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Principally ss-Supplemented Modules

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Abstract

In this paper, we introduce and study the concepts of principally ss-supplemented and principally ss-lifting modules. These two concepts are natural generalizations of the concepts of ss-supplemented and ss-lifting modules. Several properties of these modules are proven. Here, principally ss-lifting modules are focused on. New characterizations of principally ss-supplemented modules are made using principally ss-lifting modules. Here, weakly principally ss-supplemented is defined. It is proved that a module T is weakly principally ss-supplemented module if and only if it is principally ss-supplemented. One of the first results states that every strongly local module is principally ss-supplemented. It is shown that if T be a hollow module, then T is principally ss-supplemented if and only if it is strongly local. If $Rad(T)$ small in T, then T is principally ss-supplemented if and only if T is principally supplemented and $Rad(T) \subseteq Soc(T)$. Moreover, if $T = T_1 \oplus T_2$ with T_1 and T_2 principally ss-supplemented modules and T is a duo, then T is principally ss-supplemented. It is also shown that, if T is indecomposable, then T is principally ss-lifting if and only if T is a principally hollow module besides if T is a principally hollow module then T is principally ss-supplemented. In this work, the following results are proved: if T be a module with the property (ss - PD_1), then every indecomposable cyclic submodule of T is either small in T or a summand of T. Also, if T is a module over a local ring R and T has the property (ss-PD_1), then every cyclic submodule of T is either small in T, or a summand of T.

Keywords: Principally ss-supplemented module, Principally ss-lifting module, Ss-supplemented module, Ss-lifting module, Strongly local module.

Introduction

Throughout this paper, R will denote a commutative ring with identity and T be a unitary left R-module. A submodule N of an R-module T is called *small* or *superfluous* in T (we write $N \ll T$), if any submodule $X \leq T$, with $N + X = T$ involves that $X = T$. The radical of T, indicated by $Rad(T)$, is the sum of small submodules of T. T is called *hollow*, if any proper submodule of T is small in T, see for example[1,2]. A module T is named local, if it has a unique maximal submodule, i.e., a proper submodule which contains all other submodules. T is said to be *simple*, if it has no nontrivial submodule, T is called *semisimple* if T is a direct sum of simple submodules. The socle of T, indicated by $Soc(T)$, is the sum of all simple submodules in T. Let L and K be submodules of T. K is named *a supplement* of L in T if it is minimal with respect to $T = L + K$. A submodule K of T is a supplement of L in T if and only if $T = L + K$ and $L \cap K \ll K$. A module T is called *supplemented* if each $L \leq T$ has a supplement in T.

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A submodule $K \leq T$ has ample supplements in T if every $L \leq T$ with $T = K + L$ contains a supplement of K in T. A module T is called amply supplemented if each submodule of T has ample supplements in T. Local modules are hollow, and hollow modules are clearly amply supplemented[1]. A module T is called *lifting* if, for all $N \leq T$, there exists a decomposition $T = A \oplus B$ with $A \leq N$ and $N \cap B$ is small in T, see[1,3]. Kaynar, Calisici and Türkmen[4] define ss-supplemented modules as a proper generalization of semisimple modules. T is named *ss-supplemented* if any submodule U of T has a supplement V in T with $U \cap V$ is semisimple (namely, ss-supplements). Zhou and Zhang[5] generalized the notion of socle of T to $Soc_s(T)$, $Soc_s(T) = \sum\{L \ll T | L \text{ is simple}\}$. Clearly $Soc_s(T) \subseteq Rad(T)$, $Soc_s(T) \subseteq Soc(T)$, where $Soc_s(T) = Soc(T) \cap Rad(T)$. Kaynar et.al[4]. call T is strongly local if it is local and $Rad(T) \subseteq Soc(T)$, as well T is called ss-supplemented if every submodule K of T there exists M in T with $T = K + M$ and $K \cap M \subseteq Soc_s(T)$. For more details on ss-supplemented modules see[6,7]. Eryilmaz[8] calls a module T called *ss-lifting* if for any $A \leq T$, $T = T_1 \oplus T_2$ with $T_1 \leq A$ and $A \cap T_2 \subseteq Soc_s(T)$.

Inspired by these results, here, the notion of principally ss-supplemented (principally ss-lifting) modules is introduced and studied. Section 2 is devoted to introducing the notion of principally ss-supplemented modules. Here, for a projective module T with $Rad(T) \ll T$ and $Rad(T) \subseteq Soc(T)$, T is principally ss-supplemented if and only if $T/Rad(T)$ is principally semisimple are proved. Section 3 discusses principally ss-lifting modules.

Principally ss-Supplemented Modules

In this section, principally ss-supplemented modules which generalize principally supplemented modules and ss-supplemented modules are introduced.

For the local $\mathbb{Z}$-module $T = \mathbb{Z}_8$, $\text{Soc}(T) \subset Rad(T)$. So T is not a strongly local module.

Proposition 1: Every factor module of strongly local module is strongly local.

Proof: See Proposition 8 in[4]. □

Lemma 1: Suppose that T is an R-module. Then $Soc(Rad(T)) \ll T$.

Proof: See (2.8(9))[1]. □

Lemma 2: Let T be a module and $x \in T$ ($X \leq T$). Then

(1) If Rx is a semisimple submodule in T with $Rx \leq Rad(T)$. Then $Rx \ll T$.

(2) If X is a semisimple submodule in T with $X \leq Rad(T)$. Then $X \ll T$.

Proof: Suppose Rx is a semisimple submodule of T. Then $Soc(Rx) = Rx$ and since $Rx \subseteq Rad(T)$, $Soc(Rx) \subseteq Soc(Rad(T))$. By Lemma 1, $Soc(Rad(T)) \ll T$. By Lemma 5.1.3 of [9], $Rx \ll T$. □

Lemma 3: Let T be a module. Then $Soc_s(T) = Rad(T) \cap Soc(T)$.

Proof: See Lemma 2 of[4]. □

Definition 1:[10] A module T is called a principally supplemented module if for all $t \in T$ there exists $A \leq T$, $T = Rt + A$ and $(Rt) \cap A \ll A$. A module T is called a weakly principally supplemented module if for all $t \in T$ there exists a submodule A with $T = Rt + A$ and $(Rt) \cap A \ll T$.

Now one gives the definition of principally ss-supplemented modules.

Definition 2: A module T is called principally ss-supplemented if for every cyclic submodule U in T, there exists a submodule V of T with $T = U + V$, $U \cap V \ll V$ and $U \cap V$ is semisimple. Also, a module T amply principally ss-supplemented if every cyclic submodule of T has ample ss-supplements in T.

Lemma 4: Let T be a module and U, V be submodules of T. Then the following statements are equivalent:

(1) $T = U + V$ and $U \cap V \subseteq Soc_s(V)$.
(2) $T = U + V$ and $U \cap V \ll V$ and $U \cap V$ is semisimple.
(3) $T = U + V$, $U \cap V \leq Rad(V)$ and $U \cap V$ is semisimple.

Proof: See Lemma 3 of[4]. □

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By[4] one says that V an ss-supplement of U in T if the equal conditions in the above lemma are satisfied. It is clear that the following implications on submodules in a module hold[4]:

Direct summand $\Longrightarrow$ ss-supplement $\Longrightarrow$ supplement $\Longrightarrow$ Rad-supplement.

One call T ss-supplemented if every submodule of T has ss-supplement in T.

Definition 3: Let N be a cyclic submodule of T. A submodule $L \leq T$ is called a principally ss-supplement of N in T if N and L satisfy the conditions in Lemma 4 and the module T is called principally ss-supplemented if every cyclic submodule of T has a principally ss-supplement in T.

Ss-supplemented module is supplemented, so principally ss-supplemented module is principally supplemented. A module T is called a weakly principally supplemented module if for all $t \in T$ there exists a submodule A with $T = Rt + A$ and $(Rt) \cap A \ll T$ and $(Rt) \cap A$ is semisimple. By Lemma 4, T is weakly principally ss-supplemented if and only if it is principally ss-supplemented.

Example 1: Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module of rational numbers, $\mathbb{Q}$ contains no maximal submodules. If $F \leq \mathbb{Q}$ is cyclic then $F \ll \mathbb{Q}$, and so $\mathbb{Q}$ is principally ss-supplemented. Note that $\mathbb{Q}$ is not supplemented and it is not ss-supplemented.

Proposition 2: Let T be a module and U be a maximal submodule in T. $V \leq T$ is ss-supplement to U in T if and only if $T = U + V$ and V is strongly local.

Proof: Suppose V is an ss-supplement to U. Note U is local and $U \cap V = Rad(V)$ is unique maximal submodule of V. As $U \cap V$ is semisimple, now $Rad(V) \subseteq Soc(V)$. Thus V is strongly local. Conversely, since V local and $T = U + V$, one can write $U \cap V \subseteq Rad(V)$. The hypothesis give $L \cap V$ is semisimple. Thus V is an ss-supplement of U. □

Lemma 5: (Lemma 13 of[4]) Let T be an ss-supplemented module and $N \ll T$. Then $N \subseteq Soc_s(T)$.

Corollary 1: (Corollary 14 of[4]) Let T be an ss-supplemented module and $Rad(T) \ll T$. Then $Rad(T) \subseteq Soc(T)$.

Proposition 3: Every strongly local module is principally ss-supplemented.

Proof: Assume T is local as a result it is principally supplemented. Since $Rad(T) \subseteq Soc(T)$, so T is principally ss-supplemented. □

The converse of Proposition 3 is not true in general as in Example 3.

Proposition 4: (Proposition 16 of[4]) Let T be a hollow module. Then T is ss-supplemented if and only if it is strongly local.

Now, the following main result is proved:

Theorem 1: Let T be a hollow module. Then T is principally ss-supplemented if and only if it is strongly local.

Proof: Suppose T is principally ss-supplemented. Let $x \in Rad(T)$. Now, by Lemma 5.1.4 of[9], $Rx \ll T$. As T is principally ss-supplemented, by Lemma 5, $Rx \subseteq Soc_s(T)$. So $x \in Soc(T)$, and so $Rad(T) \subseteq Soc(T)$. Suppose $T = Rad(T)$. Since $T = Rad(T) = Soc(T)$, and the radical of a semisimple module is zero, one gets $T = 0$, a contradiction since T is hollow. So $T \neq Rad(T)$, that is, T is local. So T is strongly local. The converse is clear from Proposition 3. □

Example 2: For p is prime integer, $r \geq 3$. Let $T = \mathbb{Z}_{p^r}$ be a $\mathbb{Z}$-module. $Soc_s(\mathbb{Z}_{p^r}) = Soc(\mathbb{Z}_{p^r}) \cong \mathbb{Z}_p$ and $Rad(T) = p\mathbb{Z}p^r$, T is not strongly local and so it is not (principally) ss-supplemented.

Lemma 6: Let T be a principally supplemented module and $Rad(T) \subseteq Soc(T)$. Then T is principally ss-supplemented.

Proof: Let $L \leq T$. As T is principally supplemented, there exists $M \leq T$ with $T = L + M$ and $L \cap M \ll M$. Then $L \cap M \subseteq Rad(M) \subseteq Rad(T) \subseteq Soc(T)$. Thus $L \cap M$ is semisimple. So M is a principally ss-supplemented of L in T. □

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Theorem 2: If $Rad(T) \ll T$, then the following statements are equivalent.

(a) T is principally ss-supplemented,

(b) T is principally supplemented and $Rad(T)$ has a principally ss-supplement in T,

(c) T is principally supplemented and $Rad(T) \subseteq Soc(T)$.

Proof: Now (a)⟹(b) Obvious. (b)⟹(c) By Lemma 5. (c)⟹(a) By Lemma 6. □

Theorem 3: Let $T = \oplus_{i\in I} T_i$, where each T_i is a strongly local module. Then T is principally ss-supplemented.

Proof: By using Theorem 27 in[4]. □

Now we give an example of a principally ss-supplemented (ss-supplemented) module which is not strongly local.

Example 3: The $\mathbb{Z}$-module $T = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ is ss-supplemented as a sum of strongly local modules see Example 25 of[4]. So, T is principally ss-supplemented. However, M is not (strongly) local by Example 25 of[4].

A submodule $N \leq T$ is called fully invariant if for each endomorphism f to T, $f(N) \leq N$. Ozcan, Harmanci and Smith[11] call a module T is called a duo module provided any submodule of T is fully invariant. A module T is distributive if for all submodules $\mathcal{K}$, $\mathcal{X}$, and $\mathcal{P}$, $\mathcal{P} \cap (\mathcal{K} + \mathcal{X}) = \mathcal{P} \cap \mathcal{K} + \mathcal{P} \cap \mathcal{X}$ or $\mathcal{P} + (\mathcal{K} \cap \mathcal{X}) = (\mathcal{P} + \mathcal{K}) \cap (\mathcal{P} + \mathcal{X})$ (see for example[10]).

A finite direct sum of ss-supplemented modules is ss-supplemented, but this is not the case for principally ss-supplemented. However, it is the case to certain classes of modules.

Theorem 4: Let $T = T_1 \oplus T_2$ with T_1 and T_2 principally ss-supplemented modules. If T is duo, then T is principally ss-supplemented.

Proof: Analogous to proof of Theorem 9 of[10]. □

Theorem 5: Let T be a principally ss-supplemented and duo. Then any direct summand to T is a principally ss-supplemented.

Proof: Analogous to proof of Theorem 10 of[10]. □

Lemma 7: Let $T = T_1 \oplus T_2 = K + N$ and $K \leq T_1$. If T is distributive and $K \cap N \ll N$, then $K \cap N \ll T_1 \cap N$.

Proof: Let $T_1 \cap N = (K \cap N) + L$ with $L \leq T_1 \cap N$. Since T is distributive, $N = T_1 \cap N \oplus T_2 \cap N$. One has $T = K + N = K + T_1 \cap N + T_2 \cap N = K + L + (T_2 \cap N)$ and $N = K \cap N + L + (T_2 \cap N)$. Now $N/(L \oplus (T_2 \cap N)) = ((N \cap T_1) \oplus (N \cap T_2))/(L \oplus (T_2 \cap N)) \cong (N \cap T_1)/L$. Hence $N = L \oplus (T_2 \cap N)$. Thus $N = (N \cap T_1) \oplus (N \cap T_2)$ and $L \leq T_1 \cap N$ imply $L = T_1 \cap N$. So $K \cap N \ll T_1 \cap N$. □

Theorem 6: Let T be a principally ss-supplemented distributive module. Then every direct summand of T is a principally ss-supplemented module.

Proof: Assume $T = T_1 \oplus T_2$ and $x \in T_1$. Now, there is $N \leq T$, $T = Rx + N$ and $Rx \cap N$ is small in N. Then $T_1 = Rx + (T_1 \cap N)$ and by Lemma 7, $Rx \cap (T_1 \cap N)$ is small in $T_1 \cap N$. □

Proposition 5: Let T_1 and T_2 be principally ss-supplemented modules and $T = T_1 \oplus T_2$. If T is distributive, then T is principally ss-supplemented.

Proof: Let $T = T_1 \oplus T_2$ be a distributive module and $Rx \leq T$. Then $Rx = (Rx \cap T_1) \oplus (Rx \cap T_2)$. Since $Rx \cap T_1$ and $Rx \cap T_2$ are cyclic submodules of T_1 and T_2 respectively, there is $M \leq T_1$ with $T_1 = (Rx \cap T_1) + M$ and $M \cap (Rx \cap T_1) = M \cap Rx$ is small in M, $M \cap Rx$ is semisimple, and $N \leq T_2$ such that $T_2 = (Rx \cap T_2) + N$, $N \cap (Rx \cap T_2) = N \cap Rx$ is small in N, $N \cap Rx$ is semisimple. Then $T = Rx + M + N$.

One now claims that $Rx \cap (M + N) = (Rx \cap M) + (Rx \cap N)$. The inclusion $(Rx \cap M) + (Rx \cap N) \leq Rx \cap (M + N)$ always holds. For the inverse inclusion, $Rx \cap (M + N) \leq M \cap (Rx + N) + N \cap (Rx + M) = M \cap ((Rx \cap T_1) + T_2) + N \cap (T_1 + (Rx \cap T_2))$. On the other hand $M \cap ((Rx \cap T_1) + T_2) \leq (Rx \cap T_1) \cap (M + T_2) + T_2 \cap ((Rx \cap T_1) + M) = Rx \cap M$. Similarly, $N \cap (T_1 + (Rx \cap T_2)) \leq Rx \cap N$. Hence $(Rx \cap (M + N) \leq Rx \cap M + Rx \cap$

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N. Therefore, the claim $(Rx \cap (M+N) = Rx \cap M + Rx \cap N$ is defensible. Since $Rx \cap M \ll M$ and $Rx \cap N \ll N$, so, one has $Rx \cap (M+N) \ll M+N$. But $Rx \cap (M+N)$ is semisimple. Hence T is principally ss-supplemented. □

A module T is named a principally semisimple if any cyclic submodule is a direct summand to T. See[10]. However, one can say that (semisimple module $\Longrightarrow$ principally semisimple module). Any principally semisimple module is principally ss-lifting, and as a result principally ss-supplemented.

Lemma 8: Let T be a principally ss-supplemented distributive module. Then $T/Rad(T)$ is a principally semisimple module.

Proof: Take $\bar{a} \in T/Rad(T)$. There exists $N^{\alpha} \leq T$, $T = Ra + N^{\alpha}$ besides $Ra \cap N^{\alpha} \ll N^{\alpha}$, so $Ra \cap N^{\alpha} \ll T$ and $Ra \cap N^{\alpha}$ is semisimple. The distributivity of T implies $R\bar{a} \cap (N^{\alpha} + Rad(T)) = (Ra \cap T) + Ra \cap Rad(T) = Rad(T)$. Since $Ra \cap N^{\alpha} \leq Rad(T)$. Thus

$T/Rad(T) = [(Ra + Rad(T))/Rad(T)] + [(N^{\alpha} + Rad(T))/Rad(T)] = [R\bar{a}/Rad(T)] \oplus [(N^{\alpha} + Rad(T))/Rad(T)]$. Also, thus every principal submodule in $T/Rad(T)$ is a direct summand. □

A module T is called refinable if for submodules U, V of T with $T = U + V$ there is a direct summand U' of T with $U' \leq U$ and $T = U' + V$ (See for instance[10]).

Theorem 7: Let T be a projective module with $Rad(T)$ is small in T and $Rad(T) \subseteq Soc(T)$. Consider following conditions:

(1) T is principally ss-supplemented.
(2) $T/Rad(T)$ is principally semisimple.

Then (1) ⇒ (2). If T is refinable module then (2) ⇒ (1).

Proof: (1) ⇒ (2) Since T is a principally ss-supplemented module, so $T/Rad(T)$ is principally semisimple by Lemma 8.

(2)⇒ (1) Let Rt be any cyclic submodule of T. Using (2) There exists $U \leq T$ with

$T/Rad(T) = [(Rt + Rad(T))/Rad(T)] \oplus [U/Rad(T)]$.

Then $T = (Rt) + U$ and $(Rt + Rad(T)) \cap U = (Rt) \cap U + Rad(T) = Rad(T)$. Since $T = (Rt) + U$, being T refinable there exists a direct sumand A to T with $A \leq U$ and $T = (Rt) + U = (Rt) + A = B \oplus A$. $(Rt) \cap U \ll T$ so it is small in U since U is summand. But, $(Rt) \cap U \subseteq Soc(T)$. This completes the proof. □

Principally ss-Lifting Modules

In this section, principally ss-lifting modules which generalize principally lifting modules are introduced. In 2006, Clark, Lomp, Vanaja and Wisbauer[1] introduced the concept of principal lifting modules. In 2007, the principally lifting modules were considered by Kamal and Yousef[12] as follows:

Definition 4: A module T is called principally lifting (or has (PD_1)) if for all $a \in T$, $T = N \oplus S$ with $N \leq Ra$ and $Ra \cap S \ll T$, (See[1]).

Similar to (22.7)[1] the following definition is introduced.

Definition 5: A module T is called principally ss-lifting (or has (ss-PD_1) for short) if for all $a \in T$, $T = N \oplus S$ with $N \leq Ra$ and $Ra \cap S \subseteq Soc_s(T)$, where $Soc_s(T) = Soc(T) \cap Rad(T)$.

Definition 6: A nonzero module T is called a principally hollow (P-hollow) if every proper cyclic submodule is small in T. Observe that every P-hollow module satisfies the condition (ss-PD_1), (See for example[1,12]).

Similar to Proposition 2.7 of[12], the following proposition is introduced.

Proposition 6: The condition (ss-PD_1) is inherited by summands.

Proof: Let T have condition (ss-PD_1) and B a direct sumand to T, if $k \in B$, now T has a decomposition $T = N \oplus S$, $N \leq Rk$ and $Rk \cap S \ll T$. It follow that $B = N \oplus (B \cap S)$, and $Rk \cap (B \cap S) \leq Rk \cap S \ll T$, so $Rk \cap (B \cap S) \ll B$ (due to B a summand of T). So, B has (ss-PD_1). □

It is known that an indecomposable module is lifting if and only if it is a hollow module; the next Lemma gives an similarity to this fact.

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Lemma 9: The following are equivalent for an indecomposable module T:

(1) T has (ss-PD_1).
(2) T is a P-hollow module.
Proof: Follows directly from the defining condition of (ss-PD_1). □

Lemma 10: The following are equivalent for a module T.

(1) T has (ss-PD_1).
(2) Every cyclic submodule C in T can written by $C = N \oplus S$ with N is a direct summand in T and $S \ll T$.
(3) For each $a \in T$, there is principal ideals I and J in R with $Ra = Ia \oplus Ja$, where Ia is a direct summand in T and $Ja \ll T$.

Proof: (1)⇒ (2) It is obvious.
(2)⇒ (1) Lease C be a cyclic submodul of T, now by (2) $C = N \oplus S$ with N is a sumand in T besides $S \ll T$. Write $T = N \oplus N'$, it folows that $C = N \oplus C \cap N'$. Now take $\pi: N \oplus N' \to N'$ be a natural projection, one get $C \cap N' = \pi(C) = \pi(N \oplus S) = \pi(S) \ll T$. So T has (ss-PD_1).
(2)⇒ (3) Obvious. □

Clearly, all ss-supplemented module and all ss-lifting module, and so any principally ss-lifting module is principally ss-supplemented. However, there is principally ss-supplemented (ss-lifting) but not ss-supplemented.

Lemma 11: Let T be an indecomposable module. Consider following conditions:
(1) T is principally ss-lifting.
(2) T is a principally hollow module.
(3) T is principally ss-supplemented.
Then (1) ⇔ (2) and (2) ⇒ (3).
Proof: (1) ⇔ (2) Using Lemma 9.
(2)⇒ (3) Let $a \in T$. Using (2) every cyclic submodule is hollow. Now $T = Ra + T$ and $(Ra) \cap T \ll T$. □

Reminder that Lemma 11 (3) ⇒ (2) does not hold in general since there is an indecomposable principally supplemented but not principally hollow as in Example 15 of[10].

Lemma 12: Let T be a module with (ss-PD_1). Then every indecomposable cyclic submodule C of T is either small in T or a summand of T.

Proof: Similar to proof of Lemma 2.16 in[12]. □

Proposition 7: Let T be a module over a local ring R. If T has (ss-PD_1), then every cyclic submodule is either small in T, or a summand of T.

Proof: The proof follows from Lemma 12, and the fact that every cyclic module over a local ring is a local module. □

Results and Discussion

Some results are proved, such as if T_1 and T_2 are principally ss-supplemented modules with $T = T_1 \oplus T_2$ and T is distributive, then T is principally ss-supplemented.

Conclusion

In this paper, a principally ss-supplemented and principally ss-lifting modules was introduced and studied. Here, observed that if $T = T_1 \oplus T_2$ with T_1 and T_2 principally ss-supplemented modules and T is a duo, then T is principally ss-supplemented. Besides, if T is indecomposable, then T is principally ss-lifting if and only if T is a principally ss-supplemented module.

Authors' Declaration

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee at University of Thi-Qar.

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المقاسات التكميلية الرئيسية من النمط –ss

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الخلاصة

في هذا البحث، قدمنا ودرسنا مفاهيم المقاسات التكميلية الرئيسية من النمط-ss مع مقاسات الرفع من النمط-ss. هذان المفهومان هي تعميمات طبيعية للمقاسات التكميلية من النمط-ss مع مقاسات الرفع من النمط-ss. تم برهان العديد من خصائص هذه المقاسات. هنا تم التركيز على مقاسات الرفع من النمط-ss. تم الحصول على صفات جديدة للمقاسات التكميلية من النمط-ss باستخدام مقاسات الرفع من النمط-ss. هنا, عرفت مقاسات تكميلية رئيسية من نمط-ss ضعيفة. تم برهان المقاس T مقاس تكميلي رئيسي ضعيف من نمط-ss اذا وفقط اذا كان هومقاس تكميلي رئيسي من نمط-ss. واحدة من النتائج الأساسية تنص كل مقاس محلي بقوة هو تكميلي رئيسي من نمط-ss. تم اثبات اذا كان T مقاس مجوف, فأن T تكميلي رئيسي من نمط-ss اذا وفقط اذا كان محلي بقوة. اذا كان $Rad(T)$ صغير في T فان تكميلي رئيسي من نمط- ss اذا وفقط اذا T تكميلي رئيسي و $Rad(T) \subseteq Soc(T)$. بالاضافة, اذا $T = T_1 \oplus T_2$ مع T_1 و T_2مقاسان تكميليين رئيسيان من نمط-ss و T هي ديو, فأن T تكميلي رئيسي من نمط-ss. كذلك اثبت ذلك, أذا كانت T غير قابل للتحلل, فأن T رفع رئيسي من نمط-ss أذا وفقط أذا كان T مقاس مجوف رئيسي كذلك أذا كانت T مقاس مجوف رئيسي فأن T تكميلي رئيسي من نمط-ss . في هذا العمل, اثبتت النتائج التالية: أذا كانت T مقاس مع خاصية (ss -PD_1), فأن كل مقاس جزئي دوار غير قابل للتحلل في T هو اما صغيرفي T أو مجموع الى T. كذلك, أذا كانت T مقاس على حلقة محلية R و تمتلك خاصية (ss -PD_1), فأن كل مقاس جزئي دوار في T هو اما صغيرفي T أو مجموع الى T.

الكلمات المفتاحية: مقاس تكميلي رئيسي من نمط-أس أس، مقاس رفع رئيسي من نمط-أس أس، مقاس تكميلي من نمط-أس أس، مقاس رفع من نمط-أس أس، مقاس محلي بقوة.