

g-Small Intersection Graph of a Module

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Abstract

Let R be a commutative ring with identity, and M be a left R -module. The g -small intersection graph of non-trivial submodules of M , indicated by $\Gamma_g(M)$, is a simple undirected graph whose vertices are in one-to-one correspondence with all non-trivial submodules of M and two distinct vertices are adjacent if and only if the intersection of the corresponding submodules is a g -small submodule of M . In this article, the interplay among the algebraic properties of M , and the graph properties of $\Gamma_g(M)$ are studied. Properties of $\Gamma_g(M)$ such as connectedness, and completeness are considered. Besides, the girth and the diameter of $\Gamma_g(M)$ are determined, as well as presenting a formula to compute the clique and domination numbers of $\Gamma_g(M)$. The graph $\Gamma_g(M)$ is complete if, M is a generalized hollow module or M is a direct sum of two simple modules, is proved.

Keywords: Connectivity, Domination, Module, Small submodule, Small intersection graph.

Introduction

It is well identified that graphs are very useful tools in solving model problems occurring in almost all areas of our lives. This article focuses on intersection graphs. Let $\mathcal{X} = \{\mathcal{X}_i : i \in \Lambda\}$ be a random class of sets. The intersection graph $\Gamma(\mathcal{X})$ for $\mathcal{X}$ is a graph whose vertices are $\mathcal{X}_i$, $i \in \Lambda$ and there is an edge between different vertices $\mathcal{X}_i$ and $\mathcal{X}_j$ if and only if $\mathcal{X}_i \cap \mathcal{X}_j \neq \emptyset$. The studies of $\Gamma(\mathcal{X})$ whenever the elements of $\mathcal{X}$ have an algebraic structure is interesting. These revisions allow us to get representations of the classes of algebraic structure in terms of graphs and vice versa. In 2009, the idea of the intersection graph of a ring was introduced by Chakrabarty¹, et. al. Inspired by his work in 2012, Akbari², et. al. defined the intersection graphs of modules. Also, there are some graphs on groups and modules³⁻⁵. In 2021, Mahdavi and Talebi⁶ considered graph $\Gamma(M)$ on a module M with vertices as non-trivial submodules of M , where two different

vertices N , L are adjacent if and only if $N \cap L$ is small in M . Inspired by preceding revisions on the intersection graph of algebraic constructions, in this paper, $\Gamma_g(M)$ the g -small intersection graph of a module is defined.

In Section 2, certain assets of g -small submodules are introduced. In Section 3, $\Gamma_g(M)$ is complete if either M is a direct sum of two simple modules or M is a generalized hollow module are proved. Also, if M is a g -supplemented module, then $\Gamma_g(M)$ is connected and $\text{diam}(\Gamma_g(M)) \leq 2$. Besides proved that if $|\Gamma_g(M)| \geq 3$, then $\Gamma_g(M)$ is a star graph if and only if $\text{Rad}_g(M)$ is a non-zero simple g -small submodule where any pair of non-trivial submodules of M have non- g -small intersections. In addition, if $|\mathcal{S}_g(M)| \in \{1, 2\}$ and under some condition, then $\Gamma_g(M)$ is a planar graph. Also, if $|\mathcal{S}_g(M)| \geq 3$, then $\Gamma_g(M)$ is not a planar graph. In

Section 4, the main result, that is if $M = \bigoplus_{i=1}^n M_i$, with M_i is a distinct simple R -module, then $\Gamma_g(M)$ is a planar graph if and only if $n \leq 4$.

Throughout this paper R is a commutative ring with identity and M , it is a unitary left R -module. Using a non-trivial submodule of M means that it is a non-zero proper submodule of M , see⁷. A submodule N ($N \leq M$) of M is named small in M (and written $N \ll M$), if for every submodule $L \leq M$, with $N + L = M$ implies that $L = M$. $L \leq M$ is said to be essential in M , symbolized as $L \trianglelefteq M$, if $L \cap N \neq 0$ for every non-zero submodule $N \leq M$, see⁷. Kosar⁸, et. al. called a submodule K generalized small (briefly, g-small) submodule of M if, for every essential submodule T of M such that $M = K + T$ implies that $T = M$, one can write $K \ll_g M$, see⁸ (it is called an e-small submodule of M and is indicated by $K \ll_e M$ by Zhou and Zhang⁹). Small submodules are generalized small submodules nonetheless; the converse is not true generally. M is named hollow [resp., generalized hollow]^{8,10}, if all proper submodules of M are small [resp., g-small] in M . Evidently, every hollow module is generalized hollow. The converse assertion is not always true. A submodule P of a module M is maximal if it is not properly contained in any other submodule of M . M is named local if it has a unique maximal submodule. M is local if it is hollow and finitely generated⁷. $Rad(M)$ is the Jacobson radical of M , and it is the intersection of all maximal submodules of M . If T is an essential and maximal submodule in M then T is called a generalized maximal submodule of M , see Definition 2 of⁸. The intersection of all generalized maximal submodules of M is called the generalized radical of M and is given the symbol $Rad_g(M)$ that is also known as the sum of all g-small submodules in M . Since $Rad(M)$ is the sum of all small submodules of M , it follows that $Rad(M) \leq Rad_g(M)$ for a module M see⁸. The module M is named simple if M has no proper submodules, besides M is termed semisimple if M is a direct sum of simple submodules. The socle of M , is indicated by $Soc(M)$, it is the sum of all simple submodules in M . Each definition in graph theory written in the following section has appeared in Bondy and Murty work¹¹.

Let Γ be a graph, then $V(\Gamma)$ and $E(\Gamma)$ denote the set of vertices and edges in Γ , respectively. Neighborhood of v indicated by $N(v)$ which is the set of vertices adjacent to vertex v of Γ . The order of Γ is the number of vertices of Γ , it indicates using $|\Gamma|$.

If $|\Gamma| < \infty$, then Γ is finite, otherwise, Γ is infinite. If u and v are adjacent vertices of Γ , then write $u - v$, i.e. $\{u, v\} \in E(\Gamma)$. The degree of a vertex v in Γ is indicated using $\deg(v)$, which is the number of edges incident with v . Let u, v be different vertices of Γ . A u, v -path is a path that starts with vertex u and ends in vertex v . For different vertices u and v , $d(u, v)$ is the least length of a u, v -path. If Γ has no such path, then $d(u, v) = \infty$. The diameter of Γ is referred to as $\text{diam}(\Gamma)$, it is the supremum of the set $\{d(u, v) : u \text{ and } v \text{ are different vertices of } \Gamma\}$. A cycle in Γ is a path of length through at least 3 different vertices and it begins and ends at the same vertex. The girth of Γ , is indicated using $\text{gr}(\Gamma)$, it is the length of the shortest cycle in Γ , provided Γ contains a cycle; else; $\text{gr}(\Gamma) = \infty$. A graph Γ is called connected if there is a path among all pairs of vertices of Γ . A tree is a connected graph that does not contain a cycle. A star graph is a tree consisting of one vertex adjacent to all the others. A graph is complete if it is connected with a diameter that is less than or equal to one. A complete graph with n distinct vertices is indicated by K_n . A clique of Γ is its maximal complete subgraph besides the number of vertices in the largest clique of graph Γ , and it is denoted by $\omega(\Gamma)$ and is called the clique number of Γ .

g-Small Submodules

Here, some assets of g-small submodules are introduced.

Lemma 1:^{9,10} Let M be a module. Then

- (1) For submodules $\mathcal{A}, K, L$ of M with $K \leq \mathcal{A}$, we get
 - (a) If $\mathcal{A} \ll_g M$, then $K \ll_g M$ and $\mathcal{A}/K \ll_g M/K$.
 - (b) $\mathcal{A} + L \ll_g M$ if and only if $\mathcal{A} \ll_g M$ and $L \ll_g M$.
- (2) If $W \ll_g M$ and $f: M \rightarrow N$ is a homomorphism, then $f(W) \ll_g N$. Specifically, if $W \ll_g M \leq N$, then $W \ll_g N$.
- (3) Let N, K, L , and T be submodules of M . If $K \ll_g L$ and $N \ll_g T$, then $K + N \ll_g L + T$.
- (4) Let $\mathcal{F}_1 \leq \mathcal{A}_1 \leq M$, $\mathcal{F}_2 \leq \mathcal{A}_2 \leq M$ and $M = \mathcal{A}_1 \oplus \mathcal{A}_2$. Then $\mathcal{F}_1 \oplus \mathcal{F}_2 \ll_g \mathcal{A}_1 \oplus \mathcal{A}_2$ if and only if $\mathcal{F}_1 \ll_g \mathcal{A}_1$ and $\mathcal{F}_2 \ll_g \mathcal{A}_2$.

Definition 1:^{9,10} Let M be a module. Define

$$Rad_g(M) = \bigcap \{N \trianglelefteq M \mid N \text{ is maximal of } M\}.$$

If M has no maximal essential submodules, then it is indicated by $Rad_g(M) = M$.

Clearly, $Rad(M) \leq Rad_g(M)$ and $Soc(M) \leq Rad_g(M)$. For an arbitrary ring R , let $Rad_g(R) = Rad_g({}_R R)$.

Lemma 2: (Lemma 1 of^[12]) The next assertions hold for a module M .

- (1) For every $a \in Rad_g(M)$, $Ra \ll_g M$.
- (2) If $N \leq M$, at that time $Rad_g(N) \leq Rad_g(M)$.
- (3) $Rad_g(M) = \sum_{N \ll_g M} N$.

Lemma 3:^[9] Let M and N be modules. Then

- (1) If $f: M \rightarrow N$ is a homomorphism, then $f(Rad_g(M)) \leq Rad_g(N)$.
- (2) If all proper essential submodule in M is contained in a maximal submodule in M , then $Rad_g(M)$ is a unique largest g-small submodule in M .

Remark 1: It is clear that, in general, $Rad_g(M)$ need not be g-small in M . Also, if M is a finitely generated module, i.e. all proper submodule of M is contained in a maximal submodule in M , then $Rad_g(M)$ is the unique largest g-small in M by Lemma 3(2).

Lemma 4:^[10] If $M = \bigoplus_{i \in I} M_i$ then $Rad_g(M) = \bigoplus_{i \in I} Rad_g(M_i)$.

Connectivity of $\Gamma_g(M)$

In this section, g-small intersection graphs of non-trivial submodules of certain modules are connected, completed, and described. In addition, the girth and the diameter of $\Gamma_g(M)$ are fixed. Generalizing the definition of Mahdavi and Talebi^[6] considering the graph $\Gamma_g(M)$ as follows:

Definition 2: The g-small intersection graph of non-trivial submodules of an R -module M , denoted by $\Gamma_g(M)$, is a simple undirected graph whose vertices are in one-to-one correspondence with all non-trivial submodules of M and two distinct vertices N and L that are adjacent if and only if $N \cap L \ll_g M$.

Let $\Gamma(M)$ denote the graph introduced by Mahdavi and Talebi^[6]. From definition 2 one has the following corollary.

Corollary 1: Let M be an R -module. Then the graph $\Gamma(M)$ is a subgraph of $\Gamma_g(M)$.

Proof: Let $\Gamma(M)$ be a graph of M with vertex set $V(\Gamma(M))$. It is clear that $V(\Gamma(M)) = V(\Gamma_g(M))$. Now, let $N, K \in V(\Gamma_g(M))$ such that N, K are adjacent in $\Gamma(M)$. So $N \cap K \ll M$. Since every small submodule is a g-small submodule one has $N \cap K \ll_g M$. Therefore, N, K are adjacent in $\Gamma_g(M)$. Hence, $\Gamma(M)$ is a subgraph of $\Gamma_g(M)$. $\square$

Example 1: Let $R = \mathbb{Z}$ and let $M = \mathbb{Z}_{24}$. Then, $V(\Gamma(M)) = V(\Gamma_g(M)) = \{M_1 = 2\mathbb{Z}_{24}, M_2 = 3\mathbb{Z}_{24}, M_3 = 4\mathbb{Z}_{24}, M_4 = 6\mathbb{Z}_{24}, M_5 = 8\mathbb{Z}_{24} \text{ and } M_6 = 12\mathbb{Z}_{24}\}$. From Example 2, $M_i \cap M_j \ll_g M$, for all $1 \leq i, j \leq 6$ hence M_i and M_j are adjacent in $\Gamma_g(\mathbb{Z}_{24})$ for all $1 \leq i, j \leq 6$. Thus $\Gamma_g(\mathbb{Z}_{24}) \cong K_6$. Whereas $\Gamma(\mathbb{Z}_{24})$ is isomorphic to the subgraph of K_6 , since M_1 and M_3 are not small submodules of M according to^[9], so $M_1 \cap M_3 = M_3$ is not a small submodule of M . Hence M_1 and M_3 are not adjacent in $\Gamma(\mathbb{Z}_{24})$. Thus, the graph $\Gamma(\mathbb{Z}_{24})$ is not a complete graph.

Proposition 1: Let M be any module. If any one of the following holds. then $\Gamma_g(M)$ is complete:

- (1) If $M = M_1 \oplus M_2$, where M_1 and M_2 are simple R -modules.
- (2) M is a generalized hollow.

Proof: (1) Suppose $M = M_1 \oplus M_2$ where M_1 and M_2 are two simple R -modules. So, $M_1 + M_2 = M$ and $M_1 \cap M_2 = \{0\}$. Then every non-trivial submodule of M is simple. Let $N, \mathcal{A}$ be two distinct vertices of $\Gamma_g(M)$. then they are simple and minimal non-trivial submodules of M . Also, $N \cap \mathcal{A} \leq N, \mathcal{A}$ and if $N \cap \mathcal{A} \neq (0)$, using minimality of N and $\mathcal{A}$ involves that $N = N \cap \mathcal{A} = \mathcal{A}$, which is a contradiction. Thus, $N \cap \mathcal{A} = (0) \ll_g M$, and so N and $\mathcal{A}$ are adjacent in $\Gamma_g(M)$ for all two distinct vertices $N, \mathcal{A}$ of $\Gamma_g(M)$. Hence $\Gamma_g(M)$ is a complete graph.

(2) Suppose M is a generalized hollow. Assume N_1 and N_2 are two vertices of the graph $\Gamma_g(M)$. Hence N_1 and N_2 are two non-zero g-small submodules of M . As $N_1 \cap N_2 \leq N_i, i = 1, 2$, by Lemma 1(2), $N_1 \cap N_2 \ll_g M$, and so N and $\mathcal{A}$ are adjacent in $\Gamma_g(M)$ for all distinct vertices $N, \mathcal{A}$ of $\Gamma_g(M)$. Hence $\Gamma_g(M)$ is complete. $\square$

The next corollary follows from Part 2 of Proposition 1.

Corollary 2: Let M be any module. Then the following statements hold:

- (1) If $V(\Gamma_g(M))$ is a totally ordered set, then the graph $\Gamma_g(M)$ is complete.
- (2) If M is a hollow (local) and $Rad_g(M) \neq M$, then the graph $\Gamma_g(M)$ is complete.
- (3) Every non-zero g -small submodule of M is adjacent to all vertices in $\Gamma_g(M)$ and the induced subgraphs on the sets of g -small submodules of M are cliques.

Proof: (1) Assume $V(\Gamma(M))$ is a totally ordered set. Then each two non-trivial submodules of M are comparable. Clearly, every non-trivial submodule of M is small (and g -small). Hence, M is a generalized hollow module. As a result, by Proposition 1(2), $\Gamma_g(M)$ is complete.

(2) Assume that M is a hollow (or local) module and $Rad_g(M) \neq M$. then M is a generalized hollow module¹⁰. So by Proposition 1(2), $\Gamma_g(M)$ is complete.

(3) Evident. $\square$

The next example shows that $\Gamma_g(M)$ is a complete graph, whereas $M = \mathbb{Z}_{24}$ is not generalized hollow as $N = 3\mathbb{Z}_{24}$ is not a g -small submodule of $M = \mathbb{Z}_{24}$, as in Example 2.13 of Zhou and Zhang⁹.

Example 2: Let $R = \mathbb{Z}$, $M = \mathbb{Z}_{24}$ as an R -module. There are only six non-trivial submodules $M_1 = 2\mathbb{Z}_{24} \ll_g M$, $M_2 = 3\mathbb{Z}_{24}$, $M_3 = 4\mathbb{Z}_{24} \ll_g M$, $M_4 = 6\mathbb{Z}_{24} \ll_g M$, $M_5 = 8\mathbb{Z}_{24} \ll_g M$ and $M_6 = 12\mathbb{Z}_{24} \ll_g M$, as in Zhou and Zhang⁹. Clearly, $M_i \cap M_j \ll_g M$, for all $1 \leq i, j \leq 6$. Thus, $\Gamma_g(M)$ is complete with 6 vertices, i.e., $\Gamma_g(M) \cong K_6$.

Example 3: To any prime number p for any $n \in \mathbb{Z}$, $n \geq 2$. $\mathbb{Z}_{p^n}$ is local $\mathbb{Z}$ -module, at that point it is hollow and so is generalized hollow. Also, let $R = \mathbb{Z}$, p be a prime besides $M = \mathbb{Z}_{p^\infty}$, the Prüfer p -group, now every $\mathfrak{B} \leq M$, $\mathfrak{B} \neq M$, $\mathfrak{B} \ll_g M$. Also, $Rad_g(M) = M$. Thus, for all prime numbers p , $\mathbb{Z}_{p^\infty}$ is a generalized hollow $\mathbb{Z}$ -module. Using Proposition 1(2), $\Gamma_g(\mathbb{Z}_{p^n})$ and $\Gamma_g(\mathbb{Z}_{p^\infty})$ are complete graphs.

Example 4: Let $\mathcal{P}$ be a finitely generated submodule of a $\mathbb{Z}$ -module $\mathbb{Q}$. Thus $\mathcal{P} \ll \mathbb{Q}$ (and so $\mathcal{P} \ll_g \mathbb{Q}$). Then it follows from Corollary 2(3), one has that the induced subgraph on the set of finitely generated submodules of $\mathbb{Q}$ are cliques in the graph $\Gamma_g(\mathbb{Q})$. Now to clarify, let $S = \{N_i | i \in I\}$ where S is a set of nonzero g -small submodules of $\mathbb{Q}$. Since $N_i \cap N_j \leq$

$N_i \ll_g \mathbb{Q}$ for all $i, j \in I$. By Lemma 1(a), $N_i \cap N_j \ll_g \mathbb{Q}$. Hence $N_i - N_j$ and so N_i and N_j are adjacent vertices of $\Gamma_g(\mathbb{Q})$. Therefore, the induced subgraph on the set S is a clique in $\Gamma_g(\mathbb{Q})$.

Now, some descriptions of $Rad_g({}_R R)$ and certain properties of R related to $Rad_g(R)$ are given.

Remark 2: For a ring R , each of the following sets is equal to $Rad_g(R)$:

- (1) R_1 = the largest g -small left ideal of R .
- (2) R_2 = the intersection of all essential maximal left ideals of R .

Proof: It follows by putting $M = R$. $\square$

Proposition 2: Let R be an integral domain with $0 \neq Rad_g(R)$. If M is a finitely generated torsion-free module and M has a proper essential submodule. Then $\Gamma_g(M)$ is connected also $\text{diam}(\Gamma_g(M)) \leq 2$.

Proof: Suppose M is finitely generated and it has a proper essential submodule according to Remark 1, $Rad_g(M) \ll_g M$. Also, by Remark 2, $Rad_g(R)$ is the largest g -small left ideal of R . By 21.12(4)⁷, $Rad_g(R)M \leq Rad(M)$, since $Rad(M) \leq Rad_g(M)$, so $Rad_g(R)M \leq Rad_g(M)$. Hence, $Rad_g(R)M \ll_g M$. Since $Rad_g(R) \neq 0$ and M is torsion-free then $Rad_g(R)M \neq 0$. Thus $Rad_g(R)M$ is a vertex of $\Gamma_g(M)$. Since $Rad_g(R)M \ll_g M$, then $Rad_g(R)M \cap X \ll_g M$ for every $0 \neq X \leq M$ by Lemma 1. So, there exists an edge between the vertex $Rad_g(R)M$ and X of $\Gamma_g(M)$. Also, for every two vertices X, Y in the graph $\Gamma_g(M)$, there exists a path $X - Rad_g(R)M - Y$ of length 2 in $\Gamma_g(M)$. This completes the proof. $\square$

Definition 3:⁸ A submodule $A \leq M$ is called a g -supplement of a submodule $N \leq M$ if $M = N + A$ and $N \cap A \ll_g A$ (so $N \cap A \ll_g M$). A is called a g -supplement submodule if A is a g -supplement of some submodule of M . M is called a g -supplemented module if all submodules of M have a g -supplement.

Proposition 3: Let $\mathcal{U} \leq M$. then any g -supplement to $\mathcal{U}$ is adjacent to $\mathcal{U}$ in $\Gamma_g(M)$.

Proof: Let $\mathcal{A}$ be a g -supplement of $\mathcal{U}$, $\mathcal{U} \leq M$. Hence $M = \mathcal{U} + \mathcal{A}$ and $\mathcal{U} \cap \mathcal{A} \ll_g \mathcal{A}$. According to

Lemma 1(2), $\mathcal{U} \cap \mathcal{A} \ll_g M$. Thus $\mathcal{A}$ adjacent to $\mathcal{U}$ in $\Gamma_g(M)$. $\square$

Proposition 4: $\Gamma_g(M)$ is connected and $\text{diam}(\Gamma_g(M)) \leq 2$ whenever M is g -supplemented.

Proof: Let N, L be submodules of M . As M is g -supplemented, now there is $\mathcal{A} \leq M$ with $N + \mathcal{A} = M$, $N \cap \mathcal{A} \ll_g \mathcal{A}$, and $N \cap \mathcal{A} \ll_g M$ by Lemma 1(2). One can consider two possible cases for $N \cap \mathcal{A}$.

Case 1: If $N \cap \mathcal{A} = (0)$, then $N \oplus \mathcal{A} = M$.

Now, if $L \leq N$, then $L \cap \mathcal{A} \ll_g M$. Thus $L - \mathcal{A} - N$ a path of length 2 in $\Gamma_g(M)$. If $L \leq \mathcal{A}$, at that point $L \cap N \ll_g M$. As a result of $N - L$ in the graph $\Gamma_g(M)$. Hence, $\Gamma_g(M)$ is connected and $\text{diam}(\Gamma_g(M)) \leq 2$.

Case 2: If $N \cap \mathcal{A} \neq (0)$. Since $N \cap \mathcal{A} \ll_g M$, thus $N - N \cap \mathcal{A} - L$ is a path of length 2 in $\Gamma_g(M)$. This ends the proof. $\square$

Lemma 5: For a module M :

- (1) Assume N is a finitely generated submodule in M and $N \leq \text{Rad}_g(M)$. Then $N \ll_g M$.
- (2) Assume N is a semisimple submodule in M with $N \leq \text{Rad}_g(M)$. Then $N \ll_g M$.

Proof: (1) Presume that $N \leq M$ is finitely generated, as a result, $N = \sum_{i=1}^r Rn_i$ for some $n_i \in N$, $1 \leq i \leq r$. Since $Rn_i \leq \text{Rad}_g(M)$, $Rn_i \ll_g M$, by Lemma 2. As a result, $N \ll_g M$, according to Lemma 1.

(2) Let $N + K = M$ for specific essential submodule K of M . As N is semisimple, there exists a $N' \leq N$ with $N = (N \cap K) \oplus N'$. As a result, $M = N + K = [(N \cap K) \oplus N'] + K = N' + K$. Since $N' \cap K = (N' \cap N) \cap K = N' \cap (N \cap K) = 0$.

Thus $M = N' \oplus K$. By Lemma 4, $\text{Rad}_g(M) = \text{Rad}_g(N') \oplus \text{Rad}_g(K) = \text{Rad}_g(K)$ since $\text{Rad}_g(N') \leq \text{Rad}_g(N) = 0$. Then $M = N + K \leq \text{Rad}_g(M) + K \leq K$. Thus $N \ll_g M$. $\square$

Proposition 5: Let M be a R -module and $\text{Rad}_g(M) \neq (0)$. Then the next conditions hold:

- (1) If N is a non-trivial direct summand submodule for M also $(0) \neq \text{Rad}_g(M) \ll_g M$, then there is at least one cycle of length 3 in $\Gamma_g(M)$.
- (2) If N is a non-trivial semisimple or finitely generated submodule in M contained in $\text{Rad}_g(M)$. Then $d(N, \text{Rad}_g(M)) = 1$ and $d(N, L) = 1$ for any non-trivial submodule L of M .

Proof: (1) As $K \leq M$ with $N \oplus K = M$, as N is a direct summand of M . Then $\text{Rad}_g(N) \oplus \text{Rad}_g(K) = \text{Rad}_g(M)$, according to Lemma 4. Since $\text{Rad}_g(N) \leq N$ besides $N \cap \text{Rad}_g(K) \leq N \cap K = (0)$, using the modular law, $\text{Rad}_g(M) \cap N = [\text{Rad}_g(K) + \text{Rad}_g(N)] \cap N = [\text{Rad}_g(K) \cap N] + \text{Rad}_g(N) = \text{Rad}_g(N)$. Thus, $\text{Rad}_g(M) \cap N = \text{Rad}_g(N)$. At that time $\text{Rad}_g(M) \cap N \ll_g M$. Also, $\text{Rad}_g(N) = N \cap \text{Rad}_g(N) \ll_g M$ besides $\text{Rad}_g(N) = \text{Rad}_g(N) \cap \text{Rad}_g(M) \ll_g M$ and one has, $d(N, \text{Rad}_g(M)) = 1$, $d(N, \text{Rad}_g(N)) = 1$ besides $d(\text{Rad}_g(N), \text{Rad}_g(M)) = 1$. Hence, $(N, \text{Rad}_g(N), \text{Rad}_g(M))$ is a cycle. Thus, there is at least one cycle of length 3 in $\Gamma_g(M)$.

(2) Assume N is a non-trivial semisimple or finitely generated submodule in M ; $N \leq \text{Rad}_g(M)$. Using Lemma 5, $N \ll_g M$. Since $N \cap L \leq N$ so, $L \cap N \ll_g M$ for every other non-trivial submodule L of M by Lemma 1(1). Hence $d(N, \text{Rad}_g(M)) = 1$ and $d(N, L) = 1$. $\square$

Proposition 6: If M has at least one non-zero g -small submodule, then $\Gamma_g(M)$ is a connected graph and $\text{diam}(\Gamma_g(M)) \leq 2$.

Proof: Take $\mathcal{F} \in \Gamma_g(M)$ a non-zero g -small submodule. Let A and B be two non-adjacent vertices of $\Gamma_g(M)$. Obviously, $A \cap \mathcal{F} \leq \mathcal{F} \ll_g M$, and $\mathcal{F} \cap B \leq \mathcal{F} \ll_g M$. By Lemma 1(1), $A \cap \mathcal{F} \ll_g M$ and $\mathcal{F} \cap B \ll_g M$. So, $A - \mathcal{F} - B$ is a path of length 2. So $\Gamma_g(M)$ is a connected graph and $\text{diam}(\Gamma_g(M)) \leq 2$. $\square$

Corollary 3: If M has a proper essential submodule. Then $\Gamma_g(M)$ is connected, if any one of the following holds.

- (1) M is finitely generated and $\text{Rad}_g(M) \neq M$.
- (2) If there exists a non-trivial submodule of M which is finitely generated or semisimple contained in $\text{Rad}_g(M)$.

Proof: (1) Assume M is finitely generated besides M has a proper essential submodule such that $\text{Rad}_g(M) \neq M$. According to Remark 1, $0 \neq \text{Rad}_g(M) \ll_g M$. By Proposition 6, $\Gamma_g(M)$ is a connected graph.

(2) It follows from Lemma 5 and Proposition 6. $\square$

Zhou and Zhang⁹ generalized the notion of socle of M to that of $\text{Soc}_s(M)$, $\text{Soc}_s(M) = \sum \{F \ll M \mid F$

is simple}. $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$.

Proposition 7: Let M be a module with the graph $\Gamma_g(M)$ and $Soc_s(M) \neq (0)$. Then the next statements hold:

- (1) $Soc_s(M)$ is adjacent to any other vertex in $\Gamma_g(M)$.
- (2) $d(Rad(M), Soc(M)) = 1$,
- (3) $\Gamma_g(M)$ is connected and $\text{diam}(\Gamma_g(M)) \leq 2$,

Proof: (1) According to Lemma 2¹³, $Soc_s(M) = Rad(M) \cap Soc(M)$. But $Soc(Rad(M)) = Rad(M) \cap Soc(M)$ by 21.2(2)⁷. Since by 2.8(9)¹⁴, $Soc(Rad(M)) \ll M$, at this time $Soc_s(M) \ll M$, so $Soc_s(M) \ll_g M$. Thus, $Soc_s(M) \cap \mathcal{D} \ll_g M$ for any submodule $\mathcal{D}$ of M . Hence, every other vertex in $\Gamma_g(M)$ is adjacent to $Soc_s(M)$.

(2) Using the proof of (1), $Rad(M) \cap Soc(M) = Soc_s(M) \ll_g M$. Thus, $d(Rad(M), Soc(M)) = 1$.

(3) It is clear from (1). $\square$

Proposition 8: If $\Gamma_g(M)$ has no isolated vertex, then $\Gamma_g(M)$ is connected and $\text{diam}(\Gamma_g(M)) \leq 3$.

Proof: Presume A, B is non-adjacent vertices in $\Gamma_g(M)$. As $\Gamma_g(M)$ has no isolated vertex, there exist submodules A_1 and B_1 with $A \cap A_1 \ll_g M$ and $B \cap B_1 \ll_g M$. Now, if $A_1 \cap B_1 \ll_g M$, then $A - A_1 - B_1 - B$ is a path of length 3. Otherwise $A - A_1 \cap B_1 - B$ is a path of length 2. As a result, $\Gamma_g(M)$ is a connected graph besides $\text{diam}(\Gamma_g(M)) \leq 3$. $\square$

Theorem 1: Let M be a semisimple module where M is not simple. then:

- (1) $\Gamma_g(M)$ has no isolated vertex.
- (2) $\Gamma_g(M)$ is connected and $\text{diam}(\Gamma_g(M)) \leq 3$.

Proof: (1) Let $X \in V(\Gamma_g(M))$. As M is a semisimple module, then by properties (20.2)⁷ all submodules in M are direct summand to M . As a result, there is $\mathcal{A} \leq M$ and $M = X \oplus \mathcal{A}$. From now $X \cap \mathcal{A} = (0) \ll_g M$ and so, there is an edge between vertex X of $\Gamma_g(M)$ and another vertex. At that time X is not an isolated vertex.

(2) By Proposition 8 and Part (1). $\square$

For module M , now use $\mathbb{S}_g(M)$ which indicates the set of all non-zero g-small submodules of M .

Proposition 9: Assume $n \in \mathbb{Z}^+$. In R -module M with $|\mathbb{S}_g(M)| = n$ and $|\Gamma_g(M)| \geq 2$:

- (a) If $\mathcal{H} \in \mathbb{S}_g(M)$, then $\deg(\mathcal{H}) \neq 0$.
- (b) $\omega(\Gamma_g(M)) \geq n$.

Proof: (a) Clear.

(b) Let $\mathbb{S}_g(M) = \{\mathcal{H} \mid \mathcal{H} \ll_g M\}$ and let $|\mathbb{S}_g(M)| = n$. The induced subgraph on the set $\mathbb{S}_g(M)$ is a complete subgraph of $\Gamma_g(M)$. $\omega(\Gamma_g(M)) \geq n$. $\square$

Theorem 2: Let $Rad_g(M)$ be a non-zero simple g-small submodule of M and let $|\Gamma_g(M)| \geq 2$. If $\Gamma_g(M)$ is a tree, then $\Gamma_g(M)$ is a star graph.

Proof: Since $Rad_g(M) \neq 0$, then $Rad_g(M)$ is a vertex in $\Gamma_g(M)$. Now, $Rad_g(M)$ is simple g-small, so $Rad_g(M)$ a unique non-zero g-small submodule of M . But, $\mathcal{S} \cap Rad_g(M) \ll_g M$, for any $\mathcal{S} \in V(\Gamma_g(M))$. Thus $\Gamma_g(M)$ contains a vertex $Rad_g(M)$ which is adjacent to all vertices. Now, presume $n \neq Rad_g(M)$ besides $m \neq Rad_g(M)$ are two distinct vertices of $\Gamma_g(M)$. Now, if $n \cap m \ll_g M$. Then $n - Rad_g(M) - m$, which is a conflict since $\Gamma_g(M)$ is a tree. Thus, $n \cap m$ is not g-small. As a result, n, m are not adjacent vertices. As a result, $\Gamma_g(M)$ is a star with center $Rad_g(M)$. $\square$

$\chi(\Gamma)$ is the smallest number of colors needed to color the vertices, $\chi(\Gamma)$ is called the chromatic number of Γ so that no two adjacent vertices share a similar color. By Theorem 2, One has the next corollary.

Corollary 4: Let $0 \neq Rad_g(M) \ll_g M$ and let $|\Gamma_g(M)| \geq 3$. Now the next are equivalent:

- (1) $\Gamma_g(M)$ is a star,
- (2) $\Gamma_g(M)$ is a tree,
- (3) $\chi(\Gamma_g(M)) = 2$,
- (4) $Rad_g(M)$ is a simple submodule of M besides all pairs of non-trivial submodules in M , have non-g-small intersection.

Proof: (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) The implications are clear.

(3) $\Rightarrow$ (4) On the contrary, suppose $0 \neq K \leq Rad_g(M)$ besides $K \ll_g M$. If $L \in V(\Gamma_g(M))$. Evidently, $(K, Rad_g(M), L)$ is a cycle in $\Gamma_g(M)$, contradicts $\chi(\Gamma_g(M)) = 2$. So, $Rad_g(M)$ is simple. Now, assume that X, Y belong to $V(\Gamma_g(M))$ with $X \cap Y \ll_g M$. $(X, Rad_g(M), Y)$ is a cycle in $\Gamma_g(M)$, which contradicts $\chi(\Gamma_g(M)) = 2$.

(4) $\Rightarrow$ (1) It is clear that $Rad_g(M)$ is adjacent to each other vertex. Now, suppose that $N \neq Rad_g(M)$ and $L \neq Rad_g(M)$ are two distinct vertices of $\Gamma_g(M)$, and N, L are adjacent. As a result, $X \cap Y \ll_g M$, a contradiction. Hence, $\Gamma_g(M)$ is a star graph. $\square$

Proposition 10: Let M be an R -module and $|\mathbb{S}_g(M)| \geq 1$. If $\Gamma_g(M)$ does not contain a cycle, then $\Gamma_g(M) \cong K_1$ or $\Gamma_g(M)$ is a star graph.

Proof: Suppose that the graph $\Gamma_g(M)$ contains no cycle. To prove $|\mathbb{S}_g(M)| < 2$, by the contrary way, let $X \ll_g M$ besides $Y \ll_g M$. As a result, $X + Y \ll_g M$ by Lemma 1, besides, $Y - (X + Y) - X$ is cycle, which is a contradiction. Then $|\mathbb{S}_g(M)| < 2$. As $|\mathbb{S}_g(M)| \geq 1$, at that time $|\mathbb{S}_g(M)| = 1$. Hence, M has a unique non-zero g -small submodule. Let $N \in \mathbb{S}_g(M)$. For every vertex L of $\Gamma_g(M)$, if $L = N$, then $\Gamma_g(M) \cong K_1$ and if $L \neq N$, as $L \cap N \ll_g M$, now deduce $\Gamma_g(M) \cong K_2$. Let $\Psi = \{v_i \mid v_i \neq N, i \in I\}$. Then every two arbitrary distinct vertices v_i and v_j , $i \neq j$, are not adjacent, and for $i \neq j$, $v_i - N - v_j$ is a path and $\Gamma_g(M)$ is a star graph. $\square$

Proposition 11: If $|\mathbb{S}_g(M)| \geq 2$, then $\Gamma_g(M)$ contains at least one cycle and $gr(\Gamma_g(M)) = 3$.

Proof: Suppose that $|\mathbb{S}_g(M)| \geq 2$. At that point, M has at least two non-zero g -small submodules, at a guess $\mathcal{C}_1$ besides $\mathcal{C}_2$. Since $\mathcal{C}_1 \cap \mathcal{C}_2 \leq \mathcal{C}_i$, for $i = 1, 2$, by Lemma 1, $\mathcal{C}_1 \cap \mathcal{C}_2 \ll_g M$. Also, $\mathcal{C}_1 \cap (\mathcal{C}_1 \cap \mathcal{C}_2) \ll_g M$ and $\mathcal{C}_2 \cap (\mathcal{C}_1 \cap \mathcal{C}_2) \ll_g M$. One considers two possible cases for $\mathcal{C}_1 \cap \mathcal{C}_2$.

Case 1: If $\mathcal{C}_1 \cap \mathcal{C}_2 \neq (0)$, at that point $d(\mathcal{C}_1, \mathcal{C}_2) = 1$, $d(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{C}_2) = 1$ besides $d(\mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2) = 1$. Thus $(\mathcal{C}_1, \mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{C}_2)$ is a cycle of length 3. Also by Lemma 1, $\mathcal{C}_1 + \mathcal{C}_2 \ll_g M$ and since $\mathcal{C}_1 \cap (\mathcal{C}_1 + \mathcal{C}_2) \ll_g M$ besides $\mathcal{C}_2 \cap (\mathcal{C}_1 + \mathcal{C}_2) \ll_g M$, $(\mathcal{C}_1, \mathcal{C}_1 + \mathcal{C}_2, \mathcal{C}_2)$ is a cycle of length 3. Similarly, $(\mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{C}_1, \mathcal{C}_1 + \mathcal{C}_2)$ and $(\mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{C}_2, \mathcal{C}_1 + \mathcal{C}_2)$ are cycles of length 3 and $(\mathcal{C}_1, \mathcal{C}_1 + \mathcal{C}_2, \mathcal{C}_2, \mathcal{C}_1 \cap \mathcal{C}_2, \mathcal{C}_1)$ is a cycle of length 4.

Case 2: If $\mathcal{C}_1 \cap \mathcal{C}_2 = (0)$, then $(\mathcal{C}_1, \mathcal{C}_1 + \mathcal{C}_2, \mathcal{C}_2)$ is a cycle of length 3 in $\Gamma_g(M)$. Thus, $\Gamma_g(M)$ contains at least one cycle and so $gr(\Gamma_g(M)) = 3$. $\square$

Domination and Planarity of $\Gamma_g(M)$

In this section, the domination of $\Gamma_g(M)$ is fixed. And the relationship between the planarity of $\Gamma_g(M)$

and the non-zero g -small submodules of R -module M is revised.

$D \subseteq V(\Gamma)$ is called a dominating set if all vertices not in D are adjacent to a vertex in D . The domination number, $\gamma(\Gamma)$, of Γ is the minimum cardinality of a dominating set of Γ . See, for instance¹⁵. Here, $D \subseteq V(\Gamma)$ is a dominating set if and only if for any non-trivial submodule N of M there is a $L \in D$ with $N \cap L \ll_g M$.

Lemma 6: Let M be a module such that $|\Gamma_g(M)| \geq 2$, now the following hold:

- (i) If $D \subseteq V(\Gamma_g(M))$ such that D either contains at least one g -small submodule of M or there is a vertex $X \in D$ which $X \cap Y = (0)$, for $Y \in V(\Gamma_g(M)) \setminus D$. At that time D is a dominating set in $\Gamma_g(M)$.
- (ii) If M has at least one non-zero g -small submodule, at that time for all $0 \neq X \ll_g M$, $\{X\}$ is a dominating set and $\gamma(\Gamma_g(M)) = 1$.

Proposition 12: Let $M = \mathcal{H} \oplus \mathcal{F}$ be an R -module, where $\mathcal{H}$ and $\mathcal{F}$ are simple R -modules. Then $\gamma(\Gamma_g(M)) = 1$.

Proof: Assume $M = \mathcal{H} \oplus \mathcal{F}$, such that $\mathcal{H}$ and $\mathcal{F}$ are simple modules. Using Proposition 1(1), $\Gamma_g(M)$ is complete. Assume $\mathcal{X}$ is a random vertex of $\Gamma_g(M)$. At that moment for any distinct vertex $\mathcal{Y}$ of $\Gamma_g(M)$, $\mathcal{X} \cap \mathcal{Y} \ll_g M$, thus $\{\mathcal{X}\}$ is a dominating set besides $\gamma(\Gamma_g(M)) = 1$. $\square$

Proposition 13: Assume M is finitely generated and $Rad_g(M) \neq 0$. Then $\{Rad_g(M)\}$ is a dominating set of $\Gamma_g(M)$ and $\Gamma_g(M)$ is connected.

Proof: Assume $L \in \Gamma_g(M)$. Now, L is adjacent to $Rad_g(M)$ if L is g -small. Besides, if L is not g -small. Since $Rad_g(M) \neq 0$ in the finitely generated module, then $Rad_g(M) \ll_g M$. Hence, $L \cap Rad_g(M) \ll_g M$. So, L is adjacent to $Rad_g(M)$. This suggests that $\{Rad_g(M)\}$ is a dominating set to $\Gamma_g(M)$ and so $\Gamma_g(M)$ is connected as vital. $\square$

Theorem 3: Let $|\mathbb{S}_g(M)| \geq 2$ and $|\Gamma_g(M)| \geq 3$. Then the following conditions hold:

- (1) If I and J are two g -small submodules of M then there is $\mathfrak{B} \in V(\Gamma_g(M))$, with $\mathfrak{B}$ belong to $N(I) \cap N(J)$.
- (2) A graph $\Gamma_g(M)$ has at least one triangle.

Proof: It is evidence. $\square$

Proposition 14: The next are equivalent for any module M :

- (1) $\Gamma_g(M)$ has no triangle.
- (2) If $\{I, J\} \in E(\Gamma_g(M))$, then there is no $\mathfrak{B}$ belonging to $V(\Gamma_g(M))$ with $\mathfrak{B}$ belong to $N(I) \cap N(J)$.
- (3) $|\mathbb{S}_g(M)| \leq 1$ and the intersection of every pair of the non-g-small non-trivial submodules of M is not g-small.

Proof: For (1) $\Rightarrow$ (2) Suppose that the graph $\Gamma_g(M)$ has no triangle. On the other hand, consider $\mathfrak{B} \in V(\Gamma_g(M))$ such that $\mathfrak{B} \in N(I)$ and $\mathfrak{B} \in N(J)$. It follows that $(I, \mathfrak{B}, J)$ is a triangle in $\Gamma_g(M)$, which is a conflict.

Let (2) $\Rightarrow$ (3) for every two adjacent vertices of the graph $\Gamma_g(M)$, there is no $\mathfrak{B} \in V(\Gamma_g(M))$ with $\mathfrak{B} \in N(I) \cap N(J)$. Let there exist at least two submodules $0 \neq \mathcal{H}_1 \ll_g M$ and $0 \neq \mathcal{H}_2 \ll_g M$. Since $\mathcal{H}_1 \cap \mathcal{H}_2 \ll_g M$, they are adjacent vertices of the graph $\Gamma_g(M)$ and so, there is no $\mathfrak{B} \in V(\Gamma_g(M))$ with $\mathfrak{B} \in N(I) \cap N(J)$, which is a contradiction by Theorem 3(1).

(3) $\Rightarrow$ (1) Assume 0 is the only g-small submodule of M . As the intersection of all pairs of non-g-small non-trivial submodules in M is not g-small, $\Gamma_g(M)$ contains no triangles. Besides, S is the only non-zero g-small submodule of M . At that time for every three arbitrary vertices N_1, N_2 besides N_3 of $\Gamma_g(M)$, at least two of them are not g-small. Let $S = N_1$. As $N_2 \cap N_3$ is not a g-small submodule of M , then $N_2 - S - N_3$ is a path. Also, if $S \neq N_i$, for $i = 1, 2, 3$. Since $N_i \cap N_j$ is not a g-small submodule in M , for $i \neq j$, $i, j = 1, 2, 3$, then in the graph $\Gamma_g(M)$, N_1, N_2 and N_3 are not adjacent vertices. Hence, there is no triangle in $\Gamma_g(M)$. $\square$

Results and Discussion

Some results are proven, such as the graph $\Gamma_g(M)$ is connected if M is g-supplemented or $\text{Soc}_s(M) \neq (0)$.

Conclusion

In this paper, an undirected graph $\Gamma_g(M)$, the g-small intersection graph of M where $V(\Gamma_g(M))$ are non-trivial submodules of M and two different vertices N and L are adjacent if and only if $N \cap L \ll_g M$ was introduced and studied. Here, $\Gamma_g(M)$ is complete if

Remark 3: Let M be a module with $\text{Rad}_g(M) \neq (0)$ and $|\Gamma_g(M)| \geq 3$. If M is finitely generated, then $\Gamma_g(M)$ has a triangle.

Proof: Straightforward. $\square$

Definition 4: A graph Γ is called planar if Γ can be drawn in the plane so that its edges intersect only at their ends.

Lemma 7: (Theorem 10.30)¹¹ Γ is planar if and only if it has no subdivision of either K_5 or $K_{3,3}$.

Proposition 15: If $|\mathbb{S}_g(M)| = 1$ or $|\mathbb{S}_g(M)| = 2$, then $\Gamma_g(M)$ is a planar graph, whenever the intersection of all pairs of non-g-small submodules in M is not a g-small.

Proof: If $|\mathbb{S}_g(M)| = 1$, then $\Gamma_g(M)$ has a vertex I which is adjacent to another vertex. According to the assumption, if $J \neq I$ and $K \neq I$ are two distinct vertices of $\Gamma_g(M)$, then J and K are not adjacent vertices. As a result, $\Gamma_g(M)$ is a star graph with the center I . Thus, $\Gamma_g(M)$ is planar. Now, if $|\mathbb{S}_g(M)| = 2$, then $\Gamma_g(M)$ does not contain K_5 or $K_{3,3}$ and by the definition of a planar graph in Lemma 7. $\square$

Proposition 16: For any module M , if $|\mathbb{S}_g(M)| \geq 3$, then $\Gamma_g(M)$ is not a planar graph.

Proof: Assume that $|\mathbb{S}_g(M)| \geq 3$. Now there are $0 \neq F \ll_g M$ and $0 \neq N \ll_g M$ and $0 \neq P \ll_g M$. Obviously, any one of the vertices $F + N$, $N + P$, and $F + P$ are non-zero submodules as well as adjacent to each of the submodules F , N , and P in $\Gamma_g(M)$. As a result, $\Gamma_g(M)$ contains a complete graph K_5 such as the subgraph induced on $\{F, N, P, F + N, N + P\}$. Thus, by the definition of a planar graph in Lemma 7, $\Gamma_g(M)$ is not a planar graph. $\square$

M is a generalized hollow module or M is a direct sum of two simple modules. Girth, diameter, domination, and planar property of the graph $\Gamma_g(M)$ are studied.

Authors' Declaration

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Thi-Qar.

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بيان تقاطع صغير من النمط – g للمقاس

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الخلاصة

لتكن R حلقة ابدالية أحادية، وليكن M مقاساً أبسر احادي. بيان تقاطع صغير من النمط- g للمقاس M يرمز له $\Gamma_g(M)$, هو بيان غير مباشر بسيط رؤوسه تقابل كل المقاسات الجزئية غير التافهة في M وكل رأسين مختلفين متجاورين إذا وفقط إذا كان التقاطع بينهما هو صغير من النمط- g . في هذا المقال، تم دراسة العلاقة بين الخصائص الجبرية الى M والخصائص البيانية الى $\Gamma_g(M)$. أعتبر خصائص مثل الترابط، الكمال. علاوة على ذلك، يتم تحدد القطر و الطوق الى $\Gamma_g(M)$ ، وكذلك أعطى صيغة لحساب الزمرة الى جانب أعداد الهيمنة الى $\Gamma_g(M)$.

الكلمات المفتاحية: الترابط، الهيمنة، مقاس، مقاس جزئي صغير، بيان تقاطع صغير.