implies that the transversality condition is satisfied. Thus, Hopf bifurcation occurs at $a_3 = a_3^*$, $a_2 = a_2^*$. By using the transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -\omega & \frac{\sqrt{\Delta}}{\omega^2} \\ -\omega^2 & 0 & \frac{\Delta}{\omega^4} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad 24$$

and the new system can be written as

$$\begin{cases} \frac{dy_1}{dt} = -\omega\, y_2 + b_1\, y_1^2 + b_2\, y_1\, y_2 + b_3\, y_1\, y_3 + b_4\, y_2^2 + b_5\, y_2 y_3 + b_6 y_3^2, \\ \frac{dy_2}{dt} = \omega\, y_1 - \frac{\sqrt{\Delta}}{\omega^3}\left(b_1\, y_1^2 + b_2\, y_1\, y_2 + b_3\, y_1\, y_3 + b_4\, y_2^2 + b_5\, y_2 y_3 + b_6 y_3^2\right), \\ \frac{dy_3}{dt} = \frac{\sqrt{\Delta}}{\omega^2}\, y_3 - \left(b_1\, y_1^2 + b_2\, y_1\, y_2 + b_3\, y_1\, y_3 + b_4\, y_2^2 + b_5\, y_2 y_3 + b_6 y_3^2\right), \end{cases} \quad 25$$

where

$b_1 = -\frac{\omega^4(\omega^4\, a_9 - \omega^2\, a_6 + a_4)}{\omega^6 - 4a_4\, a_0 + a_1^2}$, $b_2 = -\frac{\omega^5(\omega^2 a_8 - a_5)}{\omega^6 - 4a_4\, a_0 + a_1^2}$,

$b_3 = \frac{\omega^6 a_6 + \sqrt{\Delta}\,\omega^4 a_8 - 2\omega^4 a_4 - 8\,\omega^2 a_0\, a_4 a_9 + 2\omega^2 a_1^2 a_9 - \sqrt{\Delta}\,\omega^2\, a_5 + 4a_4 a_0 a_6 - a_1^2 a_6}{\omega^6 - 4a_4\, a_0 + a_1^2}$,

$b_4 = -\frac{\omega^6\, a_7}{\omega^6 - 4a_4\, a_0 + a_1^2}$,

$b_5 = \frac{\omega\,(a_5\,\omega^4 + 2\omega^2\, a_7\sqrt{\Delta} - 4\, a_8 a_4\, a_0 + a_8 a_1^2)}{\omega^6 - 4a_4\, a_0 + a_1^2}$ and

$b_6 = -\frac{1}{\omega^4(\omega^6 - 4a_4\, a_0 + a_1^2)}\left(a_4\omega^8 + \sqrt{\Delta}\,\omega^6\, a_5 - 4\,\omega^4 a_0 a_4 a_6 - 4\,\omega^4 a_4\, a_0\, a_7 + \omega^4 a_1^2\, a_6 + \omega^4 a_1^2\, a_7 - 4\sqrt{\Delta}\,\omega^2 a_0 a_4 a_8 + \sqrt{\Delta}\,\omega^2\, a_1^2 a_8 + 16\, a_9\, a_4^2\, a_0^2 - 8\, a_9 a_4 a_0 a_1^2 + a_9 a_1^4\right)$

2024, 21(7): 2378-2394
https://doi.org/10.21123/bsj.2023.8945
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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Now, due to the computational load for calculating Liapunov quantities, it cannot be obtained them, more powerful computing devices are required. Thus, the parameter a_7 is fixed and vanished to make the calculation simpler, in this case, the value of b_4 is also disappeared. For investigating the number of limit cycles bifurcated from the singular point, the following Liapunov function is defined:

$$F(y_1, y_2, y_3) = y_1^2 + y_2^2 + \sum_{k=3}^{n}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{k-j,j-i,i}\, y_1^{k-j} y_2^{j-i}\, y_3^{i}\,.$$

which is satisfying the following equation

$$\chi(F) = \eta_1\,(y_1^2 + y_2^2) + \eta_2\,(y_1^2 + y_2^2)^2 + \cdots \qquad 26$$

where χ is the vector field of the system (Eq 25). By using the computer algebra package MAPLE, and solving Eq 26, the following linearly independent terms of Liapunov quantities are obtained:

1. $\eta_1 = 0,$
2. $\eta_2 = \frac{1}{4\omega^7(4\omega^6+\Delta)\sqrt{\Delta}}\Big(8\omega^{15} b_1 b_3 + 4\sqrt{\Delta}\,\omega^{12} b_1 b_2 - 10\sqrt{\Delta}\,\omega^{12} b_1 b_5 + 2\sqrt{\Delta}\,\omega^{12} b_2 b_3 + 8\Delta\omega^9 b_1^2 + 5\Delta\omega^9 b_1 b_3 + 3\Delta\omega^9 b_2 b_5 - 3\Delta^{\frac{3}{2}}\omega^6 b_1 b_2 - \Delta^{\frac{3}{2}}\omega^6 b_1 b_5 - \Delta^{\frac{3}{2}}\omega^6 b_2 b_3 + 2\Delta^2\omega^3 b_1^2 - \Delta^{\frac{5}{2}} b_1 b_2\Big),$
3. $\eta_3 = \frac{G_1}{96\omega^{15}(9\omega^6+\Delta)(\omega^6+\Delta)(4\omega^6+\Delta)^2\Delta^{\frac{3}{2}}}$
   where G_1 is a function of $b_1, b_2, b_3, b_5, b_6, \omega$ and Δ
4. $\eta_4 \frac{G_2}{9216\omega^{23}(\omega^6+\Delta)^2(4\omega^6+\Delta)^4(16\omega^6+\Delta)(9\omega^6+\Delta)^2(\omega^6+4\Delta)\Delta^{\frac{5}{2}}}$

   where G_2 is a function of $b_1, b_2, b_3, b_5, b_6, \omega$ and Δ.

The origin is a weak focus of order three for system (Eq 25) if and only if the following conditions are satisfied.

1. $b_2 = b_2^* = -\frac{\omega^3 b_1\left(8\omega^{12} b_3 - 10\sqrt{\Delta}\,\omega^9 b_5 + 8\Delta\omega^6 b_1 + 5\Delta\omega^6 b_3 - \Delta^{\frac{3}{2}}\omega^3 b_5 + 2\Delta^2 b_1\right)}{4\sqrt{\Delta}\,\omega^{12} b_1 + 2\sqrt{\Delta}\,\omega^{12} b_3 + 3\Delta\omega^9 b_5 - 3\Delta^{\frac{3}{2}}\omega^6 b_1 - \Delta^{\frac{3}{2}}\omega^6 b_3 - \Delta^{\frac{5}{2}} b_1},$
2. $b_6 = b_6^* = -\frac{G_3}{G4}$, where

$$\begin{aligned}
G_3 = {} & -13824\,\triangle^2\,\omega^{66} b_3^5 b_5 - 5760\Delta^3\omega^{60} b_3^5 b_5 - 768\Delta^3\omega^{60} b_3^3 b_5^3 + 28272\,\triangle^4\,\omega^{54} b_3^5 b_5 - 25920\,\triangle^4\,\omega^{54} b_3^3 b_5^3 \\
& -7056\Delta^4\omega^{54} b_3 b_5^5 + 17700\Delta^5\omega^{48} b_3^5 b_5 - 51336\Delta^5\omega^{48} b_3^3 b_5^3 - 12564\Delta^5\omega^{48} b_3 b_5^5 - 5658\Delta^6\omega^{42} b_3^5 b_5 \\
& -32082\Delta^6\omega^{42} b_3^3 b_5^3 - 5724\Delta^6\omega^{42} b_3 b_5^5 - 3534\Delta^7\omega^{36} b_3^5 b_5 - 6102\Delta^7\omega^{36} b_3^3 b_5^3 - 36\Delta^7\omega^{36} b_3 b_5^5 - 390\Delta^8 \\
& \omega^{30} b_3^5 b_5 - 198\Delta^8\omega^{30} b_3^3 b_5^3 + 180\Delta^8\omega^{30} b_3 b_5^5 + 27315\Delta^{\frac{11}{2}}\omega^{45} b_3^4 b_5^2 + 33450\Delta^{\frac{11}{2}}\omega^{45} b_3^2 b_5^4 + 14904\Delta^{\frac{13}{2}}\omega^{39} \\
& b_3^4 b_5^2 + 8367\,\triangle^{\frac{13}{2}}\,\omega^{39} b_3^2 b_5^4 + 2223\Delta^{\frac{15}{2}}\omega^{33} b_3^4 b_5^2 + 336\Delta^{\frac{15}{2}}\omega^{33} b_3^2 b_5^4 + 54\Delta^{\frac{17}{2}}\omega^{27} b_3^4 b_5^2 - 57\,\triangle^{\frac{17}{2}}\,\omega^{27} b_3^2 b_5^4 \\
& -6\Delta^9\omega^{24} b_3^5 b_5 + 6\Delta^9\omega^{24} b_3^3 b_5^3 - 29952\Delta^{\frac{5}{2}}\omega^{63} b_3^4 b_5^2 - 57888\Delta^{\frac{7}{2}}\omega^{57} b_3^4 b_5^2 + 14064\Delta^{\frac{7}{2}}\omega^{57} b_3^2 b_5^4 - 13356\Delta^{\frac{9}{2}} \\
& \omega^{51} b_3^4 b_5^2 + 39540\Delta^{\frac{9}{2}}\omega^{51} b_3^2 b_5^4 - 5\Delta^{13} b_1^4 b_3 b_5 + 10\Delta^{13} b_1^5 b_5 + 1836\Delta^{\frac{9}{2}}\omega^{51} b_5^6 + 1539\Delta^{\frac{11}{2}}\omega^{45} b_5^6 - 486\Delta^{\frac{13}{2}}\omega^{39} \\
& b_5^6 - 189\Delta^{\frac{15}{2}}\omega^{33} b_5^6 + 5\Delta^{11}\omega^{12} b_1^2 b_3 b_5^3 - 119\Delta^{12}\omega^6 b_1^4 b_3 b_5 - 21\Delta^{12}\omega^6 b_1^3 b_3^2 b_5 + 33792\Delta^{\frac{5}{2}}\omega^{63} b_1^3 b_3 b_5^2 \\
& -14400\Delta^{\frac{5}{2}}\omega^{63} b_1^2 b_3^2 b_5^2 - 34080\Delta^{\frac{5}{2}}\omega^{63} b_1 b_3^3 b_5^2 - 84320\Delta^{\frac{7}{2}}\omega^{57} b_1^3 b_3 b_5^2 + 84672\Delta^{\frac{7}{2}}\omega^{57} b_1^2 b_3^2 b_5^2 - 39376\Delta^{\frac{7}{2}}\omega^{57} \\
& b_1 b_3^3 b_5^2 - 39768\Delta^{\frac{7}{2}}\omega^{57} b_1 b_3 b_5^4 + 205392\Delta^{\frac{9}{2}}\omega^{51} b_1^3 b_3 b_5^2 + 451212\,\triangle^{\frac{9}{2}}\,\omega^{51} b_1^2 b_3^2 b_5^2 + 86334\Delta^{\frac{9}{2}}\omega^{51} b_1 b_3^3 b_5^2 \\
& -41598\Delta^{\frac{9}{2}}\omega^{51} b_1 b_3 b_5^4 + 827466\Delta^{\frac{11}{2}}\omega^{45} b_1^3 b_3 b_5^2 + 780438\Delta^{\frac{11}{2}}\omega^{45} b_1^2 b_3^2 b_5^2 + 212640\Delta^{\frac{11}{2}}\omega^{45} b_1 b_3^3 b_5^2 \\
& +33642\Delta^{\frac{11}{2}}\omega^{45} b_1 b_3 b_5^4 - 269790\Delta^6\omega^{42} b_1^3 b_3^2 b_5 - 121278\Delta^6\omega^{42} b_1^2 b_3^3 b_5 - 350082\Delta^6\omega^{42} b_1^2 b_3 b_5^3 - 38769\Delta^6
\end{aligned}$$

2024, 21(7): 2378-2394
https://doi.org/10.21123/bsj.2023.8945
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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$$\omega^{42}b_1b_3^4b_5-207267\Delta^6\omega^{42}b_1b_3^2b_5^3-442634\Delta^7\omega^{36}b_1^4b_3b_5-472261\Delta^7\omega^{36}b_1^3b_3^2b_5-219413\Delta^7\omega^{36}b_1^2b_3^3b_5$$

$$-213243\Delta^7\omega^{36}b_1^2b_3b_5^3-49714\Delta^7\omega^{36}b_1b_3^4b_5-79178\Delta^7\omega^{36}b_1b_3^2b_5^3-278155\Delta^8\omega^{30}b_1^4b_3b_5-279857\Delta^8$$

$$\omega^{30}b_1^3b_3^2b_5-100860\Delta^8\omega^{30}b_1^2b_3^3b_5-73616\Delta^8\omega^{30}b_1^2b_3b_5^3-13824\Delta^8\omega^{30}b_1b_3^4b_5-13056\Delta^8\omega^{30}b_1b_3^2b_5^3$$

$$+84480\Delta^2\omega^{66}b_1^3b_3^2b_5+21504\Delta^2\omega^{66}b_1^2b_3^3b_5+23424\Delta^2\omega^{66}b_1b_3^4b_5+143440\Delta^5\omega^{48}b_1^4b_3b_5+88352\Delta^5$$

$$\omega^{48}b_1^3b_3^2b_5+164172\Delta^5\omega^{48}b_1^2b_3^3b_5-398764\Delta^5\omega^{48}b_1^2b_3b_5^3+4608\Delta^{\frac{5}{2}}\omega^{63}b_3^6+6192\Delta^{\frac{7}{2}}\omega^{57}b_3^6-1764$$

$$\Delta^{\frac{9}{2}}\omega^{51}b_3^6-3384\Delta^{\frac{11}{2}}\omega^{45}b_3^6+279\Delta^{\frac{13}{2}}\omega^{39}b_3^6+342\Delta^{\frac{15}{2}}\omega^{33}b_3^6+27\Delta^{\frac{17}{2}}\omega^{27}b_3^6+20\Delta^{\frac{25}{2}}\omega^{3}b_1^4b_3^2+5133\Delta^{\frac{15}{2}}$$

$$\omega^{33}b_1b_3^5+32956\Delta^{\frac{17}{2}}\omega^{27}b_1^5b_3+40096\Delta^{\frac{17}{2}}\omega^{27}b_1^4b_3^2+38550\Delta^{\frac{17}{2}}\omega^{27}b_1^3b_3^3+12571\Delta^{\frac{17}{2}}\omega^{27}b_1^2b_3^4+1439\Delta^{\frac{17}{2}}$$

$$\omega^{27}b_1b_3^5+18752\Delta^{\frac{19}{2}}\omega^{21}b_1^5b_3+23160\Delta^{\frac{19}{2}}\omega^{21}b_1^4b_3^2+12640\Delta^{\frac{19}{2}}\omega^{21}b_1^3b_3^3+2339\Delta^{\frac{19}{2}}\omega^{21}b_1^2b_3^4+101\Delta^{\frac{19}{2}}$$

$$\omega^{21}b_1b_3^5+4528\Delta^{\frac{21}{2}}\omega^{15}b_1^5b_3+5396\Delta^{\frac{21}{2}}\omega^{15}b_1^4b_3^2+1776\Delta^{\frac{21}{2}}\omega^{15}b_1^3b_3^3+141\Delta^{\frac{21}{2}}\omega^{15}b_1^2b_3^4+500\Delta^{\frac{23}{2}}\omega^{9}b_1^5b_3$$

$$+554\Delta^{\frac{23}{2}}\omega^{9}b_1^4b_3^2+87\Delta^{\frac{23}{2}}\omega^{9}b_1^3b_3^3+20\Delta^{\frac{25}{2}}\omega^{3}b_1^5b_3-30720\Delta^{\frac{3}{2}}\omega^{69}b_1^3b_3^3+15360\Delta^{\frac{3}{2}}\omega^{69}b_1^2b_3^4-30720\Delta^{\frac{5}{2}}$$

$$\omega^{63}b_1^5b_3-63744\Delta^{\frac{5}{2}}\omega^{63}b_1^4b_3^2-107136\Delta^3\omega^{60}b_1^2b_3b_5^3+35232\Delta^3\omega^{60}b_1b_3^4b_5-137088\Delta^3\omega^{60}b_1b_3^2b_5^3$$

$$+191616\Delta^4\omega^{54}b_1^4b_3b_5+144832\Delta^4\omega^{54}b_1^3b_3^2b_5+195920\Delta^4\omega^{54}b_1^2b_3^3b_5-306672\Delta^4\omega^{54}b_1^2b_3b_5^3+92392\Delta^4$$

$$\omega^{54}b_1b_3^4b_5-270088\Delta^4\omega^{54}b_1b_3^2b_5^3+78954\Delta^5\omega^{48}b_1b_3^4b_5-273462\Delta^5\omega^{48}b_1b_3^2b_5^3-236656$$

$$\Delta^6\omega^{42}b_1^4b_3b_5-86141\Delta^9\omega^{24}b_1^4b_3b_5-77886\Delta^9\omega^{24}b_1^3b_3^2b_5-19278\Delta^9\omega^{24}b_1^2b_3^3b_5-11862\Delta^9\omega^{24}b_1^2b_3b_5^3$$

$$-1272\Delta^9\omega^{24}b_1b_3^4b_5-672\Delta^9\omega^{24}b_1b_3^2b_5^3-14686\Delta^{10}\omega^{18}b_1^4b_3b_5-10844\Delta^{10}\omega^{18}b_1^3b_3^2b_5-1486\Delta^{10}\omega^{18}b_1^2$$

$$b_3^3b_5-630\Delta^{10}\omega^{18}b_1^2b_3b_5^3-23\Delta^{10}\omega^{18}b_1b_3^4b_5+11\Delta^{10}\omega^{18}b_1b_3^2b_5^3-1556\Delta^{11}\omega^{12}b_1^4b_3b_5-733\Delta^{11}\omega^{12}b_1^3b_3^2b_5$$

$$-33\Delta^{11}\omega^{12}b_1^2b_3^3b_5+6270\Delta^{\frac{19}{2}}\omega^{21}b_1^2b_3^2b_5^2+218\Delta^{\frac{19}{2}}\omega^{21}b_1b_3^3b_5^2-87\Delta^{\frac{19}{2}}\omega^{21}b_1b_3b_5^4+3042\Delta^{\frac{21}{2}}\omega^{15}b_1^3b_3b_5^2$$

$$+288\Delta^{\frac{21}{2}}\omega^{15}b_1^2b_3^2b_5^2+147\Delta^{\frac{23}{2}}\omega^{9}b_1^3b_3b_5^2+788984\Delta^{\frac{13}{2}}\omega^{39}b_1^3b_3b_5^2+639312\Delta^{\frac{13}{2}}\omega^{39}b_1^2b_3^2b_5^2+165322\Delta^{\frac{13}{2}}$$

$$\omega^{39}b_1b_3^3b_5^2+47193\Delta^{\frac{13}{2}}\omega^{39}b_1b_3b_5^4+382053\Delta^{\frac{15}{2}}\omega^{33}b_1^3b_3b_5^2+261420\Delta^{\frac{15}{2}}\omega^{33}b_1^2b_3^2b_5^2+50118\Delta^{\frac{15}{2}}\omega^{33}b_1b_3^3b_5^2$$

$$+12213\Delta^{\frac{15}{2}}\omega^{33}b_1b_3b_5^4+119190\Delta^{\frac{17}{2}}\omega^{27}b_1^3b_3b_5^2+56388\Delta^{\frac{17}{2}}\omega^{27}b_1^2b_3^2b_5^2+6024\Delta^{\frac{17}{2}}\omega^{27}b_1b_3^3b_5^2+405\Delta^{\frac{17}{2}}\omega^{27}$$

$$b_1b_3b_5^4+25054\Delta^{\frac{19}{2}}\omega^{21}b_1^3b_3b_5^2+48896\Delta^3\omega^{60}b_1^4b_3b_5+131328\Delta^3\omega^{60}b_1^3b_3^2b_5+68352\Delta^3\omega^{60}b_1^2b_3^3b_5-1403$$

$$\Delta^{\frac{13}{2}}\omega^{39}b_1b_3^5+3404\Delta^{\frac{15}{2}}\omega^{33}b_1^5b_3+4630\Delta^{\frac{15}{2}}\omega^{33}b_1^4b_3^2+41457\Delta^{\frac{15}{2}}\omega^{33}b_1^3b_3^3+21381\Delta^{\frac{15}{2}}\omega^{33}b_1^2b_3^4-468408$$

$$\Delta^{\frac{9}{2}}\omega^{51}b_1^3b_3^3-243696\Delta^{\frac{9}{2}}\omega^{51}b_1^2b_3^4-9204\Delta^{\frac{9}{2}}\omega^{51}b_1b_3^5-36064\Delta^{\frac{11}{2}}\omega^{45}b_1^5b_3-71128\Delta^{\frac{11}{2}}\omega^{45}b_1^4b_3^2-258936$$

2024, 21(7): 2378-2394
https://doi.org/10.21123/bsj.2023.8945
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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$$\Delta^{\frac{11}{2}}\omega^{45}b_1^3b_3^3-161740\Delta^{\frac{11}{2}}\omega^{45}b_1^2b_3^4-23786\Delta^{\frac{11}{2}}\omega^{45}b_1b_3^5-51296\Delta^{\frac{13}{2}}\omega^{39}b_1^5b_3-43032\Delta^{\frac{13}{2}}\omega^{39}b_1^4b_3^2-47326$$
$$\Delta^{\frac{13}{2}}\omega^{39}b_1^3b_3^3-29900\Delta^{\frac{13}{2}}\omega^{39}b_1^2b_3^4-43392\Delta^{\frac{5}{2}}\omega^{63}b_1^3b_3^3-46016\Delta^{\frac{5}{2}}\omega^{63}b_1^2b_3^4-12832\Delta^{\frac{5}{2}}\omega^{63}b_1b_3^5$$
$$-46592\Delta^{\frac{7}{2}}\omega^{57}b_1^5b_3-265216\Delta^{\frac{7}{2}}\omega^{57}b_1^4b_3^2-283328\Delta^{\frac{7}{2}}\omega^{57}b_1^3b_3^3-184240\Delta^{\frac{7}{2}}\omega^{57}b_1^2b_3^4-3448\Delta^{\frac{7}{2}}\omega^{57}b_1b_3^5$$
$$-15488\Delta^{\frac{9}{2}}\omega^{51}b_1^5b_3-246736\Delta^{\frac{9}{2}}\omega^{51}b_1^4b_3^2-233120\,\triangle^4\,\omega^{54}b_1^3b_5^3+15192\Delta^4\omega^{54}b_1b_5^5+10944\Delta^5\omega^{48}b_1^5b_5$$
$$+167312\Delta^5\omega^{48}b_1^3b_5^3+5526\Delta^5\omega^{48}b_1b_5^5-91632\Delta^6\omega^{42}b_1^5b_5+234462\Delta^6\omega^{42}b_1^3b_5^3-12528\Delta^6\omega^{42}b_1b_5^5$$
$$-112824\Delta^7\omega^{36}b_1^5b_5-4150\Delta^7\omega^{36}b_1^3b_5^3-2088\Delta^7\omega^{36}b_1b_5^5-42498\Delta^8\omega^{30}b_1^5b_5-73028\Delta^8\omega^{30}b_1^3b_5^3$$
$$+936\Delta^8\omega^{30}b_1b_5^5+4386\Delta^9\omega^{24}b_1^5b_5-28164\Delta^9\omega^{24}b_1^3b_5^3+162\Delta^9\omega^{24}b_1b_5^5+7548\Delta^{10}\omega^{18}b_1^5b_5-3914\Delta^{10}$$
$$\omega^{18}b_1^3b_5^3+2124\Delta^{11}\omega^{12}b_1^5b_5-166\Delta^{11}\omega^{12}b_1^3b_5^3+246\Delta^{12}\omega^6b_1^5b_5-13824\Delta^{\frac{5}{2}}\omega^{63}b_1^4b_5^2-32832\Delta^{\frac{7}{2}}\omega^{57}b_1^4b_5^2$$
$$-163056\Delta^{\frac{7}{2}}\omega^{57}b_1^2b_5^4+186112\Delta^{\frac{9}{2}}\omega^{51}b_1^4b_5^2-281260\Delta^{\frac{9}{2}}\omega^{51}b_1^2b_5^4+3157708\Delta^{\frac{11}{2}}\omega^{45}b_1^4b_5^2-79956\Delta^{\frac{11}{2}}\omega^{45}b_1^2$$
$$b_5^4+15360\Delta^3\omega^{60}b_1^5b_5-210432\Delta^3\omega^{60}b_1^3b_5^3+46336\Delta^4\omega^{54}b_1^5b_5+88520\Delta^{\frac{13}{2}}\omega^{39}b_1^4b_5^2+77454\Delta^{\frac{13}{2}}\omega^{39}b_1^2b_5^4$$
$$-34711\Delta^{\frac{15}{2}}\omega^{33}b_1^4b_5^2+46255\Delta^{\frac{15}{2}}\omega^{33}b_1^2b_5^4-6729\Delta^{\frac{17}{2}}\omega^{27}b_1^4b_5^2+7137\Delta^{\frac{17}{2}}\omega^{27}b_1^2b_5^4+9298\Delta^{\frac{19}{2}}\omega^{21}b_1^4b_5^2$$
$$+57\Delta^{\frac{19}{2}}\omega^{21}b_1^2b_5^4+3898\Delta^{\frac{21}{2}}\omega^{15}b_1^4b_5^2-31\Delta^{\frac{21}{2}}\omega^{15}b_1^2b_5^4+537\Delta^{\frac{23}{2}}\omega^9b_1^4b_5^2+23\Delta^{\frac{25}{2}}\omega^3b_1^4b_5^2$$

and

$$G_4=\omega^3b_1(-4096\Delta^{\frac{7}{2}}\omega^{54}b_1^4+19629\Delta^{\frac{11}{2}}\omega^{42}b_5^4-20736\Delta^{\frac{3}{2}}\omega^{66}b_3^4-21408\Delta^{\frac{5}{2}}\omega^{60}b_3^4-16704\Delta^{\frac{7}{2}}\omega^{54}b_3^4$$
$$+50112\Delta^{\frac{7}{2}}\omega^{54}b_5^4-14310\Delta^{\frac{9}{2}}\omega^{48}b_3^4+67284\Delta^{\frac{9}{2}}\omega^{48}b_5^4+5073\Delta^{\frac{11}{2}}\omega^{42}b_3^4+21\Delta^{\frac{17}{2}}\omega^{24}b_3^4+27\Delta^{\frac{17}{2}}\omega^{24}b_5^4$$
$$+459\Delta^{\frac{15}{2}}\omega^{30}b_5^4+567\Delta^{\frac{15}{2}}\omega^{30}b_3^4+2889\Delta^{\frac{13}{2}}\omega^{36}b_5^4+3897\Delta^{\frac{13}{2}}\omega^{36}b_3^4+576\Delta^{\frac{23}{2}}\omega^6b_1^4+140256\Delta^{\frac{15}{2}}\omega^{30}b_1^4$$
$$+111780\Delta^{\frac{17}{2}}\omega^{24}b_1^4+37632\Delta^{\frac{19}{2}}\omega^{18}b_1^4+6552\Delta^{\frac{21}{2}}\omega^{12}b_1^4+41472\Delta^{\frac{9}{2}}\omega^{48}b_1^4-154368\Delta^{\frac{11}{2}}\omega^{42}b_1^4-29232$$
$$\Delta^{\frac{13}{2}}\omega^{36}b_1^4-110592\Delta^{\frac{5}{2}}\omega^{60}b_1^4-22522\Delta^7\omega^{33}b_1b_5^3-262680\Delta^8\omega^{27}b_1^3b_5-870\Delta^8\omega^{27}b_1b_5^3-193536\Delta^2$$
$$\omega^{63}b_1^3b_5+230040\Delta^5\omega^{45}b_1^3b_5+4\Delta^{12}\omega^3b_1^3b_5+85920\Delta^{\frac{17}{2}}\omega^{24}b_1^3b_3+22910\Delta^{\frac{17}{2}}\omega^{24}b_1^2b_3^2+144548\Delta^{\frac{15}{2}}$$
$$\omega^{30}b_1^2b_5^2+13386\Delta^{\frac{15}{2}}\omega^{30}b_1b_3^3+187168\Delta^{\frac{15}{2}}\omega^{30}b_1^3b_3+83736\Delta^{\frac{15}{2}}\omega^{30}b_1^2b_3^2+153\Delta^{\frac{21}{2}}\omega^{12}b_1^2b_3^2-23\Delta^{\frac{21}{2}}$$
$$\omega^{12}b_1^2b_5^2+96\Delta^{\frac{23}{2}}\omega^6b_1^3b_3+659185\Delta^{\frac{13}{2}}\omega^{36}b_1^2b_5^2+34942\Delta^{\frac{13}{2}}\omega^{36}b_1b_3^3+32\Delta^{\frac{19}{2}}\omega^{18}b_1^2b_5^2+98\Delta^{\frac{19}{2}}\omega^{18}b_1b_3^3$$
$$+204124\Delta^{\frac{13}{2}}\omega^{36}b_1^3b_3+156665\Delta^{\frac{13}{2}}\omega^{36}b_1^2b_3^2+2228\Delta^{\frac{21}{2}}\omega^{12}b_1^3b_3+55296\Delta^{\frac{3}{2}}\omega^{66}b_1^3b_3-82944\Delta^{\frac{3}{2}}\omega^{66}b_1^2b_3^2$$

2024, 21(7): 2378-2394
https://doi.org/10.21123/bsj.2023.8945
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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$$-41472\Delta^{\frac{3}{2}}\omega^{66}b_1b_3^3-387840\Delta^{\frac{5}{2}}\omega^{60}b_1^3b_3-339328\Delta^{\frac{5}{2}}\omega^{60}b_1^2b_3^2-470016\Delta^{\frac{5}{2}}\omega^{60}b_1^2b_5^2-91456\Delta^{\frac{5}{2}}\omega^{60}$$

$$b_1b_3^3-477760\Delta^{\frac{7}{2}}\omega^{54}b_1^3b_3-381152\Delta^{\frac{7}{2}}\omega^{54}b_1^2b_3^2-397888\Delta^{\frac{7}{2}}\omega^{54}b_1^2b_5^2-188608\Delta^{\frac{7}{2}}\omega^{54}b_1b_3^3+43568\Delta^{\frac{9}{2}}$$

$$\omega^{48}b_1^3b_3-91800\Delta^{\frac{9}{2}}\omega^{48}b_1^2b_3^2+865840\Delta^{\frac{9}{2}}\omega^{48}b_1^2b_5^2-169620\Delta^{\frac{9}{2}}\omega^{48}b_1b_3^3+163356\Delta^{\frac{11}{2}}\omega^{42}b_1^3b_3+125914$$

$$\Delta^{\frac{11}{2}}\omega^{42}b_1^2b_3^2+1320708\Delta^{\frac{11}{2}}\omega^{42}b_1^2b_5^2-7604\Delta^{\frac{11}{2}}\omega^{42}b_1b_3^3+12414\Delta^{\frac{17}{2}}\omega^{24}b_1^2b_5^2+1934\Delta^{\frac{17}{2}}\omega^{24}b_1b_3^3+19844$$

$$\Delta^{\frac{19}{2}}\omega^{18}b_1^3b_3+3046\triangle^{\frac{19}{2}}\omega^{18}b_1^2b_3^2-707792\Delta^4\omega^{51}b_1b_5^3+48768\Delta^3\omega^{57}b_1^3b_5-284832\Delta^3\omega^{57}b_1b_5^3+705248$$

$$\Delta^4\omega^{51}b_1^3b_5+74256\Delta^3\omega^{57}b_3^3b_5-142416\Delta^3\omega^{57}b_3b_5^3+60960\Delta^4\omega^{51}b_3^3b_5-325368\Delta^4\omega^{51}b_3b_5^3-54759\Delta^5$$

$$\omega^{45}b_3^3b_5-235881\Delta^5\omega^{45}b_3b_5^3-41052\Delta^6\omega^{39}b_3^3b_5-57780\Delta^6\omega^{39}b_3b_5^3-8064\Delta^2\omega^{63}b_3^3b_5-8106\Delta^7\omega^{33}$$

$$b_3^3b_5-5094\Delta^7\omega^{33}b_3b_5^3-444\Delta^8\omega^{27}b_3^3b_5-252\Delta^8\omega^{27}b_3b_5^3+9\Delta^9\omega^{21}b_3^3b_5-9\Delta^9\omega^{21}b_3b_5^3+231666\Delta^{\frac{9}{2}}\omega^{48}$$

$$b_3^2b_5^2+159222\Delta^{\frac{11}{2}}\omega^{42}b_3^2b_5^2+35778\Delta^{\frac{13}{2}}\omega^{36}b_3^2b_5^2+2490\Delta^{\frac{15}{2}}\omega^{30}b_3^2b_5^2+12\Delta^{\frac{17}{2}}\omega^{24}b_3^2b_5^2-82656\triangle^{\frac{5}{2}}\omega^{60}b_3^2$$

$$b_5^2+23088\Delta^{\frac{7}{2}}\omega^{54}b_3^2b_5^2-589938\Delta^5\omega^{45}b_1b_5^3-48844\Delta^9\omega^{21}b_1^3b_5-108\Delta^9\omega^{21}b_1b_5^3-4090\Delta^{10}\omega^{15}b_1^3b_5$$

$$-8\Delta^{10}\omega^{15}b_1b_5^3-78\Delta^{11}\omega^{9}b_1^3b_5-700946\Delta^6\omega^{39}b_1^3b_5-188730\Delta^6\omega^{39}b_1b_5^3-685886\Delta^7\omega^{33}b_1^3b_5$$

$$-514187\Delta^7\omega^{33}b_1^2b_3b_5-120322\Delta^7\omega^{33}b_1b_3^2b_5-50272\Delta^3\omega^{57}b_1b_3^2b_5+64880\Delta^4\omega^{51}b_1^2b_3b_5+79744\Delta^4$$

$$\omega^{51}b_1b_3^2b_5-564628\Delta^5\omega^{45}b_1^2b_3b_5-262374\Delta^5\omega^{45}b_1b_3^2b_5-897505\Delta^6\omega^{39}b_1^2b_3b_5-312932\Delta^6\omega^{39}b_1b_3^2b_5$$

$$+69120\Delta^2\omega^{63}b_1^2b_3b_5-182016\Delta^2\omega^{63}b_1b_3^2b_5+195904\triangle^3\omega^{57}b_1^2b_3b_5-134402\Delta^8\omega^{27}b_1^2b_3b_5-18618$$

$$\Delta^8\omega^{27}b_1b_3^2b_5-15506\Delta^9\omega^{21}b_1^2b_3b_5-832\Delta^9\omega^{21}b_1b_3^2b_5-493\Delta^{10}\omega^{15}b_1^2b_3b_5+22\Delta^{10}\omega^{15}b_1b_3^2b_5+17\Delta^{11}\omega^{9}$$

$$b_1^2b_3b_5+977580\Delta^{\frac{9}{2}}\omega^{48}b_1b_3b_5^2+876702\Delta^{\frac{11}{2}}\omega^{42}b_1b_3b_5^2+46494\Delta^{\frac{15}{2}}\omega^{30}b_1b_3b_5^2+2046\Delta^{\frac{17}{2}}\omega^{24}b_1b_3b_5^2$$

$$-20\Delta^{\frac{19}{2}}\omega^{18}b_1b_3b_5^2-44352\Delta^{\frac{5}{2}}\omega^{60}b_1b_3b_5^2+327424\triangle^{\frac{7}{2}}\omega^{54}b_1b_3b_5^2+315326\Delta^{\frac{13}{2}}\omega^{36}b_1b_3b_5^2+20\Delta^{\frac{25}{2}}b_1^4).$$

Since the Jacobian determinant of the functions η_2,η_3 with respect to b_2,b_6 is given by

$$\begin{vmatrix}\frac{\partial\eta_2}{\partial b_2} & \frac{\partial\eta_2}{\partial b_6}\\ \frac{\partial\eta_3}{\partial b_2} & \frac{\partial\eta_3}{\partial b_6}\end{vmatrix}=\Omega\neq 0,$$

where

$$\Omega=-(b_1^2(67284\Delta^{\frac{9}{2}}\omega^{48}b_5^4+5073\Delta^{\frac{11}{2}}\omega^{42}b_3^4+19629\Delta^{\frac{11}{2}}\omega^{42}b_5^4-20736\Delta^{\frac{3}{2}}\omega^{66}b_3^4-21408\Delta^{\frac{5}{2}}\omega^{60}b_3^4+3897$$

$$\Delta^{\frac{13}{2}}\omega^{36}b_3^4-16704\Delta^{\frac{7}{2}}\omega^{54}b_3^4+50112\Delta^{\frac{7}{2}}\omega^{54}b_5^4-14310\Delta^{\frac{9}{2}}\omega^{48}b_3^4+2889\Delta^{\frac{13}{2}}\omega^{36}b_5^4+567\Delta^{\frac{15}{2}}\omega^{30}b_3^4+459$$

$$\Delta^{\frac{15}{2}}\omega^{30}b_5^4-110592\Delta^{\frac{5}{2}}\omega^{60}b_1^4-4096\Delta^{\frac{7}{2}}\omega^{54}b_1^4+111780\Delta^{\frac{17}{2}}\omega^{24}b_1^4+37632\Delta^{\frac{19}{2}}\omega^{18}b_1^4+140256\Delta^{\frac{15}{2}}\omega^{30}b_1^4$$

2024, 21(7): 2378-2394
https://doi.org/10.21123/bsj.2023.8945
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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$$+6552\Delta^{\frac{21}{2}}\omega^{12}b_1^4+576\Delta^{\frac{23}{2}}\omega^{6}b_1^4+41472\Delta^{\frac{9}{2}}\omega^{48}b_1^4-154368\Delta^{\frac{11}{2}}\omega^{42}b_1^4-29232\Delta^{\frac{13}{2}}\omega^{36}b_1^4+21\Delta^{\frac{17}{2}}\omega^{24}b_3^4$$

$$+27\Delta^{\frac{17}{2}}\omega^{24}b_5^4+12\Delta^{\frac{17}{2}}\omega^{24}b_3^2b_5^2+12414\Delta^{\frac{17}{2}}\omega^{24}b_1^2b_5^2+1934\Delta^{\frac{17}{2}}\omega^{24}b_1b_3^3+19844\Delta^{\frac{19}{2}}\omega^{18}b_1^3b_3-23\Delta^{\frac{21}{2}}\omega^{12}$$

$$b_1^2b_5^2-235881\Delta^5\omega^{45}b_3b_5^3-41052\Delta^6\omega^{39}b_3^3b_5-57780\Delta^6\omega^{39}b_3b_5^3-8064\Delta^2\omega^{63}b_3^3b_5+74256\Delta^3\omega^{57}b_3^3b_5$$

$$-142416\Delta^3\omega^{57}b_3b_5^3+60960\Delta^4\omega^{51}b_3^3b_5-325368\Delta^4\omega^{51}b_3b_5^3-54759\Delta^5\omega^{45}b_3^3b_5+9\Delta^9\omega^{21}b_3^3b_5-9\Delta^9\omega^{21}$$

$$b_3b_5^3+96\Delta^{\frac{23}{2}}\omega^{6}b_1^3b_3+3046\Delta^{\frac{19}{2}}\omega^{18}b_1^2b_3^2+32\Delta^{\frac{19}{2}}\omega^{18}b_1^2b_5^2+98\Delta^{\frac{19}{2}}\omega^{18}b_1b_3^3+2228\Delta^{\frac{21}{2}}\omega^{12}b_1^3b_3+153\Delta^{\frac{21}{2}}\omega^{12}$$

$$b_1^2b_3^2+163356\Delta^{\frac{11}{2}}\omega^{42}b_1^3b_3+125914\Delta^{\frac{11}{2}}\omega^{42}b_1^2b_3^2+1320708\Delta^{\frac{11}{2}}\omega^{42}b_1^2b_5^2-7604\Delta^{\frac{11}{2}}\omega^{42}b_1b_3^3+55296\Delta^{\frac{3}{2}}\omega^{66}$$

$$b_1^3b_3-82944\Delta^{\frac{3}{2}}\omega^{66}b_1^2b_3^2-41472\Delta^{\frac{3}{2}}\omega^{66}b_1b_3^3-387840\Delta^{\frac{5}{2}}\omega^{60}b_1^3b_3-339328\Delta^{\frac{5}{2}}\omega^{60}b_1^2b_3^2-470016\Delta^{\frac{5}{2}}\omega^{60}b_1^2$$

$$b_5^2-91456\Delta^{\frac{5}{2}}\omega^{60}b_1b_3^3-477760\Delta^{\frac{7}{2}}\omega^{54}b_1^3b_3+204124\Delta^{\frac{13}{2}}\omega^{36}b_1^3b_3+156665\Delta^{\frac{13}{2}}\omega^{36}b_1^2b_3^2+659185\,\triangle^{\frac{13}{2}}\omega^{36}b_1^2$$

$$b_5^2+34942\Delta^{\frac{13}{2}}\omega^{36}b_1b_3^3-381152\Delta^{\frac{7}{2}}\omega^{54}b_1^2b_3^2-397888\Delta^{\frac{7}{2}}\omega^{54}b_1^2b_5^2-188608\Delta^{\frac{7}{2}}\omega^{54}b_1b_3^3+43568\Delta^{\frac{9}{2}}\omega^{48}b_1^3b_3$$

$$-91800\Delta^{\frac{9}{2}}\omega^{48}b_1^2b_3^2+865840\Delta^{\frac{9}{2}}\omega^{48}b_1^2b_5^2-169620\Delta^{\frac{9}{2}}\omega^{48}b_1b_3^3+187168\Delta^{\frac{15}{2}}\omega^{30}b_1^3b_3+83736\Delta^{\frac{15}{2}}\omega^{30}b_1^2b_3^2$$

$$+144548\Delta^{\frac{15}{2}}\omega^{30}b_1^2b_5^2+13386\Delta^{\frac{15}{2}}\omega^{30}b_1b_3^3+85920\Delta^{\frac{17}{2}}\omega^{24}b_1^3b_3+22910\Delta^{\frac{17}{2}}\omega^{24}b_1^2b_3^2-444\Delta^8\omega^{27}b_3^3b_5-252$$

$$\Delta^8\omega^{27}b_3b_5^3-8106\Delta^7\omega^{33}b_3^3b_5-5094\Delta^7\omega^{33}b_3b_5^3-193536\Delta^2\omega^{63}b_1^3b_5-108\Delta^9\omega^{21}b_1b_5^3-4090\Delta^{10}\omega^{15}b_1^3b_5$$

$$-8\Delta^{10}\omega^{15}b_1b_5^3-78\Delta^{11}\omega^{9}b_1^3b_5+4\Delta^{12}\omega^{3}b_1^3b_5+48768\Delta^3\omega^{57}b_1^3b_5-685886\Delta^7\omega^{33}b_1^3b_5-22522\Delta^7\omega^{33}b_1b_5^3$$

$$-262680\Delta^8\omega^{27}b_1^3b_5-870\Delta^8\omega^{27}b_1b_5^3-48844\Delta^9\omega^{21}b_1^3b_5-284832\Delta^3\omega^{57}b_1b_5^3+705248\Delta^4\omega^{51}b_1^3b_5-707792$$

$$\Delta^4\omega^{51}b_1b_5^3+230040\Delta^5\omega^{45}b_1^3b_5-589938\Delta^5\omega^{45}b_1b_5^3-700946\Delta^6\omega^{39}b_1^3b_5-188730\,\triangle^6\,\omega^{39}b_1b_5^3+2490\Delta^{\frac{15}{2}}$$

$$\omega^{30}b_3^2b_5^2+35778\Delta^{\frac{13}{2}}\omega^{36}b_3^2b_5^2-82656\Delta^{\frac{5}{2}}\omega^{60}b_3^2b_5^2+23088\Delta^{\frac{7}{2}}\omega^{54}b_3^2b_5^2+231666\Delta^{\frac{9}{2}}\omega^{48}b_3^2b_5^2+159222\Delta^{\frac{11}{2}}$$

$$\omega^{42}b_3^2b_5^2-514187\,\triangle^7\,\omega^{33}b_1^2b_3b_5-120322\Delta^7\omega^{33}b_1b_3^2b_5+22\Delta^{10}\omega^{15}b_1b_3^2b_5+17\,\triangle^{11}\,\omega^{9}b_1^2b_3b_5+69120\Delta^2\omega^{63}$$

$$b_1^2b_3b_5-20\Delta^{\frac{19}{2}}\omega^{18}b_1b_3b_5^2-897505\Delta^6\omega^{39}b_1^2b_3b_5-312932\Delta^6\omega^{39}b_1b_3^2b_5+195904\Delta^3\omega^{57}b_1^2b_3b_5-50272\Delta^3\omega^{57}$$

$$b_1b_3^2b_5+64880\Delta^4\omega^{51}b_1^2b_3b_5+79744\Delta^4\omega^{51}b_1b_3^2b_5-564628\Delta^5\omega^{45}b_1^2b_3b_5-262374\Delta^5\omega^{45}b_1b_3^2b_5+46494\Delta^{\frac{15}{2}}$$

$$\omega^{30}b_1b_3b_5^2+2046\Delta^{\frac{17}{2}}\omega^{24}b_1b_3b_5^2+315326\Delta^{\frac{13}{2}}\omega^{36}b_1b_3b_5^2+977580\Delta^{\frac{9}{2}}\omega^{48}b_1b_3b_5^2+876702\Delta^{\frac{11}{2}}\omega^{42}b_1b_3b_5^2$$

$$-832\Delta^9\omega^{21}b_1b_3^2b_5-493\Delta^{10}\omega^{15}b_1^2b_3b_5-44352\Delta^{\frac{5}{2}}\omega^{60}b_1b_3b_5^2+327424\Delta^{\frac{7}{2}}\omega^{54}b_1b_3b_5^2-134402\Delta^8\omega^{27}b_1^2b_3b_5$$

$$-18618\Delta^8\omega^{27}b_1b_3^2b_5-15506\Delta^9\omega^{21}b_1^2b_3b_5-182016\Delta^2\omega^{63}b_1b_3^2b_5+20\Delta^{\frac{25}{2}}b_1^4))/(96\omega^7(9\omega^6+\Delta)(\omega^6+\triangle)$$

2024, 21(7): 2378-2394
https://doi.org/10.21123/bsj.2023.8945
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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$$(4\omega^6+\Delta)^3\Delta^2\left(4\sqrt{\Delta}\omega^{12}b_1+2\sqrt{\Delta}\omega^{12}b_3+3\Delta\omega^9b_5-3\Delta^{\frac{3}{2}}\omega^6b_1-\Delta^{\frac{3}{2}}\omega^6b_3-\Delta^{\frac{5}{2}}b_1\right)^2) \qquad 27$$

and η_4 at the values $b_2=b_2^*$ and $b_6=b_6^*$ is non-zero. Then, by suitable perturbation of the coefficients of Liapunov quantities, three limit cycles can be bifurcated from the origin of system (Eq 4) in the neighborhood of the singular point. ■

Theorem 2: For system (Eq 4) when $\Delta>0$ and $a_4=0$, three limit cycles can be bifurcated from the *singular* point at $\left(-\frac{a_0}{a_1},0,0\right)$; $a_1\neq 0$ when the parameters satisfy the conditions of Proposition 4 with

i. $\frac{b_2^3\left(54\,\omega^{18}\,b_2-171\,\omega^{18}\,b_5-71\,\omega^{12}a_1^2b_2-18\,\omega^{12}\,a_1^2b_5+20\,\omega^6a_1^4b_2+9\omega^6\,a_1^4\,b_5+a_1^6b_2\right)}{96\,\omega^4\left(4\omega^6+a_1^2\right)^2(9\omega^6+a_1^2)}\neq 0,$

ii. $\eta_4|_{\{b_3=b_3^*,b_6=b_6^*\}}\neq 0.$

Proof: At first, the singular point is moved to the origin by applying the translation $\rightarrow x+\frac{-a_0}{a_1}$, then system (4) can be changed into

$$\begin{cases}\dot{x}=y,\\ \dot{y}=z,\\ \dot{z}=a_0+a_1\left(x-\dfrac{a_0}{a_1}\right)+a_2\,y+a_3\,z+a_5\left(x-\dfrac{a_0}{a_1}\right)y+a_6\left(x-\dfrac{a_0}{a_1}\right)z\\ \qquad +a_7\,y^2+a_8yz+a_9\,z^2,\end{cases} \qquad 28$$

where $\omega=\sqrt{\frac{a_5a_0-a_1a_2}{a_1}}$

The Jacobian matrix of the system above at the origin is

$$J=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ a_1 & \dfrac{-a_0\,a_5}{a_1}+a_2 & \dfrac{-a_0a_6}{a_1}+a_3\end{bmatrix}$$

and its characteristic equation is

$$\lambda^3+\frac{(a_6a_0-a_3\,a_1)}{a_1}\lambda^2+\frac{(a_5a_0-a_2\,a_1)}{a_1}\lambda-a_1=0. \qquad 29$$

and Eq.29 has a simple pair of purely imaginary roots, $\pm i\omega$, with a non-zero root $\frac{a_1}{\omega^2}$. An application of the implicit function theorem yields the following transversality condition:

$$\frac{dRe(\lambda_{1,2})}{da_3}\Big|_{\{a_3=a_3^*,a_2=a_2^*,\lambda=i\,\omega\}}=\frac{\omega^6}{2\omega^6+2a_1^2}>0. \qquad 30$$

The conditions of Proposition 4 leads to

$a_3=a_3^*=\frac{a_0^2\,a_5\,a_6+a_0\,(\omega^2\,a_1-a_5\,a_0)a_6+a_1^3}{a_1^2\omega^2}$ and

$a_2=a_2^*=-\frac{\omega^2\,a_1-a_5\,a_0}{a_1}$

Thus, Hopf bifurcation occurs at $a_3=a_3^*$, $a_2=a_2^*$. Introducing the transformation

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1 & 0 & 1\\ 0 & -\omega & \dfrac{a_1}{\omega^2}\\ -\omega^2 & 0 & \dfrac{a_1^2}{\omega^4}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} \qquad 31$$

and the new system is given by

$$\begin{cases}\dfrac{dy_1}{dt}=-\omega\,y_2+b_1\,y_1^2+b_2\,y_1\,y_2+b_3\,y_1\,y_3+b_4\,y_2^2+b_5\,y_2y_3+b_6y_3^2,\\ \dfrac{dy_2}{dt}=\omega\,y_1-\dfrac{a_1}{\omega^3}\,(b_1\,y_1^2+b_2\,y_1\,y_2+b_3\,y_1\,y_3+b_4\,y_2^2+b_5\,y_2y_3+b_6y_3^2),\\ \dfrac{dy_3}{dt}=\dfrac{a_1}{\omega^2}\,y_3-\,(b_1\,y_1^2+b_2\,y_1\,y_2+b_3\,y_1\,y_3+b_4\,y_2^2+b_5\,y_2y_3+b_6y_3^2),\end{cases} \qquad 32$$

Where

$b_1=-\frac{\omega^6\,(\omega^2a_9-a_6)}{\omega^6+a_1^2},$

$b_2=-\frac{\omega^5\,(\omega^2a_8-a_5)}{\omega^6+a_1^2},$

$$b_3 = \frac{\omega^6 a_6 + \omega^4 a_1 a_8 + 2\omega^2 a_1^2 a_9 - \omega^2 a_1 a_5 - a_1^2 a_6}{\omega^6 + a_1^2}$$

$b_4 = -\frac{\omega^6 a_7}{\omega^6 + a_1^2}$, $b_5 = \frac{\omega(\omega^4 a_5 + 2\omega^2 a_1 a_7 + a_1^2 a_8)}{\omega^6 + a_1^2}$ and $b_6 = -\frac{a_1(\omega^6 a_5 + \omega^4 a_1 a_6 + \omega^4 a_1 a_7 + \omega^2 a_1^2 a_8 + a_1^3 a_9)}{\omega^4(\omega^6 + a_1^2)}$.

At present, due to the computational load for computing Liapunov quantities, it cannot be found them, more powerful computing devices are needed. Therefore, to make the calculation easier, the parameters a_6, a_7 and a_9 are fixed and vanished, in this case, the value of b_1 and b_4 are also vanished. For studying the number of periodic orbits bifurcated from the singular point, the following Liapunov function is introduced

$$F(y_1, y_2, y_3) = y_1^2 + y_2^2 + \sum_{k=3}^{n}\sum_{j=0}^{k}\sum_{i=0}^{j} C_{k-j,j-i,i}\, y_1^{k-j} y_2^{j-i} y_3^{i}.$$

which is satisfying the following equation

$$\chi(F) = \eta_1(y_1^2 + y_2^2) + \eta_2(y_1^2 + y_2^2)^2 + \cdots \quad 33$$

where χ is the vector field of the system (Eq 32). By solving Eq 33 and using computer algebra MAPLE, the following linearly independent terms of Liapunov quantities are obtained:

1. $\eta_1 = 0$,
2. $\eta_2 = \frac{b_2(2\omega^6 b_3 + 3\omega^3 a_1 b_5 - a_1^2 b_3)}{4\omega(4\omega^6 + a_1^2)}$,
3. $\eta_3 = \frac{b_2 F_1}{96\omega^9 a_1 (9\omega^6 + a_1^2)(\omega^6 + a_1^2)(4\omega^6 + a_1^2)^2}$

where F_1 is a polynomial of b_2, b_3, b_5, b_6, a_1 and ω of degree 33.

4. $\eta_4 = \frac{b_2 F_2}{9216\, a_1^2 \omega^{17}(16\omega^6 + a_1^2)(\omega^6 + a_1^2)^2 (4\omega^6 + a_1^2)^4 (\omega^6 + 4a_1^2)(9\omega^6 + a_1^2)^2}$

where F_2 is a polynomial of b_2, b_3, b_5, b_6, a_1 and ω of degree 78.

The origin is a weak focus of order three for Eq 32 under our conditions if and only if the following conditions are held.

1. $b_3 = b_3^* = \frac{-3\omega^3 a_1 b_5}{2\omega^6 - a_1^2}$,
2. $b_6 = b_6^* = \frac{a_1 b_5 F_3}{\omega^3 b_2 F_4}$, where

$F_3 = 120\omega^{30} b_2^2 - 1020\omega^{30} b_2 b_5 + 1812\omega^{30} b_5^2 - 152\omega^{24} a_1^2 b_2^2 + 372\omega^{24} a_1^2 b_2 b_5 + 549\omega^{24} a_1^2 b_5^2 - 98\omega^{18} a_1^4 b_2^2 + 117\omega^{18} a_1^4 b_2 b_5 + 36\omega^{18} a_1^4 b_5^2 + 144\omega^{12} a_1^6 b_2^2 + 6\omega^{12} a_1^6 b_2 b_5 + 3\omega^{12} a_1^6 b_5^2 - 32\omega^6 a_1^8 b_2^2 - 15\omega^6 a_1^8 b_2 b_5 - 2a_1^{10} b_2^2$

and

$F_4 = 216\omega^{30} b_2 - 684\omega^{30} b_5 - 500\omega^{24} a_1^2 b_2 + 612\omega^{24} a_1^2 b_5 + 418\omega^{18} a_1^4 b_2 - 63\omega^{18} a_1^4 b_5 - 147\omega^{12} a_1^6 b_2 - 54\omega^{12} a_1^6 b_5 + 16\omega^6 a_1^8 b_2 + 9\omega^6 a_1^8 b_5 + a_1^{10} b_2$.

Since the Jacobian determinant of the functions η_2, η_3 with respect to b_3, b_6 is given by

$$\begin{vmatrix} \frac{\partial \eta_2}{\partial b_3} & \frac{\partial \eta_2}{\partial b_6} \\ \frac{\partial \eta_3}{\partial b_3} & \frac{\partial \eta_3}{\partial b_6} \end{vmatrix} = \Omega \neq 0,$$

where

$$\Omega = \frac{b_2^3(54\omega^{18} b_2 - 171\omega^{18} b_5 - 71\omega^{12} a_1^2 b_2 - 18\omega^{12} a_1^2 b_5 + 20\omega^6 a_1^4 b_2 + 9\omega^6 a_1^4 b_5 + a_1^6 b_2)}{96\omega^4(4\omega^6 + a_1^2)^2(9\omega^6 + a_1^2)} \quad 34$$

and $\eta_4|_{\{b_3 = b_3^*, b_6 = b_6^*\}}$ is non-zero. Then by suitable perturbation of the coefficients of Liapunov quantities, three limit cycles can be bifurcated from the origin of system (Eq 4) in the neighborhood of the singular point. ■

Remark: In this research, a computer with the following specifications was used: CPU = Core i7-8700K 3.70GHz, Memory (RAM) = 48GB, and Total GPU = 22 GB.

A numerical example for a special case of the quadratic Jerk system (Eq 4) was carried out by fixing some of the parameters, such as $a_0 = 0, a_1 = 1, a_2 = -1, a_4 = 1, a_7 = 0, a_8 = 0$ and $a_9 = 1$ to verify the analytical results. As a result, the Jacobian matrix of system (4) at the origin has two pairs of purely imaginary with non-zero eigenvalues when $a_3 = 1$, which makes the origin of the system a Hopf point. To investigate how many limit cycles can bifurcate at the origin, the Liapunov quantities technique is applied. It can be seen that all the conditions of Theorem 1 are satisfied. The origin is a weak focus of order three if the following condition holds.

1. $a_5 = \left(6 + \sqrt{31}\right)(a_6 - 2)$,
2. $a_6 = \frac{(21671+5165\sqrt{31})}{9173}$..

At these values $\eta_2 = \eta_3 = 0$ and $\eta_4$with the Jacobian determinant of the functions η_2, η_3 with respect to a_5, a_6 are non-zero. This indicates that three limit cycle can bifurcate from the origin of system (4).

Results and Discussion

The three-dimensional quadratic jerk system is investigated and the local stability of singular points of the system is examined in the paper. The existence of Hopf bifurcation is studied and the number of limit cycles that can bifurcate from the Hopf points is determined by computing the Liapunov quantities. It is confirmed that three limit cycles can bifurcate from the Hopf point for some special cases of the quadratic jerk system in the neighborhood of the point. An example is presented to verify their results. However, some parameters are fixed to simplify the calculation of the Liapunov quantities. The behavior of the system in other regions of the phase space could be investigated by future studies.

Conclusion

In this paper, the three-dimensional quadratic jerk system has been investigated. The local stability of singular points of the system has been examined by analyzing the characteristic equations. The existence of Hopf bifurcation has also been studied under certain conditions on the parameters. Additionally, to investigate the cyclicity of the system, the Liapunov quantities have been computed to determine the number of limit cycles that can bifurcate from the Hopf points. Due to the computational load required for computing the Liapunov quantities, some parameters have been fixed to simplify the calculation. It has been confirmed that three limit cycles can bifurcate from the Hopf point for some special cases of the quadratic jerk system in the neighborhood of the point. Finally, an example is presented to verify the results that were obtained.

Authors' Declaration

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee at Soran University.

Authors' Contribution Statement

This work was carried out in collaboration between all authors. R.H.S. and T.I.R. contributed to the design and implementation of the research, the analysis of the results, and the writing of the manuscript. All authors read and approved the final manuscript.

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2024, 21(7): 2378-2394
https://doi.org/10.21123/bsj.2023.8945
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

تشعب هوبف لنظام رعشة تربيعي ثلاثي الأبعاد

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الخلاصة

هذا البحث مخصص لبحث تشعب هوبف لنظام الرعشة التربيعي ثلاثي الأبعاد. تم دراسة الاستقرار للنقاط المنفردة وظهور تشعب هوبف والدورات الحدية للنظام. بالإضافة إلى ذلك، يتم استخدام تقنية كميات ليابانوف لدراسة دورية النظام ومعرفة عدد دورات الحد التي يمكن تشعبها من نقاط هوبف. نظرًا للحمل الحسابي المطلوب لحساب كميات ليابانوف، تم تثبيت بعض المعلمات. حاليًا، يُظهر التحليل أنه يمكن تشعب ثلاث دورات حدية من نقاط هوبف. تم استخدام برمجة Maple للتحقق من جميع النتائج المعروضة في هذه الدراسة.

الكلمات المفتاحية: تشعب هوبف، نظام الرعشة، دورة الحد، كميات ليابانوف، الاستقرارية.