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Some Results about Acts over Monoid and Bounded Linear Operators

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Abstract

This study delves into the properties of the associated act V over the monoid S of sinshT. It examines the relationship between faithful, finitely generated, and separated acts, as well as their connections to one-to-one and onto operators. Additionally, the correlation between acts over a monoid and modules over a ring is explored. Specifically, it is established that V_{sinshT} functions as an act over S if and only if V_{sinshT} functions as module, where T represents a nilpotent operator. Furthermore, it is proved that when T is onto operator and V_{sinshT} is finitely generated, V is guaranteed to be finite-dimensional. Prove that for any bounded operator the following, V_{sinshT} is acting over S if and only if V_{sinshT} is a module where T is a nilpotent operator, V_{sinshT} is a faithful act over S, where T is any bounded linear operator, if T is any bounded operator, then V_{sinshT} is separated, if V_{sinshT} is separated act over S, Then T is injective, if a basis $K = \{v_j, j \in \wedge\}$ for V, then every element w of V_{sinshT} can be composed as $w = \lim_{n\to\infty}(\sum_{i=0}^{n}\frac{(T)^i}{i\,!} + \sum_{i=0}^{n}\frac{(-T)^i}{i\,!})\sum_{j\in\wedge} a_j v_j = \lim_{n\to\infty}(p_n(T) + p_n(-T)).\ v$, for some v in V, and put T as similar to any operator $\mathfrak{D}$ from $\mathfrak{N}$ to $\mathfrak{N}$, and V as a finite dimensional normed space, then V_{sinshT} is Noetherian act over S if S is Noetherian.

Keywords: Associated act V over monoid of sinshT, Bounded linear operator, Faithful act over monoid, One-to-one operator, Separated acts over monoid.

Introduction

Consider a nonempty set A and a monoid S. Let μ: A × S ⟶A be defined as μ(a, s) = (a, s) ⟼ as, such that (as)t = a(st). This leads to a = a . 1, where s, t ∈ S, and a ∈ A. A_S being a right act[1-3]. Moving forward, let us examine a Hilbert space H over a field F (where F can be either real or complex), and let T be a bounded linear operator on H. The exponential operator e^T is $e^T = \sum_{n=0}^{\infty}\frac{T^n}{n\,!}$, where T^0 = I, the identity operator on H. The exponential operator is well-defined, the sum exists[4]. Flowing[5], consider a polynomial ring R = F[x] with coefficients in F. Define the function Ø: $R \times V \longrightarrow V$ such that p(T)v = p . v = Ø(p, v), where T is a linear operator. Here, Ø transforms V into a left R-module, denoted as V_T. When a bounded operator T on a Banach-space act as V over a field F, and $S = \{e^x : x \in R\}$ represents the semi-group, the function $\mu : S \times V \to V$ is $e^T(v) = \mu(e^x, v)$, which establishes V as a left act over the monoid S, denoted as V_T[5]. In the context of an act A_S, if for any x, y, the equation xc = yc implies x = y for any

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right cancellable element c ∈ S, then A_S is torsion-free. Moreover, if S satisfies the ascending chain condition for right ideals, it is equivalent to being Noetherian. This condition translates to the existence of every ascending chain $\mathfrak{V}_1 \subseteq \mathfrak{V}_2 \subseteq \mathfrak{V}_3 \subseteq \ldots \subseteq \mathfrak{V}_n \subseteq \mathfrak{V}_{n+1} \subseteq \ldots$, of its right sub acts, there exists n ∈ N such that $\mathfrak{V}_n = \mathfrak{V}_{n+1} = \ldots$[1]. Recall that an operator T is considered nilpotent if $T^n = 0$ for some integer[2].

Results and Discussion

Definition 1: Let T be a bounded operator on a Banach-space V over a field F, and consider the semigroup $S = \{sinshx : x \in R\}$. Define the function μ from S × V into V as $\mu(\text{sinsh}\, x, v) = \text{sinsh}\, T(v)$, then V is called a left act over S, denoted as V_{sinshT}.

Put $p_n(T) = \sum_{i=0}^{n} \frac{(T)^i}{i\,!} = I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots + \frac{(T)^n}{n!}$ and $p_n(-T) = \sum_{i=0}^{n} \frac{(-T)^i}{i\,!} = I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \cdots + \frac{(-T)^n}{n!}$.

Proposition 1: If a basis K = {v_j, j ∈ Λ} for V, then every element w of V_{sinshT} can be composed as

$$w = \lim_{n\to\infty}\Big(\sum_{i=0}^{n} \frac{(T)^i}{i\,!} + \sum_{i=0}^{n} \frac{(-T)^i}{i\,!}\Big)\sum_{j\in\wedge} a_j\, v_j = \lim_{n\to\infty}(p_n(T) + p_n(-T)).$$

v, for some v in V.

Proof: Define μ: S×V →V, by μ (sinsh x, v) = sinshT (v) $= \frac{1}{2}(e^{T} - e^{-T})(v) =$

$\frac{1}{2}\Big(\sum_{i=0}^{\infty} \frac{(T)^i}{i\,!} - \sum_{i=0}^{\infty} \frac{(-T)^i}{i\,!}\Big)$ (v). For w ∈ $V_{\text{sinh}\,T}$, then $w = \frac{1}{2}\Big(\sum_{i=0}^{\infty} \frac{(T)^i}{i\,!} - \sum_{i=0}^{\infty} \frac{(-T)^i}{i\,!}\Big)(v) = \frac{1}{2}\Big[I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots + \frac{(T)^n}{n!} - \Big[I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \cdots + \frac{(-T)^n}{n!}\Big]\Big](v)$.

Since K = {v_j, j ∈ ∧} is a basis for V then

$$w = \frac{1}{2}\Big(\Big(\sum_{i=0}^{\infty} \frac{(T)^i}{i\,!} - \sum_{i=0}^{\infty} \frac{(-T)^i}{i\,!}\Big)\Big)\big(\textstyle\sum_{j\in\wedge} a_j\, v_j\big)$$

$$= \frac{1}{2}\Big[I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots + \frac{(T)^n}{n!} + \cdots - \Big[I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \cdots + \frac{(-T)^n}{n!} \cdots\Big]\Big]\big(\textstyle\sum_{j\in\wedge} a_j\, v_j\big)$$

$$= \frac{1}{2}\Big[I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots + \frac{(T)^n}{n!} + \cdots - I - (-T) - \frac{(-T)^2}{2!} - \frac{(-T)^3}{3!} - \cdots - \frac{(-T)^n}{n!} - \cdots\Big]\big(\textstyle\sum_{j\in\wedge} a_j\, v_j\big)$$

$$= \frac{1}{2}\Big[I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots + \frac{(T)^n}{n!} + \cdots - I + (T) - \frac{(T)^2}{2!} + \frac{(T)^3}{3!} - \cdots - \frac{(-T)^n}{n!} - \cdots\Big]\big(\textstyle\sum_{j\in\wedge} a_j\, v_j\big) = \frac{1}{2}\Big[2T + 2\frac{(T)^3}{3!} + 2\frac{(T)^5}{5!} + \cdots + 2\frac{(T)^{2n+1}}{(2n+1)!}\Big]\big(\textstyle\sum_{j\in\wedge} a_j\, v_j\big)$$

$$= T\big(\textstyle\sum_{j\in\wedge} a_j\, v_j\big) + \frac{(T)^3}{3!}\big(\textstyle\sum_{j\in\wedge} a_j\, v_j\big) + \frac{(T)^5}{5!}\big(\textstyle\sum_{j\in\wedge} a_j\, v_j\big) + \cdots + \frac{(T)^{2n+1}}{(2n+1)!}\big(\textstyle\sum_{j\in\wedge} a_j\, v_j\big) +$$

$$= \sum_{i=0}^{\infty} \frac{(T)^{2i+1}}{(2i+1)!}\big(\textstyle\sum_{j\in\wedge} a_j\, v_j\big)$$

But when T ∈ B(H) the series $\sum_{i=0}^{\infty} \frac{T^i}{i\,!}$ converges in B(H)[2]. Therefore, $\sum_{i=0}^{\infty} \frac{(T)^{2i+1}}{(2i+1)\,!}$ is converge. Thus $w = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{(T)^{2i+1}}{(2i+1)\,!}\big(\sum_{j\in\wedge} a_j\, v_j\big) = \lim_{n\to\infty}(\, p_n(T) - p_n(-T)).V$, where B(H) is the set of all bounded operator on H

Lemma 1: V_{sinshT} is act over S if and only if V_{sinshT} is a module where T is a nilpotent operator.

Proof: This follows the same method as the proof of Lemma 2.4 [6].

Proposition 2: If *T* and *S* are similar bounded operators, then V_{sinshS} and V_{sinshT} are isomorphic.

Proof: This follows the same method as the proof of Proposition 2.5 [6].

Proposition 3: V_{sinshT} is a faithful act over S, where T is any bounded linear operator**.**

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Proof: To show that any bounded operator V_{sinshT} is faithful, consider $sinsh\ x_1 . sinshT(v) = sinshx_2 . sinshT(v)$. As sinshT is an operator, thus, sinshT is linear transformation, and $sinsh\ T(sinsh\ x_1 . v) = sinshT(sinsh\ x_2 . v)$. But sinshT is one-to-one [2]. So that $sinsh\ x_1 . v = sinsh\ x_2 . v$, this implies that $sinsh\ x_1 = sinsh\ x_2, \forall x_1, x_2 \neq 0$. Therefore V_{sinshT} is faithful act over S.

Proposition 4: If T is an onto operator and V_{sinshT} is finitely generated (f . g), then V is finite dimensional.

Proof: This follows the same method as the proof of Proposition 2.8 [6].

Proposition 5: For any bounded operator T, then,

1- If T is any bounded operator, then V_{sinshT} is separated,

2. If, V_{sinshT} is separated act over S, Then T is injective.

Proof: (1) Let $p \neq q$ in V_{sinshT}. To prove that V_{sinshT} is separated, it must be shown that there is m, n ∈ S, m≠ n, with n as the identity element, such that ma ≠ mb. Suppose ma = mb, n ≠ m ϵ S, such that m = sinsh x, n is the identity element, p, q $\epsilon\ V_{sinshT}$, this gives $sinsh\ x . sinshT(v_1) = sinsh\ x . sinshT(v_2) \ni v_1, v_2 \in V, x \in R$, as sinshT is an operator, sinshT is linear transformation, this give $sinsh\ x . sinshT(v_1) = sinsh\ x. sinshT(v_2)$s, thus $sinshT(sinsh\ x . v_1) = sinsh\ T(sinsh\ x . v_2)$, but sinshT is one to one [2]. Therefore, $sinsh\ x . v_1 = sinsh\ x . v_2$, thus $(v_1 - v_2) sinshx = 0$, since $sinsh\ x \neq 0$. Thus $v_2 = v_1$, thus either $sinshT(v_1) \neq sinshT(v_2)$ or $sinshT(v_1) = sinshT(v_2)$, but if $sinshT(v_1) \neq sinshT(v_2)$, this give $v_1 \neq v_2$ this contradcts with $v_1 = v_2$, then $sinshT(v_1) = sinshT(v_2)$, means p = q which is in contradiction, then V_{sinshT} is separated act.

(2) Assume that V_{sinshT} is separated, to prove the operator T is 1-1. Put $v_1 \neq v_2$, must show that $T(v_1) \neq T(v_2)$. Because $v_1 \neq v_2$, thus either $sinshT(v_2) \neq sinshT(v_1)$ or $sinshT(v_2) = sinshT(v_1)$. If $sinshT(v_2) = sinshT(v_1)$, this contradicts with $v_2 \neq v_1$(because sinshT is 1-1[2]), hence $sinshT(v_1) \neq sinshT(v_2)$, but V_{sinshT} is separated act over S then ∃ e ≠ s, s = sinsh x ∈ S such that, $sinsh\ x . sinsh\ T(v_1) \neq sinsh\ x . sinshT(v_2)$, because sinshT is an operator, then sinshT is linear transformation, thus $sinshT(sinsh\ x . v_1) \neq sinshT(sinsh\ x . v_2)$, thus,

$$\left(I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots - \left[I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \cdots\right]\right)(sinsh\ x . v_1) \neq \left(I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots - \left[I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \cdots\right]\right)(sinsh\ x . v_2),$$

i.e., $T(sinsh\ x . v_1) + \frac{(T)^3}{3!}(sinsh\ x . v_1) + \frac{(T)^5}{4!}(sinsh\ x . v_1) + \cdots \neq T(sinsh\ x . v_2) + \frac{(T)^3}{3!}(sinsh\ x . v_2) + \frac{(T)^4}{4!}(sinsh\ x . v_2) + \cdots$, hence $T(sinsh\ x . v_1) \neq T(sinsh\ x . v_2)$, thus $v_2 \neq v_1$ and $\frac{(T)^3}{3!}(sinsh\ x . v_1) \neq \frac{(T)^3}{3!}(sinsh\ x . v_2)$, implies that $\frac{T}{3!}(T^2(sinsh\ x . v_1) \neq \frac{T}{3!}(T^2(sinsh\ x . v_2))$, thus $(T(T(sinsh\ x . v_1)) \neq T(T(sinsh\ x . v_2))$.

Proof: Put $sinsh\ x\ sinshT(v_1) = sinsh\ x\ sinsh\ T(v_2)$, for all cancellable element in S, $sinsh\ x$, thus $sinsh\ T(sinsh\ x . v_1) = sinsh\ T(sinsh\ x . v_2)$, (because sinshT is 1-1). Therefore, $sinsh\ x. v_1 = sinsh\ x . v_2$, then $v_1 = v_2$. Hence, either $sinsh\ T(v_1) = sinshT(v_2)$ or $sinshT(v_1) \neq sinshT(v_2)$, if $sinshT(v_1) \neq sinshT(v_2)$, this contradiction with $v_1 = v_2$, then $sinshT(v_1) = sinshT(v_2)$, thus $V_{sinshT}$is torsion free. $(T(sinsh\ x. v_2)) \neq (T(sinsh\ x. v_1))$, by the same argument, hence $T(v_2) \neq T(v_1)$. Therefore, T is one to one.

Proposition 6: Let T be a bounded linear operator, then V_{sinshT} is torsion free act over monoid.

Proof: Put $sinsh\ x\ sinshT(v_1) = sinsh\ x\ sinsh\ T(v_2)$, for all cancellable element in S, $sinsh\ x$, thus $sinsh\ T(sinsh\ x. v_1) = sinsh\ T(sinsh\ x . v_2)$, (because sinshT is 1-1)[2], therefore $sinsh\ x . v_1 = sinsh\ x . v_2$, then $v_1 = v_2$. Hence, either $sinshT(v_1) = sinshT(v_2)$ or $sinshT(v_1) \neq sinshT(v_2)$, if $sinshT(v_1) \neq sinshT(v_2)$, this contradicts with $v_1 = v_2$, then $sinshT(v_1) = sinshT(v_2)$. Therefore, V_{sinshT} is torsion free.

If A is f . g act and S is Noetherian, then A is Noetherian act over S [7-10].

Proposition 7: Put T as similar to any operator $\mathfrak{D}$ from $\mathfrak{N}$ to $\mathfrak{N}$, and V as a finite dimensional normed space, then V_{sinshT} is Noetherian act over S if S is Noetherian.

Proof: Since V is finite dimension, it is a f . g act over S[2], and S is Noetherian, then it is Noetherian S-act. Put any ascending sequence ideals of S as $\mathfrak{V}_1 \subseteq \mathfrak{V}_2 \subseteq \mathfrak{V}_3 \subseteq \ldots \subseteq \mathfrak{V}_n \subseteq \mathfrak{V}_{n+1} \subseteq \ldots$, thus it is a sequence of sub-acts of S_S denoted by S $\mathfrak{D}$, for any operator $\mathfrak{D}$ from $\mathfrak{N}$ to $\mathfrak{N}$, because T is similar to $\mathfrak{D}$. Therefore, by Proposition 2, V_{sinshT} is isomorphic $S_{sinsh\mathfrak{D}}$, making $S_{sinsh\mathfrak{D}}$ as Noetherian act over S, thus S is Noetherian.

Conclusion

In this work, we have introduced and established the concept of associated act over the monoid S of sinshT. The following relationships are proven: If T and S are similar bounded operators. Then V_{sinshS} and V_{sinshT} are isomorphic. When operator T is similar to any operator $\mathfrak{D}$ from $\mathfrak{N}$ to $\mathfrak{N}$, and V is a F.D.N.S, then V_{sinshT} is Noetherian act over S if and only if S is Noetherian. The S-act V_{sinshT} is separated, where T is any bounded linear operator.

Authors' Declaration

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee at University of Baghdad.

Authors' Contribution Statement

This work described in this study was performed in collaboration among the authors. U. S. A. proposed the concept of associated act V over monoid with reference to sinshT. N. M. J. I. and M. J. M. A. contributed to the composition and editing of the manuscript, as well as the investigation of the properties detailed in the study.

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بعض النتائج حول الآثار والمؤثرات المقيدة

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الخلاصة

الأثر V بالنسبة إلى sinshT و خواصه قد تم دراسته في هذا البحث حيث تم دراسة علاقة الأثر المخلص والاثر المنتهى التولد والاثر المنفصل وربطها بالمؤثرات المتباينة حيث تم بهنة العلاقات التالية ان الاثر اذا وفقط اذا مقاس في حالة كون المؤثر هو عديم القوة وكذلك في حالة كون المؤثر شامل فان الاثر هو منتهي التولد اي ان الغضاء هو منتهي التولد وايضا تم برهن ان الاثر مخلص لكل مؤثر مقيد وك\لك قد تم التحقق من انه لاي مؤثر مقيد فان الاثر منفصل وفي حالة كون الاثر منفصل فان المؤثر سوف يكون متباين وايضا تم برهنة انه في حالة كون الفضاء يمتلك قاعدة فان الاثر سوف يكون كل عنصر فيه يمكن كتابته كتركيبة خطية ومن العلاقات المهمة التي قد تم برهانها انه اذا كان المؤثر مشابه لمؤثر اخر فان الاثر سوف يكون نوثيرين بوجد شرط على الفضاء .

الكلمات المفتاحية: أثر V بالنسبة إلى sinshT، المؤثرات المفيدة، الأثر المخلص، المؤثر المتباينات، الأثر المنفصل.