

## Some Results about Acts over Monoid and Bounded Linear Operators

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Received 23/05/2023, Revised 10/09/2023, Accepted 12/09/2023, Published Online First 20/02/2024



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### Abstract

This study delves into the properties of the associated act  $V$  over the monoid  $S$  of  $\text{sinsh}T$ . It examines the relationship between faithful, finitely generated, and separated acts, as well as their connections to one-to-one and onto operators. Additionally, the correlation between acts over a monoid and modules over a ring is explored. Specifically, it is established that  $V_{\text{sinsh}T}$  functions as an act over  $S$  if and only if  $V_{\text{sinsh}T}$  functions as module, where  $T$  represents a nilpotent operator. Furthermore, it is proved that when  $T$  is onto operator and  $V_{\text{sinsh}T}$  is finitely generated,  $V$  is guaranteed to be finite-dimensional. Prove that for any bounded operator the following,  $V_{\text{sinsh}T}$  is acting over  $S$  if and only if  $V_{\text{sinsh}T}$  is a module where  $T$  is a nilpotent operator,  $V_{\text{sinsh}T}$  is a faithful act over  $S$ , where  $T$  is any bounded linear operator, if  $T$  is any bounded operator, then  $V_{\text{sinsh}T}$  is separated, if  $V_{\text{sinsh}T}$  is separated act over  $S$ , Then  $T$  is injective, if a basis  $K = \{v_j, j \in \Lambda\}$  for  $V$ , then every element  $w$  of  $V_{\text{sinsh}T}$  can be composed as  $w = \lim_{n \rightarrow \infty} \left( \sum_{i=0}^n \frac{(T)^i}{i!} + \sum_{i=0}^n \frac{(-T)^i}{i!} \right) \sum_{j \in \Lambda} a_j v_j = \lim_{n \rightarrow \infty} (p_n(T) + p_n(-T)) \cdot v$ , for some  $v$  in  $V$ , and put  $T$  as similar to any operator  $\mathfrak{D}$  from  $\mathfrak{N}$  to  $\mathfrak{N}$ , and  $V$  as a finite dimensional normed space, then  $V_{\text{sinsh}T}$  is Noetherian act over  $S$  if  $S$  is Noetherian.

**Keywords:** Associated act  $V$  over monoid of  $\text{sinsh}T$ , Bounded linear operator, Faithful act over monoid, One-to-one operator, Separated acts over monoid.

### Introduction

Consider a nonempty set  $A$  and a monoid  $S$ . Let  $\mu: A \times S \rightarrow A$  be defined as  $\mu(a, s) = (a, s) \mapsto as$ , such that  $(as)t = a(st)$ . This leads to  $a = a \cdot 1$ , where  $s, t \in S$ , and  $a \in A$ .  $A_s$  being a right act<sup>1-3</sup>. Moving forward, let us examine a Hilbert space  $H$  over a field  $F$  (where  $F$  can be either real or complex), and let  $T$  be a bounded linear operator on  $H$ . The exponential operator  $e^T$  is  $e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!}$ , where  $T^0 = I$ , the identity operator on  $H$ . The exponential operator is well-defined, the sum exists<sup>4</sup>. Flowing<sup>5</sup>, consider a polynomial ring  $R = F[x]$  with

coefficients in  $F$ . Define the function  $\emptyset: R \times V \rightarrow V$  such that  $p(T)v = p \cdot v = \emptyset(p, v)$ , where  $T$  is a linear operator. Here,  $\emptyset$  transforms  $V$  into a left  $R$ -module, denoted as  $V_T$ . When a bounded operator  $T$  on a Banach-space act as  $V$  over a field  $F$ , and  $S = \{e^x: x \in R\}$  represents the semi-group, the function  $\mu: S \times V \rightarrow V$  is  $e^T(v) = \mu(e^x, v)$ , which establishes  $V$  as a left act over the monoid  $S$ , denoted as  $V_T$ <sup>5</sup>. In the context of an act  $A_S$ , if for any  $x, y$ , the equation  $xc = yc$  implies  $x = y$  for any right cancellable element  $c \in S$ , then  $A_S$  is torsion-free. Moreover, if  $S$  satisfies the ascending chain

condition for right ideals, it is equivalent to being Noetherian. This condition translates to the existence of every ascending chain  $\mathfrak{B}_1 \subseteq \mathfrak{B}_2 \subseteq \mathfrak{B}_3 \subseteq \dots \subseteq \mathfrak{B}_n \subseteq \mathfrak{B}_{n+1} \subseteq \dots$ , of its right sub acts,

## Results and Discussion

**Definition 1:** Let  $T$  be a bounded operator on a Banach-space  $V$  over a field  $F$ , and consider the semigroup  $S = \{\sinh x : x \in R\}$ . Define the function  $\mu$  from  $S \times V$  into  $V$  as  $\mu(\sinh x, v) = \sinh T(v)$ , then  $V$  is called a left act over  $S$ , denoted as  $V_{\sinh T}$ .

$$\text{Put } p_n(T) = \sum_{i=0}^n \frac{(T)^i}{i!} = I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \dots + \frac{(T)^n}{n!} \quad \text{and} \quad p_n(-T) = \sum_{i=0}^n \frac{(-T)^i}{i!} = I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \dots + \frac{(-T)^n}{n!}.$$

**Proposition 1:** If a basis  $K = \{v_j, j \in \Lambda\}$  for  $V$ , then every element  $w$  of  $V_{\sinh T}$  can be composed as

$$w = \lim_{n \rightarrow \infty} \left( \sum_{i=0}^n \frac{(T)^i}{i!} + \sum_{i=0}^n \frac{(-T)^i}{i!} \right) \sum_{j \in \Lambda} a_j v_j = \lim_{n \rightarrow \infty} (p_n(T) + p_n(-T)).$$

$v$ , for some  $v$  in  $V$ .

**Proof:** Define  $\mu: S \times V \rightarrow V$ , by  $\mu(\sinh x, v) = \sinh T(v) = \frac{1}{2}(e^T - e^{-T})(v) =$

$$\frac{1}{2} \left( \sum_{i=0}^{\infty} \frac{(T)^i}{i!} - \sum_{i=0}^{\infty} \frac{(-T)^i}{i!} \right) (v). \quad \text{For } w \in V_{\sinh T}, \text{ then } w = \frac{1}{2} \left( \sum_{i=0}^{\infty} \frac{(T)^i}{i!} - \sum_{i=0}^{\infty} \frac{(-T)^i}{i!} \right) (v) =$$

$$\frac{1}{2} \left[ I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \dots + \frac{(T)^n}{n!} - \left[ I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \dots + \frac{(-T)^n}{n!} \right] \right] (v).$$

Since  $K = \{v_j, j \in \Lambda\}$  is a basis for  $V$  then

$$w = \frac{1}{2} \left( \left( \sum_{i=0}^{\infty} \frac{(T)^i}{i!} - \sum_{i=0}^{\infty} \frac{(-T)^i}{i!} \right) \right) \left( \sum_{j \in \Lambda} a_j v_j \right)$$

$$= \frac{1}{2} \left[ I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \dots + \frac{(T)^n}{n!} + \dots - \left[ I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \dots + \right. \right.$$

there exists  $n \in \mathbb{N}$  such that  $\mathfrak{B}_n = \mathfrak{B}_{n+1} = \dots$ <sup>1</sup>. Recall that an operator  $T$  is considered nilpotent if  $T^n = 0$  for some integer<sup>2</sup>.

$$\left. \frac{(-T)^n}{n!} \dots \right] \left( \sum_{j \in \Lambda} a_j v_j \right) = \frac{1}{2} \left[ I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \dots + \frac{(T)^n}{n!} + \dots - I - (-T) - \frac{(-T)^2}{2!} - \frac{(-T)^3}{3!} - \dots - \frac{(-T)^n}{n!} - \dots \right] \left( \sum_{j \in \Lambda} a_j v_j \right)$$

$$= \frac{1}{2} \left[ I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \dots + \frac{(T)^n}{n!} + \dots - I + (T) - \frac{(T)^2}{2!} + \frac{(T)^3}{3!} - \dots - \frac{(-T)^n}{n!} - \dots \right] \left( \sum_{j \in \Lambda} a_j v_j \right) = \frac{1}{2} [2T + 2 \frac{(T)^3}{3!} + 2 \frac{(T)^5}{5!} + \dots + 2 \frac{(T)^{2n+1}}{(2n+1)!}] \left( \sum_{j \in \Lambda} a_j v_j \right)$$

$$= T \left( \sum_{j \in \Lambda} a_j v_j \right) + \frac{(T)^3}{3!} \left( \sum_{j \in \Lambda} a_j v_j \right) + \frac{(T)^5}{5!} \left( \sum_{j \in \Lambda} a_j v_j \right) + \dots + \frac{(T)^{2n+1}}{(2n+1)!} \left( \sum_{j \in \Lambda} a_j v_j \right) +$$

$$= \sum_{i=0}^{\infty} \frac{(T)^{2i+1}}{(2i+1)!} \left( \sum_{j \in \Lambda} a_j v_j \right)$$

But when  $T \in B(H)$  the series  $\sum_{i=0}^{\infty} \frac{T^i}{i!}$  converges in  $B(H)^2$ . Therefore,  $\sum_{i=0}^{\infty} \frac{(T)^{2i+1}}{(2i+1)!}$  is converge. Thus

$$w = \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{(T)^{2i+1}}{(2i+1)!} \left( \sum_{j \in \Lambda} a_j v_j \right) = \lim_{n \rightarrow \infty} (p_n(T) - p_n(-T)).V, \text{ where } B(H) \text{ is the set of all bounded operator on } H$$

**Lemma 1:**  $V_{\sinh T}$  is act over  $S$  if and only if  $V_{\sinh T}$  is a module where  $T$  is a nilpotent operator.

**Proof:** This follows the same method as the proof of Lemma 2.4<sup>6</sup>.

**Proposition 2:** If  $T$  and  $S$  are similar bounded operators, then  $V_{\sinh S}$  and  $V_{\sinh T}$  are isomorphic.

**Proof:** This follows the same method as the proof of Proposition 2.5<sup>6</sup>.

**Proposition 3:**  $V_{\sinh T}$  is a faithful act over  $S$ , where  $T$  is any bounded linear operator.

**Proof:** To show that any bounded operator  $V_{\sinh T}$  is faithful, consider  $\sinh x_1 \cdot \sinh T(v) = \sinh x_2 \cdot \sinh T(v)$ . As  $\sinh T$  is an operator, thus,  $\sinh T$  is linear transformation, and  $\sinh T(\sinh x_1 \cdot v) = \sinh T(\sinh x_2 \cdot v)$ . But  $\sinh T$  is one-to-one<sup>2</sup>. So that  $\sinh x_1 \cdot v = \sinh x_2 \cdot v$ , this implies that  $\sinh x_1 = \sinh x_2, \forall x_1, x_2 \neq 0$ . Therefore  $V_{\sinh T}$  is faithful act over  $S$ .

**Proposition 4:** If  $T$  is an onto operator and  $V_{\sinh T}$  is finitely generated ( $f \cdot g$ ), then  $V$  is finite dimensional.

**Proof:** This follows the same method as the proof of Proposition 2.8<sup>6</sup>.

**Proposition 5:** For any bounded operator  $T$ , then,

1- If  $T$  is any bounded operator, then  $V_{\sinh T}$  is separated,

2. If,  $V_{\sinh T}$  is separated act over  $S$ , Then  $T$  is injective.

**Proof:** (1) Let  $p \neq q$  in  $V_{\sinh T}$ . To prove that  $V_{\sinh T}$  is separated, it must be shown that there is  $m, n \in S, m \neq n$ , with  $n$  as the identity element, such that  $ma \neq mb$ . Suppose  $ma = mb, n \neq m \in S$ , such that  $m = \sinh x, n$  is the identity element,  $p, q \in V_{\sinh T}$ , this gives  $\sinh x \cdot \sinh T(v_1) = \sinh x \cdot \sinh T(v_2) \ni v_1, v_2 \in V, x \in R$ , as  $\sinh T$  is an operator,  $\sinh T$  is linear transformation, this give  $\sinh x \cdot \sinh T(v_1) = \sinh x \cdot \sinh T(v_2)$ , thus  $\sinh T(\sinh x \cdot v_1) = \sinh T(\sinh x \cdot v_2)$ , but  $\sinh T$  is one to one<sup>2</sup>. Therefore,  $\sinh x \cdot v_1 = \sinh x \cdot v_2$ , thus  $(v_1 - v_2) \sinh x = 0$ , since  $\sinh x \neq 0$ . Thus  $v_2 = v_1$ , thus either  $\sinh T(v_1) \neq \sinh T(v_2)$  or  $\sinh T(v_1) = \sinh T(v_2)$ , but if  $\sinh T(v_1) \neq \sinh T(v_2)$ , this give  $v_1 \neq v_2$  this contradicts with  $v_1 = v_2$ , then  $\sinh T(v_1) = \sinh T(v_2)$ , means  $p = q$  which is in contradiction, then  $V_{\sinh T}$  is separated act.

(2) Assume that  $V_{\sinh T}$  is separated, to prove the operator  $T$  is 1-1. Put  $v_1 \neq v_2$ , must show that  $T(v_1) \neq T(v_2)$ . Because  $v_1 \neq v_2$ , thus either  $\sinh T(v_2) \neq \sinh T(v_1)$  or  $\sinh T(v_2) = \sinh T(v_1)$ . If  $\sinh T(v_2) = \sinh T(v_1)$ , this contradicts with  $v_2 \neq v_1$  (because  $\sinh T$  is 1-1<sup>2</sup>), hence  $\sinh T(v_1) \neq \sinh T(v_2)$ , but  $V_{\sinh T}$  is

separated act over  $S$  then  $\exists e \neq s, s = \sinh x \in S$  such that,  $\sinh x \cdot \sinh T(v_1) \neq \sinh x \cdot \sinh T(v_2)$ , because  $\sinh T$  is an operator, then  $\sinh T$  is linear transformation, thus  $\sinh T(\sinh x \cdot v_1) \neq \sinh T(\sinh x \cdot v_2)$ , thus,

$$\begin{aligned} & \left( I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \dots - \left[ I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \dots \right] \right) (\sinh x \cdot v_1) \neq \left( I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \dots - \left[ I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \dots \right] \right) (\sinh x \cdot v_2), \\ & \text{i.e., } T(\sinh x \cdot v_1) + \frac{(T)^3}{3!} (\sinh x \cdot v_1) + \frac{(T)^5}{4!} (\sinh x \cdot v_1) + \dots \neq T(\sinh x \cdot v_2) + \frac{(T)^3}{3!} (\sinh x \cdot v_2) + \frac{(T)^4}{4!} (\sinh x \cdot v_2) + \dots, \text{ hence } T(\sinh x \cdot v_1) \neq T(\sinh x \cdot v_2), \\ & \text{thus } v_2 \neq v_1 \text{ and } \frac{(T)^3}{3!} (\sinh x \cdot v_1) \neq \frac{(T)^3}{3!} (\sinh x \cdot v_2), \\ & \text{implies that } \frac{T}{3!} (T^2(\sinh x \cdot v_1)) \neq \frac{T}{3!} (T^2(\sinh x \cdot v_2)), \text{ thus } (T(T(\sinh x \cdot v_1))) \neq T(T(\sinh x \cdot v_2)). \end{aligned}$$

**Proof:** Put  $\sinh x \sinh T(v_1) = \sinh x \sinh T(v_2)$ , for all cancellable element in  $S$ ,  $\sinh x$ , thus  $\sinh T(\sinh x \cdot v_1) = \sinh T(\sinh x \cdot v_2)$ , (because  $\sinh T$  is 1-1). Therefore,  $\sinh x \cdot v_1 = \sinh x \cdot v_2$ , then  $v_1 = v_2$ . Hence, either  $\sinh T(v_1) = \sinh T(v_2)$  or  $\sinh T(v_1) \neq \sinh T(v_2)$ , if  $\sinh T(v_1) \neq \sinh T(v_2)$ , this contradiction with  $v_1 = v_2$ , then  $\sinh T(v_1) = \sinh T(v_2)$ , thus  $V_{\sinh T}$  is torsion free.  $(T(\sinh x \cdot v_2)) \neq (T(\sinh x \cdot v_1))$ , by the same argument, hence  $T(v_2) \neq T(v_1)$ . Therefore,  $T$  is one to one.

**Proposition 6:** Let  $T$  be a bounded linear operator, then  $V_{\sinh T}$  is torsion free act over monoid.

**Proof:** Put  $\sinh x \sinh T(v_1) = \sinh x \sinh T(v_2)$ , for all cancellable element in  $S$ ,  $\sinh x$ , thus  $\sinh T(\sinh x \cdot v_1) = \sinh T(\sinh x \cdot v_2)$ , (because  $\sinh T$  is 1-1)<sup>2</sup>, therefore  $\sinh x \cdot v_1 = \sinh x \cdot v_2$ , then  $v_1 = v_2$ . Hence, either  $\sinh T(v_1) = \sinh T(v_2)$  or  $\sinh T(v_1) \neq \sinh T(v_2)$ , if  $\sinh T(v_1) \neq \sinh T(v_2)$ , this contradicts with  $v_1 = v_2$ , then  $\sinh T(v_1) = \sinh T(v_2)$ . Therefore,  $V_{\sinh T}$  is torsion free.

If  $A$  is  $f.g$  act and  $S$  is Noetherian, then  $A$  is Noetherian act over  $S^{7-10}$ .

**Proposition 7:** Put  $T$  as similar to any operator  $\mathfrak{D}$  from  $\mathfrak{N}$  to  $\mathfrak{N}$ , and  $V$  as a finite dimensional normed space, then  $V_{\text{sinsh}T}$  is Noetherian act over  $S$  if  $S$  is Noetherian.

**Proof:** Since  $V$  is finite dimension, it is a  $f.g$  act over  $S^2$ , and  $S$  is Noetherian, then it is Noetherian

## Conclusion

In this work, we have introduced and established the concept of associated act over the monoid  $S$  of  $\text{sinsh}T$ . The following relationships are proven: If  $T$  and  $S$  are similar bounded operators. Then  $V_{\text{sinsh}S}$  and  $V_{\text{sinsh}T}$  are isomorphic. When operator

$S$ -act. Put any ascending sequence ideals of  $S$  as  $\mathfrak{B}_1 \subseteq \mathfrak{B}_2 \subseteq \mathfrak{B}_3 \subseteq \dots \subseteq \mathfrak{B}_n \subseteq \mathfrak{B}_{n+1} \subseteq \dots$ , thus it is a sequence of sub-acts of  $S_S$  denoted by  $S\mathfrak{D}$ , for any operator  $\mathfrak{D}$  from  $\mathfrak{N}$  to  $\mathfrak{N}$ , because  $T$  is similar to  $\mathfrak{D}$ . Therefore, by Proposition 2,  $V_{\text{sinsh}T}$  is isomorphic  $S_{\text{sinsh}\mathfrak{D}}$ , making  $S_{\text{sinsh}\mathfrak{D}}$  as Noetherian act over  $S$ , thus  $S$  is Noetherian.

$T$  is similar to any operator  $\mathfrak{D}$  from  $\mathfrak{N}$  to  $\mathfrak{N}$ , and  $V$  is a F.D.N.S, then  $V_{\text{sinsh}T}$  is Noetherian act over  $S$  if and only if  $S$  is Noetherian. The  $S$ -act  $V_{\text{sinsh}T}$  is separated, where  $T$  is any bounded linear operator.

## Authors' Declaration

- Conflicts of Interest: None.

- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

## Authors' Contribution Statement

This work described in this study was performed in collaboration among the authors. U. S. A. proposed the concept of associated act  $V$  over monoid with reference to  $\text{sinsh}T$ . N. M. J. I. and M. J. M. A.

contributed to the composition and editing of the manuscript, as well as the investigation of the properties detailed in the study.

## References

1. Abdul-Kareem SA. On Hollow acts. IOP Conf. Ser.: Mater. Sci. Eng. 2020; 928(042004): 1-14. <https://doi.org/10.1088/1757-899X/928/4/042004>
2. Ashtiani B. T., Rasoulia H., Tehraniana A., Barzegarb H., On lifting acts over monoids, Journal of Linear and Topological Algebra 2022; 11( 04), 265-269. <https://doi.org/10.30495/JLTA.2023.699063>
3. Ashtiani B. T., Rasouli H., Tehranian A., and Barzegar H., On the notion of supplement in acts over monoids, 2023; 22(03), 2350066, <https://doi.org/10.1142/S0219498823500664>.
4. Mohammed RI, Ahmed FA. Exponential Function of a bounded Linear Operator on a Hilbert Space. Baghdad Sci. J. 2014;11(3): 1267-1273. <https://doi.org/10.21123/bsj.11.3.1267-1273>
5. Kadhim SN, Abed Z. The Act of Linear Operators. Baghdad Sci. J. 2016; 13(2): 388-393. DOI: <http://dx.doi.org/10.21123/bsj.2016.13.2.0388>
6. Kadhim SN, Ali MJM, Atiya ZA. A Note On S-acts And Bounded Linear Operators. Ital J Pure Appl Math. 2018; 39: 16-22.
7. Abdul-Kareem SA. Generalization of injective S-acts. Common Algebra. 2023; 51(4): 1743-1751. DOI: <https://doi.org/10.1080/00927872.2022.2141766>
8. Abdul-Kareem SA, Minimal Quasi Injective S-Acts, Int J Math Comput Sci. 2020; 15(2), 515-524
9. Abdul-Kareemi A. S., Salih S. R., Fuzzy Injective S-act On Monoids, TWMS J App Eng Math. 2021; 11(4): 1072-1079.
10. Al-Saadi S.A., On Locally Injective Acts over Monoids, J Southwest Jiaotong Univ, 2019; 54(6): 1-10. <https://doi.org/10.35741/issn.0258-2724.54.6.54>

## بعض النتائج حول الآثار والمؤثرات المقيدة

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### الخلاصة

الآثر  $V$  بالنسبة إلى  $\sinshT$  و خواصه قد تم دراسته في هذا البحث حيث تم دراسة علاقة الآثر المخلص والآثر المنتهي التولد والآثر المنفصل وربطها بالمؤثرات المتباينة حيث تم بهنة العلاقات التالية ان الآثر اذا وفقط اذا مقاس في حالة كون المؤثر هو عديم القوة وكذلك في حالة كون المؤثر شامل فان الآثر هو منتهي التولد اي ان الغضاء هو منتهي التولد وايضا تم برهن ان الآثر مخلص لكل مؤثر مقيد ولكل  $\lambda$  قد تم التحقق من انه لاي مؤثر مقيد فان الآثر منفصل وفي حالة كون الآثر منفصل فان المؤثر سوف يكون متباين وايضا تم برهنة انه في حالة كون الغضاء يمتلك قاعدة فان الآثر سوف يكون كل عنصر فيه يمكن كتابته كتركيب خطية ومن العلاقات المهمة التي قد تم برهانها انه اذا كان المؤثر مشابه لمؤثر اخر فان الآثر سوف يكون نوثيرين بوجد شرط على الغضاء .

**الكلمات المفتاحية:** أثر  $V$  بالنسبة إلى  $\sinshT$ ، المؤثرات المفيدة، الآثر المخلص، المؤثر المتباينات، الآثر المنفصل.