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New Approximating Results by Weak Convergence of Forked Sequences

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Abstract

The modular function spaces are natural generalization of spaces like Lebesgue space, Orlicz space, Lorentz p-space, Orlicz–Lorentz space, Musielak–Orlicz space, et al. The function modulars lack basic and flexible properties that norm functions have, as they are functional lacks homogeneity and subadditivity and, therefore, it might be surprising to use techniques involving asymptotic centers, normal structure and uniform convexity to obtain fixed point theorems. The purpose of this paper is to give a new accelerated iterative algorithm for multi valued\ single valued mappings in modular function spaces and to prove some results about their convergence (strong or weak) to a fixed point (or a common fixed point). Through the work, the modular function satisfies (UUC1) property and -condition. Sometimes the work required the use of the Opial's property or demi-closed condition. The intent of this manuscript is proving the existence and uniqueness of fixed point inducing from weak convergence of a forked iterative scheme. This scheme is constructed by five-step iterative for (λ, ρ) -firmly nonexpansive (multi\ single) mappings in modular spaces with respect to modular ρ satisfies (UUC1) property and Δ2-condition. To obtain these results and other finding, the definitions of weak convergence, demi-closeness and Opial's condition format for the case of double sequences. Note that the authors presented a previous study on the strong convergence of forked double sequences including important results, see references.

Keywords: Double sequence, Firmly nonexpansive, Fixed point, Strong convergence, Weak convergence.

Introduction

Fixed point theory in general is a thriving field for researchers whose purpose is to work on the existence of iterative scheme to reach the fixed point as quickly as possible in different spaces. There are many applied sciences as well as engineering, that can be formulated in the form of an integral equation or differential equation, and this equation can easily be transferred to the fixed point theory, as here lies the importance of the fixed point topic to prove the existence and unique of the solution[1]. In addition, the fixed point theory is included in the field of physics, game theory and economics[2], as well as, many researchers used fixed point theory to study the stability of the differential equation see[3], for whoever is looking for more applications, [4] about existences solution for differential equations. In general, to solve fixed point problems analytically is almost impossible,

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therefore, resorting to the approximate solution by using iterative scheme for, see[5], over the years the fixed point problem has evolved and many iterative schemes have emerged to solve the fixed point, research is still ongoing in order to develop algorithms and obtain faster and more efficient algorithms[6]. The notion of modular spaces, as a generalization of metric spaces introduces by Nakano and redefined and generalized by Musielak and Orlicz that have been studied by many researchers[7,8]. Khamsi *et.al* [9] the first to discuss the concept for a fixed point in modular function spaces. While Kozilowski developed the fixed point topic extensively in modular function spaces see[10-12], since then the theory of fixed point has become prevalent, culminating in the publication when the researchers worked on the fixed point in different spaces see[13,14]. Recently, Salman and Abed gave various results for new iterative schemes that suitable with $(\lambda,\rho)-$ firmly nonexpansive multivalued mappings[15]. Here, a five-step iterative scheme is introduced that, at first glance, seems forked, but it's not hard. This scheme is constructed for (λ,ρ)- firmly nonexpansive (multi\ single) mappings in modular function spaces. Many different ρ-weak convergence results are proved for double scheme of that under consideration.

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of L_p. let ρ be a nontrivial ring subsets of Ω, which means that ρ is closed with respect to forming finite union, and countable intersections and differences, Assume further that $E\cap A\in\rho$ for any $E\in\rho$ and $A\in\Sigma$, let us assume that there exists an increasing sequence of sets $K_n\in\rho$ such that $\Omega=\cup K_n$. Now $E:=$ the linear space of all simple functions with supports from ρ and M_∞ :=the space of all extended measurable functions.

In this study, $L_p$will be a modular function spaces with respect to $\rho\in\Re$ and L_ρ^* be its dual of L_p. Recalling the following

Definition 1[9]: If ρ is convex modular in X, then is called modular spaces

$$L_\rho=\{f\in M:\rho(\lambda f)\longrightarrow 0\ as\ \lambda\longrightarrow 0\}$$

The modular spaces L_p it could be in the form an F-norm define by

$$\|f\|_\rho=\inf\{\alpha>0:\ \rho(\frac{f}{\alpha})\le\alpha\}$$

If ρ is convex and modular F-norm is define

$$\|f\|_\rho=\inf\{\alpha>0:\ \rho(\frac{f}{\alpha})\le 1\}$$

F-norm is called Luxemburg norm.

Definition 2[15]: Let $\rho: M\longrightarrow[0,\infty]$ possesses the below properties

1- $\rho(0)=0$ if and only if, $f=0,\rho-a.e$

2- $\rho(\alpha f)=\rho(f)$, for α any scalar.

3- $\rho(\alpha x+\beta y)\le\rho(x)+\rho(y)$ for every $\alpha,\beta\ge 0$ with $\alpha+\beta=1$.

ρ is called a convex modular.

Definition 3[16,17]: Let $\rho\in\Re$

1- The sequence $\{f_n\}$ is called ρ-convergent to f if $\rho(f_n-f)\longrightarrow 0$

2-A sequence $\{f_n\}$ is ρ-Cauchy sequence if $\rho(f_n-f_m)\longrightarrow 0$ as $n,m\to\infty$

3-A set $B\sqsubset L_p$ is called ρ-closed if for any $f_n\in L_p$ the convergence $\rho(f_n-f)\longrightarrow 0$ and f belongs to B.

4-A set $B\sqsubset L_p$ is called ρ-compact if every $f_n\in B$, there exists a subsequence $\{f_{n_k}\}$ and f in $\rho(f_{n_k}-f)\to 0$.

Definition 4[18]: A duality pairing in modular function spaces and denoted by ρ- duality pairing is define as $\langle .,.\rangle: L_\rho\times L_\rho^*\longrightarrow R$ such that $\langle u\setminus h\rangle=h(u)$, where $u\in L_\rho$ and $h\in L_\rho^*$.

Proposition 1[18]: Let $\langle .,.\rangle$ is the by ρ- duality pairing on $L_\rho\times L_\rho^*$ then

1-$\langle\alpha u+\beta v\setminus h\rangle=\alpha\langle u\setminus h\rangle+\beta\langle v\setminus h\rangle$

2-$\langle u\setminus\alpha h_1+\beta h_2\rangle=\alpha\langle u\setminus h_1\rangle+\beta\langle u\setminus h_2\rangle$

3- $\langle u\setminus h\rangle=0$ for all $u\in L_\rho$, $h=0$

4-$\langle u\setminus h\rangle=0$ for all $h\in L_\rho^*$, u=0.

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Definition 5[18]: In modular spaces let E_ρ^* the dual for L_ρ, then $h: L_\rho \longrightarrow 2^{L_\rho^*}$ is called ρ-normalized duality mapping if $H(u) = \{h \in L_\rho^*, \langle u \setminus h\rangle = \rho^2(u) = \rho^{*^2}(u)\}$.

Lemma 1[7]: Let $\{\rho_n\}_{n=1}^{\infty}$, $\{\theta_n\}_{n=1}^{\infty}$and $\{\zeta_n\}_{n=1}^{\infty}$ nonnegative sequence such that

$$\rho_{n+1} \le (1-\theta_n)\rho_n + \zeta_n$$

Where $\{\theta_n\}$ sequence in $(0,1)$ and $\{\zeta_n\}$ sequence in real number such that

$\sum_{n=1}^{\infty}\theta_n < \infty$ and $\sum_{n=1}^{\infty}\zeta_n < \infty$, then $\lim_{n\to\infty}\rho_n$ is exists.

Definition 6[19]: Let ρ be a nonzero convex regular modular defined on Ω let $r>0, \epsilon>0$ define $D(r,\epsilon) = \{(f,g): f,g \in L_P, \rho f \le r, \rho f - g \ge \epsilon r\}$

Let$\xi_1(r,\epsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) : (f,g) \in D(r,\epsilon)\right\}$

if $D(r,\epsilon) \ne \emptyset$ and $\xi_1(r,\epsilon) = 1$, If $D(r,\epsilon) = \emptyset$

Note that, ρ satisfy (UC1) if for every r> $0, \epsilon>0$ $\xi_1(r,\epsilon) > 0$ then $D(r,\epsilon) \ne \emptyset$.

Note that**:** ρ satisfy (UUC1) $\delta \ge 0, \epsilon > 0$ there exists $\eta_1(r,\epsilon) > 0$ depending only on δ and ϵ such that $\xi_1(r,\epsilon) > \eta_1(r,\epsilon) > 0$ for any $r > \delta$.

Definition 7[8,20]: A set $E \subset L_p$ is said to be ρ-proximinal if for each $f \in L_p$ exists an element g in E then $(f-g) = dist_p(f,E) = \inf\{\rho(f - h): h \; in \; E\}$.

Here, $P_p(E)$ denotes the family of nonempty ρ-proximinal, ρ-bounded subset of E,

$C_p(E)$ denotes the family of nonempty ρ-closed, ρ-bounded subset of E,

$H_p(.,.)$

$H_p(A,B) = \max\left\{ sup_{f\in A}\, dist_p\,(f,B),\; sup_{g\in B}\, dist_p\,(g,A)\right\}$ $A,B \in C_p(L_p)$

where $dist_p(f,B) = \inf\{\rho(f-g), g \in B\}$. As it is known $H_p(.,.)$ refers to ρ- Hausdorff distance on $C_p(E)$.

Definition 8[21]: Let $\rho \in \Re$ then ρ has Δ_2-condition if $\sup \rho(2f_n, D) \longrightarrow 0$ as $k \longrightarrow \infty$ and $D \longrightarrow \emptyset$, and $\sup \rho(f_n, D) \longrightarrow 0$.

Lemma 2[22]**:** Let $\rho \in \Re$ and ρ is (UUC1), let $\{t_n\}$ in $(0,1)$ be bounded away from 0 and 1, if exists constant $m > 0$ such that

$$\lim sup_{n\to\infty}\rho(f_n) \le m, \lim sup_{n\to\infty}\rho(g_n) \le m$$

and $lim_{n\to\infty}\rho(t_n f_n + (1-t_n)g_n) = m$, then $\lim_{n\to\infty}\rho(f_n - g_n) = 0$.

Lemma 3[8]**:** Let $\rho \in \mathcal{R}$ and $A,B \in P_p(L_p)$ for each $f \; in \; A$ there exists $g \; in \; B$ then $\rho(f-g) \le H_p(A,B)$.

Definition 9[21]**:** $\subset L_p$, let $T: E \longrightarrow 2^E$ called satisfy condition (I) if there exists no decreasing function $\emptyset: [0,\infty) \longrightarrow [0,\infty)$ with $\emptyset(0) = 0, \emptyset(r) > 0$ for all $r \in [0,\infty]$ and $\rho(f - Tf) \ge \emptyset(dist_\rho\left(f, F_p(t)\right))$ for all $f \in E$.

Preliminaries

Salman and Abed[15] mentioned the definition of (λ,ρ)-firmly nonexpansive mapping in multivalued mapping for modular spaces

Definition 10: Let $T: E \longrightarrow 2^E$ said to be (λ,ρ) -firmly nonexpansive multivalued mapping if for λ in (0,1)

$$H_p(Tf,Tg) \le \rho[(1-\lambda)(f-g) + \lambda(u-v)]$$

$u \in Tf, v \in Tg$.

Definitions 11: A double sequence $f_{k,n}$ an modular spaces in L_p is called ρ-strongly convergence to any point z in L_P, if $\lim_{n\to\infty}\rho(f_{k,n} - z) \le \in$, and write $f_{k,n} \to z$.

Definitions 12:A double sequence $f_{k,n}$ an modular spaces in L_p is called ρ-weakly convergence to any point z in L_P, if there exists Λ in L_ρ^* such that $\lim_{n\to\infty}\rho(\Lambda f_{k,n} - \Lambda z) \le \in$, and write $f_{k,n} \rightharpoonup z$.

Lemma 4: Let $f_{k,n}$ be a double sequence in modular function spaces than every ρ-strongly convergence is ρ-weakly convergence.

Proof: let $f_{k,n} \longrightarrow z$ and A in L_ρ^* then

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$$\lim_{n\to\infty} \rho(\Lambda f_{k,n} - \Lambda z) \leq \lim_{n\to\infty} \rho(\Lambda(f_{k,n} - z))$$

$$\leq \Lambda \lim_{n\to\infty} \rho(f_{k,n} - z)$$

$$\leq \in$$

Hence, $f_{k,n} \rightharpoonup z$

Note that: The concept (λ, ρ) - firmly nonexpansive multivalued mapping denoted by (λ, ρ)-FNMM

Definition 13: Let $\rho \in \mathcal{R}$ and E in L_p, E is called satisfying ρ-Opials condition if for any double sequence $f_{k,n}$ in E ρ-weakly convergence to a then for all b in E

$$\lim_{n\to\infty} \inf \rho(f_{k,n} - a) \leq \lim_{n\to\infty} \inf \rho(f_{k,n} - b), \quad \text{with } a \neq b$$

The definition of demi-closeness in accordance with the double sequences is below

Definition 14: Let $\rho \in \mathcal{R}$ and E in L_p, E and $T: E \longrightarrow 2^E$ said to be demi-closed with respect to b in E, if for any double sequences $f_{k,n}$ in E and $f_{k,n}$ ρ-weakly convergence to a and $T(f_{k,n})$ ρ-strongly convergence to b then a in E and $T(a) = b$.

Or, $(I - T)$ is demi closed, if the double sequence $f_{k,n}$ in E is ρ-weakly convergence to a in E and $(I - T)$ ρ-weakly convergence to 0, then $(I - T)(a) = 0$.

Now, define $T_k: E \longrightarrow 2^E$ and E nonempty convex subset of $L_p$the following equation

$$T_k f = (1 - \eta_k)Tf + \eta_k w \quad 1$$

where η_k in $(0,1)$ and $f, w \in E$

let $T: E \longrightarrow 2^E$, and E nonempty convex subset of L_p sequence, here, the sequence $\{f_{k,n}\}$ introduced by the following algorithm

$$u_{k,n} = \frac{1}{n+1} r_{k,n}$$

$$h_{k,n} = (1 - \beta_n) f_{k,n} + \beta_n u_{k,n}$$

$$g_{k,n} = v_{k,n}$$

$$J_{k,n} = (1 - \alpha_n) g_{k,n} + \alpha_n w_{k,n}$$

$$f_{k,n+1} = m_{k,n}, n \epsilon N \quad 2$$

Where $r_{k,n} \in P_\rho^{T_k}(f_{k,n}), v_{k,n} \in P_\rho^{T_k}(h_{k,n}), w_{k,n} \in P_\rho^{T_k}(g_{k,n})$, and $m_{k,n} \in P_\rho^{T_k}(J_{k,n})$, also $\{\alpha_n\}$ and $\{\beta_n\}$ in (0,1).

In this paper study Eq.1 when the value of $w = 0$.

Lemma 5: Let $h: L_\rho \longrightarrow 2^{L_\rho^*}$ be the ρ-normalized duality mapping, there for any $f, g \in E$ then for all $h(f + g) \in H(f + g)$ then $\rho^2(f + g) = \rho^2(f) + \rho^2(g)$

Proof: by Proposition 1 and Definition 5

$$\rho^2(f + g) = \langle f + g \setminus h \rangle = \langle f \setminus h \rangle + \langle g \setminus h \rangle$$

$$= \rho^2(f) + \rho^2(g)$$

Lemma 6: Let $h: L_\rho \longrightarrow 2^{L_\rho^*}$ be the ρ-normalized duality mapping and let f, g two function in modular spaces if $\rho(f) \leq \rho(f + \alpha g)$ then exists $h \in H(f)$ and $h(g) \geq 0$ where α in [0,1].

Proof: By Lemma 5 and Definition 5

$$\rho(f) \leq \rho(f + \alpha g) \text{ then } \rho(f)^2 \leq \rho(f + \alpha g)^2$$

$$\leq \rho(f)^2 + \rho(\alpha g)^2$$

$$\leq \rho(f)^2 + \alpha \rho(g)^2$$

So $\rho(f)^2 \leq \rho(f)^2 + \alpha h(g)$, clear $h(g) \geq 0$.

Definition 15: Let $\rho \in \mathfrak{R}$, E in L_p and E is ρ-closed and convex said to be ρ-weakly lower semi continues if every sequence $\{f_{k,n}\}$ in E ρ-weakly convergence to f This implies to $\rho(f) \leq \lim_{n\to\infty} inf \rho(f_{k,n})$.

Lemma 7: Let $\rho \in \mathfrak{R}$, E in L_p and E is ρ-closed and convex satisfies ρ-weakly lower semi continues and $\{f_{k,n}\}$ sequence in E such that $\lim_{n\to\infty} \rho(\alpha f_{k,n} + (1 - \alpha)s_1 - s_2)$ exists for $\alpha \in [0,1]$ then $s_1 = s_2$.

Proof: Let exists f_{k,n_j}, f_{k,n_r} two subsequence of $f_{k,n}$ such that $f_{k,n_j} \longrightarrow s_1$ and $f_{k,n_r} \longrightarrow s_2$ then

$$\alpha f_{k,n_j} + (1 - \alpha)s_1 - s_2 \rightharpoonup s_1 - s_2$$

By ρ- is weakly lower semi continues Definition 15

$\rho(s_1 - s_2) \le \lim_{n\to\infty} \inf \rho(\alpha f_{k,n_j} + (1-\alpha)s_1 - s_2)$

$= \lim_{n\to\infty} \inf \rho(\alpha(f_{k,n_j} - s_1) + s_1 - s_2)$

$\le \lim_{n\to\infty} \inf \rho(\alpha(f_{k,n} - s_1) + s_1 - s_2)$

$\le \lim_{n\to\infty} \inf \rho(\alpha(f_{k,n_r} - s_1) + s_1 - s_2)$

Let $h = (f_{k,n_r} - s_1)$

By Lemma 2-8 there exists $h \in H(s_1 - s_2)$ such that $h(f_{k,n_r} - s_1) \ge 0$

Now, $h(f_{k,n_r} - s_1) = \lim_{n\to\infty} h(s_2 - s_1) = -h(s_1 - s_2)$

By Definition 5, then $-\rho^2(s_1 - s_2) \ge 0$, hence $\rho^2(s_1 - s_2) \le 0$ and $s_1 = s_2$.

Lemma 8: Let $\rho \in \Re$ and ρ is (UUC1), Δ_2-condition, let E be nonempty ρ-bounded, convex and ρ-closed, $E \subset L_p$ and $T, T_k: E \longrightarrow 2^E$ are (λ,ρ)-FNMM, let $\{f_{k,n}\}$ a double sequence define by Eq. 2 then $\lim_{n\to\infty} \rho(f_{k,n} - s)$ exists for all s fixed point.

Proof: by Eq. 2, convexity of ρ, Definitions 10, Lemma 3 implies that

$$\rho(f_{k,n+1} - s) = \rho(m_{k,n} - s) \le H_p(P_p^{T_k}(J_{k,n}), P_p^{T_k}(s))$$

$$\le (1-\eta_k)\rho(J_{k,n} - s) \qquad 3$$

$$\rho(J_{k,n} - s) \le \rho\left((1-\alpha_n)g_{k,n} + \alpha_n w_{k,n}\right) - s)$$

$$\le (1-\alpha_n)\rho(g_{k,n} - s) + \alpha_n H_p(P_p^{T_k}(g_{k,n}), P_p^{T_k}(s))$$

$$\le [(1-\alpha_n) + \alpha_n(1-\eta_k)]\rho(g_{k,n} - s) \qquad 4$$

Also, $\rho(g_{k,n} - s) = \rho(v_{k,n} - s) \le H_p(P_p^{T_k}(h_{k,n}), P_p^{T_k}(s))$

$$\le (1-\eta_k)\rho(h_{k,n} - s) \qquad 5$$

Similarly, $\rho(h_{k,n} - s) = \rho(\beta_n u_{k,n} + (1-\beta_n) f_{k,n} - s)$

$$\le \beta_n \rho\left(\frac{1}{n+1} r_{k,n} - s\right) + (1-\beta_n)\rho(f_{k,n} - s)$$

$$\le \beta_n H_p(P_p^{T_k}(f_{k,n}), P_p^{T_k}(s)) + (1-\beta_n)\rho(f_{k,n} - s)$$

$$\le [\beta_n(1-\eta_k) + (1-\beta_n)]\rho(f_{k,n} - s) \qquad 6$$

By Eq. 3, Eq. 4, Eq. 5 and Eq. 6,

$$\rho(f_{k,n+1} - s) \le \mu_n \rho(f_{k,n} - s)$$

$$\mu_n = [(1-\eta_k)^2(1-\beta_n)(1-\alpha_n) + (1-\eta_k)^3\alpha_n(1-\beta_n) + (1-\eta_k)^3(1-\alpha_n)\beta_n + (1-\eta_k)^4\alpha_n\beta_n]$$

By Lemma 1-7, $lim_{n\to\infty}\rho(f_{k,n} - s)$ exists for all $s \in F_p(T)$.

Note that: $lim_{n\to\infty}\rho(f_{k,n} - s_k)$ is also exists when $s_k \in F_p(T_k)$ it is possible to prove it in the same way.

Results and Discussion

Below ρ satisfies (UUC1) and Δ_2-condition and E be nonempty ρ-bounded, convex and ρ-closed $E \subset L_p$ as in ([5] and [6])

Theorem 1: Let $\rho \in \Re$, ρ is (UUC1) and Δ_2-condition, let E be nonempty ρ-bounded, convex and ρ-closed $E \subset L_p$ and $, T_k: E \longrightarrow 2^E$, are be (λ,ρ)-FNMM, let $\{f_{k,n}\}$ in E define by Eq. 2 then $lim_{n\to\infty}\, dist_\rho \rho(f_{k,n}, P_p^{T_k}(f_{k,n})) = 0$

Proof: By Lemma 8 $lim_{n\to\infty}\rho(f_{k,n} - s)$ exists

Let $lim_{n\to\infty}\rho(f_{k,n} - s) = k$, where $k \ge 0$ 7

By Eq. 4, Eq. 5, and Eq. 6 the following hold

$$\rho(h_{k,n} - s) \le (1-\eta_k)\rho(f_n - s) \le \rho(f_n - s) \Rightarrow$$

$$lim_{n\to\infty}\rho(h_{k.n} - s) \le k \qquad 8$$

$$lim_{n\to\infty}\rho(g_{k,n} - s) \le k \qquad 9$$

$$lim_{n\to\infty}\rho(J_{k,n} - s) \le k \qquad 10$$

$$\rho(v_{k,n} - s) \le H_p\left(P_p^{T_k}(h_{k,n}), P_p^{T_k}(s)\right) \le (1-\eta_k)\rho(h_{k,n} - s)$$

$$\le \rho(f_{k,n} - s)$$

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$\lim_{n\to\infty} \rho(v_{k,n} - s) \leq \lim_{n\to\infty} \rho(f_{k,n} - s) \leq k$ 11

$\rho(u_{k,n} - s) \leq H_p\left(P_p^{T_k}(f_{k,n}), P_p^{T_k}(s)\right) \leq (1-\eta_k)\rho(f_{k,n} - s)$

$\leq (f_{k,n} - s)$

then $\lim_{n\to\infty} \rho(u_{k,n} - s) \leq k$ 12

$\rho(w_{k,n} - s) \leq H_p\left(P_p^{T_k}(g_{k,n}), P_p^{T_k}(s)\right) \leq (1-\eta_k)\rho(g_{k,n} - s)$

$\leq \rho(g_{k,n} - s) \leq (f_{k,n} - s)$

then $\lim_{n\to\infty} \rho(w_{k,n} - s) \leq k$ 13

$$\rho(m_{k,n} - s) \leq H_p(P_p^{T_k}(J_{k,n}), P_p^{T_k}(s)) \leq (1-\eta_k)\rho(J_{k,n} - s)$$

$$\leq \rho(f_{k,n} - s)$$

then $\lim_{n\to\infty} \rho(m_{k,n} - s) \leq k$
14

Let $\lim_{n\to\infty} \alpha_n = \alpha$

$\rho(f_{k,n+1} - s) = \rho(m_{k,n} - s) \leq H_p\left(P_p^{T_k}(J_{k,n}), P_p^{T_k}(s)\right)$

$$\leq (1-\eta_k)\rho(J_{k,n} - s) \leq \rho(J_{k,n} - s)$$

$\leq \rho(\alpha_n w_{k,n} + (1-\alpha_n)g_{k,n} - s) \leq \alpha_n\rho(w_{k,n} - s) + (1-\alpha_n)\rho(g_{k,n} - s)$.

so, $\lim_{n\to\infty} inf\, \rho(f_{k,n+1} - s) \leq \lim_{n\to\infty} \inf[\alpha_n\rho(w_{k,n} - s) + (1-\alpha_n)\rho(g_{k,n} - s)]$

then, $k \leq \lim_{n\to\infty} \inf \alpha_n\rho(w_{k,n} - s) + (1-\alpha)k \Rightarrow \alpha k \leq \alpha \lim_{n\to\infty} \inf \rho(w_n - s)$

hence, $k \leq \lim_{n\to\infty} \inf \rho(w_{k,n} - s)$ 15

By Eq. 13 and Eq. 14, $\lim_{n\to\infty} \rho(w_{k,n} - s) = k$
16

$\rho(w_{k,n} - s) \leq H_p(P_p^{T_k}(g_{k,n}), P_p^{T_k}(s)) \leq \rho(g_{k,n} - s)$

then, $k \leq \rho(g_{k,n} - s)$ 17

By Eq. 9 and Eq. 17, $\lim_{n\to\infty} \rho(g_{k,n} - s) = k$
18

Since, $\rho(g_{k,n} - s) = \rho(v_{k,n} - s)$, so, $\lim_{n\to\infty} \rho(v_{k,n} - s) = k$ 19

$$\rho(v_{k,n} - s) \leq H_p\left(P_p^{T_k}(h_{k,n}), P_p^{T_k}(s)\right) \leq (1-\eta_k)\rho(h_{k,n} - s)$$

$\leq \rho(h_{k,n} - s)$

$\lim_{n\to\infty} \rho(v_{k,n} - s) \leq \lim_{n\to\infty} \rho(h_{k,n} - s)$

so, $k \leq \lim_{n\to\infty} \rho(h_{k,n} - s)$
20

By Eq. 8 and Eq. 20, then $\lim_{n\to\infty} \rho(h_{k,n} - s) = k$
21

By Eq. 21, $\lim_{n\to\infty} \rho(h_{k,n} - s) = k \Rightarrow \lim_{n\to\infty} \rho(\beta_n u_{k,n} + (1-\beta_n)f_{k,n} - s) = k$

$\lim_{n\to\infty} \rho(\beta_n(r_{k,n} - s) + (1-\beta_n)(f_{k,n} - s) = k$
22

By Eq. 9, Eq. 12, Eq. 22 and Lemma 2, $\lim_{n\to\infty} \rho(f_{k,n} - u_{k,n}) = 0$ then $u_{k,n} \in P_p^{T_k}(f_{k,n})$. Since $dist_\rho \rho(f_{k,n}, P_p^{T_k}(f_n)) \leq \lim_{n\to\infty} \rho(f_{k,n} - u_{k,n})$, $lim_{n\to\infty}\, dist_\rho \rho(f_{k,n}, P_p^{T_k}(f_{k,n}))$ =0. This completes the proof.

Theorem 2: Let ,$T_k: E \longrightarrow 2^E$, are be (λ, ρ)-FNMM, let $\{f_{k,n}\}$ in E define by Eq. 2 and s_1, s_2 fixed point of T in E then $\lim_{n\to\infty} \rho(\alpha f_{k,n} + (1-\alpha)s_1 - s_2)$ exists.

Proof: To prove $\lim_{n\to\infty} \rho(\alpha f_{k,n} + (1-\alpha)s_1 - s_2)$ exists

Let $\gamma_{k,n}(\alpha) = \rho(\alpha f_{k,n} + (1-\alpha)s_1 - s_2)$

$$\gamma_n(0) = (s_1 - s_2), \gamma_n(1) = (f_{k,n} - s_2)$$

Define $R_n: E \longrightarrow 2^E$ for all $n \in N$

$$R_n(f_{k,n}) = P_p^{T_k}[(1-\alpha_n)f_{k,n} + \alpha_n u_{k,n}] = P_p^{T_k}(h_{k,n}) = v_{k,n}$$

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$$\rho\left(R_n(f_{k,n,1}) - R_n(f_{k,n,2})\right) = \rho(v_{k,n,1} - v_{k,n,2})$$

By Lemma 3 $\leq H_p(P_p^{T_{k1}}(h_{k,n,1}), P_p^{T_{k2}}(h_{k,n,2}))$

$$\leq \rho(h_{k,n,1} - h_{k,n,2}) \quad 23$$

By Definitions 3, convexity of ρ, and Lemmas 2, 3, hence

$$\rho(h_{k,n,1} - h_{k,n,2}) = \rho[(1-\beta_n)f_{k,n,1} + \frac{\beta_n}{n+1} r_{k,n,1} - (1-\beta_n)f_{k,n,2} + \frac{\beta_n}{n+1} r_{k,n,2}$$

$$\leq (1-\beta_n)(f_{k,n,1} - f_{k,n,2}) + \beta_n(r_{k,n,1} - r_{k,n,2})$$

$$\leq (1-\beta_n)(f_{k,n,1} - s) + (1-\beta_n)(f_{k,n,2} - s) + \beta_n(r_{k,n,1} - s) + \beta_n(r_{k,n,2} - s)$$

$$\leq (1-\beta_n)(f_{k,n,1} - s) + (1-\beta_n)(f_{k,n,2} - s) + \beta_n H_p(P_p^{T_{k1}}(f_{k,n,1}), P_p^{T_{k1}}(s)) + \beta_n H_p(P_p^{T_{k2}}(f_{k,n,2}), P_p^{T_{k2}}(s))$$

$$\leq (f_{k,n,1} - s) + (f_{k,n,2} - s) \quad 24$$

Let $I_{(k,n)m} = R_{(k,n)+m}\, R_{(k,n)+m-1}\, R_{(k,n)+m-2} \ldots R_{(k,n)}$

And $I_{(k,n)m}(f_{k,n}) = f_{(k,n)+m}$, $I_{(k,n)m}(s) = s$

By Eq. 23, Eq. 24 and convexity of ρ become

$$\rho\left(I_{(k,n)m}(f_{k,n,1}) - I_{(k,n)m}(f_{k,n,2})\right) \leq \rho(I_{(k,n)m}(f_{k,n,1}) - s) + \rho(I_{(k,n)m}(f_{k,n,2}) - s)$$

$$\leq (f_{k,n,1} - s) + (f_{k,n,2} - s) \quad 25$$

Let $b_{(k,n)m} = \rho(I_{(k,n)m}(\alpha f_{k,n} + (1-\alpha)s_1) - (\alpha I_{(k,n)m}(f_{k,n}) + (1-\alpha)s_1))$ for all $k, n, m \in N$

By convexity of ρ

$$b_{(k,n)m} = \rho(I_{(k,n)m}[\alpha f_{k,n} + (1-\alpha)s_1] - s_1) - \rho(\alpha I_{(k,n)m}(f_{k,n}) + (1-\alpha)s_1 - s_1))$$

$$\leq \rho(\alpha f_{k,n} - \alpha s_1) - \rho(\alpha f_{k,n} - \alpha s_1) = 0 \quad 26$$

Now,

$$\gamma_{(k,n)+m} = \rho(\alpha f_{(k,n)+m} + (1-\alpha)s_1 - s_2)$$

$$= \rho(\alpha I_{(k,n)m} f_{(k,n)} + (1-\alpha)s_1 - s_2)$$

$$= \rho(\alpha I_{(k,n)m} f_{(k,n)} + (1-\alpha)s_1 - s_2 + I_{(k,n)m}[\alpha f_{(k,n)} + (1-\alpha)s_1] - I_{(k,n)m}[\alpha f_{(k,n)} + (1-\alpha)s_1])$$

$$\leq b_{(k,n)m} + \rho(I_{(k,n)m}[\alpha f_{(k,n)} + (1-\alpha)s_1] - s_2)$$

$$\leq b_{(k,n)m} + \rho(\alpha f_{(k,n)} + (1-\alpha)s_1 - s_2)$$

$$= b_{(k,n)m} + \gamma_{k,n}(\alpha)$$

Then $\gamma_{(k,n)+m}(\alpha) \leq \gamma_{k,n}(\alpha)$

So, $\lim_{n\to\infty} \gamma_{(k,n)+m}(\alpha) \leq \lim_{n\to\infty} \gamma_{k,n}(\alpha)$

Hence, $\lim_{n\to\infty} \rho(\alpha f_{k,n} + (1-\alpha)s_1 - s_2)$ exists.

Theorem 3: Let $\rho \in \Re$ satisfy $(I-T)$ dim closed, let E be ρ- compact satisfying ρ-Opials condition and , $T_k: E \longrightarrow 2^E$, be (λ, ρ)-FNMM, then $\{f_{k,n}\}$ in E define by Eq. 2 ρ-weakly convergence to s, for s unique fixed point of T in E.

Proof: $s \in F_p(T)$, by Lemma 8 $\lim_{n\to\infty} \rho(f_{k,n} - s)$ exists

Since E is ρ- compact $f_{k,n}$ has two convergence subsequence f_{k,n_j}, f_{k,n_r}

Let $f_{k,n}$ ρ-weakly convergence to s_1 and s_2

s_1, s_2 in E weak limit of f_{k,n_j} and f_{k,n_r}, $(I-T)$ dim closed at zero

$(I-T)(s_1) = 0$ then $T(s_1) = s_1$, $s_1 \in F_p(T)$

Similarity $(I-T)(s_2) = 0$ then $T(s_2) = s_2$, $s_2 \in F_p(T)$

To prove $s_1 = s_2$

Assume that $s_1 \neq s_2$, by ρ-Opials condition

$$\lim_{n\to\infty} \rho(f_{k,n} - s_1) = \lim_{n\to\infty} \rho\left(f_{k,n_j} - s_1\right)$$

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$$\leq \lim_{n\to\infty} \rho\left(f_{k,n_j} - s_2\right) \leq \lim_{n\to\infty} \rho(f_{k,n} - s_2)$$

$$= \lim_{n\to\infty} \rho\left(f_{k,n_r} - s_2\right)$$

$$\leq \lim_{n\to\infty} \rho\left(f_{k,n_r} - s_1\right)$$

$$= \lim_{n\to\infty} \rho(f_{k,n} - s_1).$$

Contradiction, then $s_1 = s_2$, so, $f_{k,n}$ ρ-weakly convergence to unique fixed point s_1 for T in E.

Theorem 4: Let $\rho \in \Re$ and $(I - T)$ dim closed at zero let E be ρ- compact satisfying ρ-weakly lower semi continues and $,T_k: E \longrightarrow 2^E$, are be (λ, ρ)-FNMM, then $\{f_{k,n}\}$ in E define by Eq. 2 ρ-weakly convergence to s, for s unique fixed point of T in E.

Proof: Let $f_{k,n}$ ρ-weakly convergence to s_1 and s_2

$(I - T)$ dim closed at zero

$(I - T)(s_1) = 0$ then $T(s_1) = s_1$, $s_1 \in F_p(T)$, Similarity $(I - T)(s_2) = 0$ then $T(s_2) = s_2$, $s_2 \in F_p(T)$

Since E is ρ- compact, $f_{k,n}$ has subsequence f_{k,n_j} ρ-weakly convergence to s_1.

$f_{k,n}$ has another subsequence f_{k,n_r} ρ-weakly convergence to s_2

By Theorem 2 $\lim_{n\to\infty} \rho(\alpha f_{k,n} + (1 - \alpha)s_1 - s_2)$ exists

And by Lemma 7 $s_1 = s_2$

Then $f_{k,n}$ ρ-weakly convergence to unique fixed point s_1 for T in E.

Theorem 5: Let $,T_k: E \longrightarrow 2^E$, are (λ, ρ)-FNMM and satisfy condition (I), then $\{f_{k,n}\}$ in E defined by Eq. 2 ρ-weakly convergence to s_k, for all s_k fixed point of T_k in E.

Proof: By Lemma 8 $lim_{n\to\infty}\rho(f_{k,n} - s_k)$ exists for all s_k is fixed point, if $lim_{n\to\infty}\rho(f_{k,n} - s_k) = 0$, nothing to prove, if $lim_{n\to\infty}\rho(f_{k,n} - s_k) = k, k \geq 0$

Since $\rho(f_{k,n+1} - s_k) \leq \rho(f_{k,n} - s_k)$, then $dist_\rho(f_{k,n+1}, F_p(T_k)) \leq dist_\rho(f_n, F_p(T_k))$

So $lim_{n\to\infty}dist_\rho(f_n, F_p(T_k))$ exists, by applying condition (I) and Theorem 1

$$lim_{n\to\infty}\emptyset(dist_\rho\left(f_n, F_p(T_k)\right) \leq lim_{n\to\infty}\, dist_\rho\rho(f_n, P_p^{T_k}(f_n)) = 0$$

Since $\emptyset(0) = 0$, hence $lim_{n\to\infty}dist_\rho(f_n, F_p(T_k))$ = 0

By Lemma 8 $lim_{n\to\infty}\rho(f_{k,n} - s_k)$ exists, then $lim_{n\to\infty}\rho(f_{k,n} - F_p(T_k))$ exists and $s_k \in F_p(T_k)$

Suppose that f_{k,n_j} subsequence of $f_{k,n}$, and $z_{k,n}$ sequence in $F_p(T_k)$

$\rho(f_{k,n} - z_{k,n}) \leq \frac{1}{2^k}$ since $\lim inf_{n\to\infty}\, dist_p(f_{k,n}, F_p(T_k)) = 0$

$$\rho(f_{k,n_j} - z_{k,n}) \leq \rho(f_{k,n} - z_{k,n}) \leq \frac{1}{2^k}$$

$$\rho(z_{(k,n)+1} - z_{k,n}) \leq \rho\left(z_{(k,n)+1} - f_{k,n_j}\right) + \rho(f_{k,n_j} - z_{k,n})$$

$$\leq \frac{1}{2^{k+1}} + \frac{1}{2^k}$$

$$\leq \frac{1}{2^{k-1}}$$

$\rho(z_{(k,n)+1} - z_{k,n}) \longrightarrow 0$ as $k, n \longrightarrow \infty$

$z_{k,n}$ is ρ-Cauchy, $F_p(T_k)$, Since Δ_2 condition , then ρ-cauchy$\Leftrightarrow$ ρ-converge,

So, $z_{k,n}$ is ρ-converge to $F_p(T_k)$, then $\rho(z_{k,n} - s_k) \longrightarrow 0$

Now,

$\rho\left(f_{k,n_j} - s_k\right) \leq \rho\left(f_{k,n_j} - z_{k,n}\right) + \rho(z_{k,n} - s_k)$, hence, $f_{k,n}$ ρ-strongly converge to fixed point $s_k$in $F_p(T_k)$

By Lemma 4 $f_{k,n}$ ρ-weakly convergence to s_k

Conclusion

The iterative scheme in Eq. 2 suggested by double sequence, where prove later that iterative scheme has weak convergence to the unique fixed point as in Theorem 3 and Theorem 4. While the iterative scheme in Eq. 2 strong and weak convergence to fixed point provided that the Condition (I) as in Theorem 5, it is possible for researchers to deal with this iterative scheme with different class of mapping and reach the results.

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Authors' Declaration

- Conflicts of Interest: None.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Baghdad.

Authors' Contribution Statement

This work was carried out in collaboration between all authors. S S, the owner of the research idea she reviewed and processed the work. B S, wrote and proved the results. All authors read and approved the final manuscript.

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نتائج تقريبية جديدة بواسطة التقارب الضعيف لمتتابعات متشعبة

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الخلاصة

تعتبر الفضاءات المعيارية هي تعميم طبيعي لبعض الفضاءات مثل فضاء Lebesgue و فضاء Orlicz و فضاء Lorentz P وفضاء Orlicz-Lorentz و فضاء Musielak- Orlicz وغيرها . الدوال المعيارية تفتقر الى الخصائص الأساسية والمرنة التي يمتلكها دوال المعيار. لأنها دوال تفتقر الى التجانس والترابط الفرعي وبالتالي قد يكون من المدهش ان نكون قادرون على استخدام تقنيات تتضمن مراكز التقارب, الهيكل الطبيعي والتحدب المنتظم للحصول على نظريات النقطه الصامده. الغرض من هذه الورقة هو اعطاء خوارزميه تكراريه جديده مسرعة للتطبيق متعددة القيم- ذات القيمة الواحدة في فضاءات مودلر واثبات بعض النتائج حول تقاربها (ضعيف-قوي) مع نقطه صامده (او نقطه صامده مشتركه) من خلال العمل فضاءات مودلر تحقق (UUC1) وكذلك شرط Δ_2 و في بعض الاحيان يتطلب العمل خاصيه أوبيل وشبه الانغلاق. الهدف من هذا البحث هو اثبات الوجود والوحدانية للنقاط الصامدة الناتجة عن التقارب الضعيف لمخطط تكراري متشعب. تم انشاء هذا المخطط من خلال خمسه خطوات تكراريه ل (λ,ρ)- دوال غير ممتدة (متعددة و احاديه) بقوه في فضاء مودلر. و ρ تحقق(UUC1) و شرط Δ_2-. للحصول على هذه النتائج والحقائق الاخرى, نقوم بأجراء تنسيقات بين التعاريف هي التقارب الضعيف, شبه القرب, و شرط اوبيل لحاله المتتابعات المزدوجة. لاحظ أن المؤلفين قدموا دراسة سابقة حول التقارب القوي لمتتابعات مزدوجة متشعبة متضمنة نتائج مهمة, راجع المصادر.

الكلمات المفتاحية: متتابعة مزدوجة، دوال غير ممتدة بشدة، تقارب ضعيف، تقارب قوي، النقطه الصامدة.