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A New Invariant Regarding Irreversible k-Threshold Conversion Processes on Some Graphs

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Abstract

An irreversible k-threshold conversion (k-conversion in short) process on a graph $G = (V, E)$ is a specific type of graph diffusion problems which particularly studies the spread of a change of state of the vertices of the graph starting with an initial chosen set while the conversion spread occurs according to a pre -determined conversion rule. Irreversible k-conversion study the diffusion of a conversion of state (from 0 to 1) on the vertex set of a graph $G = (V, E)$. At the first step $t = 0$, a set $S_0 \subseteq V$.is selected and for $t \in \{1,2, \dots, \}$; S_t is obtained by adding all vertices that have k or more neighbors in S_{t-1} to S_{t-1}. S_0 is called the seed set of the process and a seed set is called an irreversible k-threshold conversion set (IkCS) of G if the following condition is achieved: Starting from S_0 and for some $t \geq 0$; $S_t = V(G)$. The minimum cardinality of all the IkCSs of G is called the k- conversion number of G (denoted as $(C_k(G))$. In this paper, a new invariant called the irreversible k-threshold conversion time (denoted by $(CT_k(G))$ is defined. This invariant retrieves the minimum number of steps (t) that the minimum IkCS needs in order to convert $V(G)$ entirely. $CT_k(G)$ is studied on some simple graphs such as paths, cycles and star graphs. $C_k(G)$ and $CT_k(G)$ are also determined for the tensor product of a path P_m and a cycle C_n (which is denoted by $P_m \times C_n$) for some values of k, m, n. Finally, $C_k(G)$ of the Ladder graph L_n is determined for $n \geq 2$.

Keywords: Graph conversion process, k-Threshold conversion number, k-Threshold conversion time, Ladder graph, Seed set, Tensor product.

AMS Subject Classification: 68R10, 05C69, 05C90

Introduction

Let $G(V, E)$ be a graph with $|V| = n$ vertices and $|E| = m$ edges. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V: uv \in E\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v (denoted by $deg(v)$) is the number of all vertices that are adjacent to v. Therefore, $deg(v) = |N(v)|$ while $\Delta(G) = max\{deg(v): v \in V(G)\}$.

The distance $d(u, v)$ between two vertices u and v of a finite graph is the minimum length of the paths connecting them[1]. The diameter of a graph (denoted by $diam(G)$) is the length $max_{u,v} d(u, v)$ between any two vertices u and v of the graph[2]. For any undefined term in this paper, Bickle[3] is recommended.

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An independent vertex set of a graph $G(V,E)$ is a subset of V such that no two vertices in the subset represent an edge of G. The independence number, denoted by $\alpha(G)$, is the cardinality of the largest independent vertex set of G.

An Irreversible k-threshold conversion process on a graph G is a sequence of subsets $S_0, S_1, \ldots$ of $V(G)$ such that for $t = 1,2,\ldots$,

$$S_t = S_{t-1} \cup \{v \in V - S_{t-1} : |N(v) \cap S_{t-1}| \geq k\}.$$

The set S_0 is called the seed set for the process, and if $S_t = V(G)$ for some finite t then the seed set is called an irreversible k-threshold conversion set (IkCS) of G.

Vertices in S_t are called "converted" and vertices in $V - S_t$ are called "unconverted", and if a vertex v belongs to both $V - S_{t-1}$ and S_t, then v is said to be "converted at time t". The minimum cardinality of all the IkCSs of G is called the k- conversion number of G (denoted as $(C_k(G))$. It is obvious that $C_1(G) = 1$ for connected graphs.

The problem was introduced for the first time by Dreyer in his doctoral dissertation. Dreyer and Roberts[4] found $C_2(G)$ for trees. The problem then was further studied by Wodlinger[5], Mynhardt et al.[6], and Shaheen et al.[7].

The tensor product[8] of a path P_m and a cycle C_n (denoted by $P_m \times C_n$) has the vertex set $V(P_m \times C_n) = \{(i,j): 1 \leq i \leq m,\ 1 \leq j \leq n\}$ when $(i,j), (l,r)$ are adjacent if $(i,l) \in E(P_m)$ and $(j,r) \in E(C_n)$. The Ladder graph L_n is the cartesian product $P_n \square P_2$[9].

In this paper, a new invariant called the irreversible k-threshold conversion time (denoted by $(CT_k(G))$ is defined. This invariant retrieves the minimum number of steps (t) that the minimum IkCS needs in order to convert $V(G)$ entirely. $CT_k(G)$ is studied on some simple graphs such as the path P_n, the cycle C_n and the star graph $K_{1,n}$. Then $C_k(P_m \times C_n)$ and $CT_k(P_m \times C_n)$ are determined for some values of k, m, n. Finally, $CT_k(G)$ of the Ladder graph L_n is determined for $n \geq 2$.

The two following Theorems are preliminary work that is used in the proofs later in this paper:

Theorem 1[3]: For $m \geq 2$ and $n \geq 3$: $\alpha(P_m) = \left\lceil \frac{m}{2} \right\rceil$; $\alpha(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem 2[10]: If G, H are either a path or a cycle; $\alpha(G \times H) = \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$.

Theorem 3[4]: For $m \geq 2$ and $n \geq 3$: $C_2(P_m) = \left\lceil \frac{m+1}{2} \right\rceil$; $C_2(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Definition 1[5]: A nonempty set U of vertices of G is k −immune if, for all $v \in V$, $|N(v) - U| < k$.

The following are some helpful notes for better understanding of the next section of the paper:

Note 1: As an immediate consequence of the definition of IkCS, $C_k(G) \geq k$ for any graph G.

Note 2: As an immediate consequence of the definition of IkCS, when studying an Irreversible k-threshold conversion process on a graph $G = (V, E)$ all vertices $\{v \in V\,; deg(v) < k\}$ must be included in the seed set S_0, otherwise the process will fail because none of these vertices can satisfy the conversion rule. These vertices are called k-immune vertices[7].

Note 3: It is obvious that a k −threshold conversion process fails if there exists a k −immune set (U) for which $U \cap S_0 = \emptyset$.

Note 4: Throughout this paper, the rows of $P_m \times C_n$ are denoted by $R_i: 1 \leq i \leq m$ and $R_l = \{(l,j): 1 \leq j \leq n\}$. Meanwhile, the columns are denoted by $CO_j: 1 \leq j \leq n$; $CO_l = \{(i,l): 1 \leq i \leq m\}$. The same notations are used for the ladder graph L_n.

Note 5: In every figure of this article, the black color is assigned to the converted vertices while the white color is assigned to the unconverted ones. All edges that extend beyond the border of the $P_m \times C_n$ grid are assumed to wrap around to the opposite side.

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Results and Discussion

Remark 1: In this paper, optimization is used for two parameters (the size of the seed set and the number of steps). However, these two parameters are not equally important. The priority is always to minimize the seed set S_0 (by determining $C_k(G)$), then the parameter $CT_k(G)$ retrieves the minimum number of steps of all the processes initiated by seed sets of cardinality $C_k(G)$.

Definition 2: Let S_0 be an IkCS of a graph $G = (V, E)$. The time of the irreversible k-Threshold process that is initiated by S_0 is the number of steps needed for the conversion to reach every vertex of $V(G)$ starting from S_0.

Definition 3: The irreversible k-Threshold conversion time of a graph G (denoted by $CT_k(G)$) is the minimum time of all IkCSs with cardinality $C_k(G)$ of G.

Proposition 1: For any graph $G = (V, E)$ with $|V| = n$ and for any non-trivial conversion threshold $(1 \le k \le \Delta(G))$, the irreversible k -Threshold conversion time $CT_k(G)$ is well defined because it is bounded by the following lower and upper bounds:

$$1 \le CT_k(G) \le n - k \quad 1$$

Proof:

Due to the definition of Irreversible k-threshold conversion processes, the following remarks can be made:

i. $CT_k(G) \ge 1$ because when $k \le \Delta(G)$, $S_0 \ne V(G)$ when $|S_0| = C_k(G)$.

ii. $CT_k(G) \le n - k$ because the process fails if there exists a step t for which $S_t = S_{t-1} \ne V(G)$, therefore it is necessary to add at least one vertex to S_0 during every step of the process, but considering that $|S_0| \ge k$, then $|S_1| \ge k + 1$; $|S_2| \ge k + 2$ and the argument applies for any step t which means:
For any step $1 \le t \le CT_k(G)$; $|S_t| \ge k + t$

Therefore, $|S_{CT_k(G)}| \ge k + CT_k(G)$.
However, $S_{CT_k(G)} = V(G)$ which means $|V(G)| \ge k + CT_k(G)$, thus $CT_k(G) \le n - k$.

From i and ii the requested is concluded.□

Proposition 2: For any connected graph $G = (V, E)$; $CT_k(G) = 1$ when $k = \Delta(G)$.

Proof:

Since all vertices of degree $deg(v) < \Delta(G)$ are k-immune, they must be included in the seed set S_0 or else the process automatically fails. Let a and b be two adjacent vertices from $V(G)$ with $deg(a) = deg(b) = \Delta(G)$. If neither a nor b belongs to S_0, then $\{a, b\}$ forms an unconvertable set because neither a nor b can be adjacent to k converted vertices at any step of the process. This means that there cannot be two adjacent vertices of $V - S_0$. Therefore, for any vertex $x \in V(G)$ the following cases must be considered:

Case 1: $deg(x) < \Delta(G)$. This means $x \in S_0$.

Case 2: $deg(x) = \Delta(G)$. Then following two sub-cases must be discussed:

Case 2.a: $x \in S_0$.

Case 2.b: $x \notin S_0$ and x is adjacent to k vertices from S_0. Therefore, x satisfies the conversion rule and is converted at step $t = 1$.

From the previous cases; $V(G)$ is entirely converted at the end of step $t = 1$. Therefore, $CT_k(G) = 1$.□

Observation 1: As an immediate consequence of Proposition 2, $CT_k(G) = 1$ for all k-regular graphs.

Observation 2: For all connected graphs, $CT_1(G) \le diam(G)$.

Observation 3:

1. $CT_1(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$. Where P_n is the path graph of order $n \ge 2$ and $S_0 = \{a_{\left\lfloor \frac{n+1}{2} \right\rfloor}\}$.
2. $CT_1(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$. Where C_n is the cycle graph of order $n \ge 3$. S_0 contains only one arbitrary vertex.

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3. $CT_1(K_{1,n}) = 1$. Where $K_{1,n}$ is the star graph of order $n \geq 3$ and S_0 contains only the central vertex (the vertex of degree).

Proposition 3: For $n \geq 3$:

i. $C_k(P_2 \times C_n) = \begin{cases} 1 \text{ if } k = 1 \text{ and } n \text{ is odd}; \\ 2 \text{ if } k = 1 \text{ and } n \text{ is even}; \\ n \text{ if } k = 2. \end{cases}$

ii. $CT_k(P_2 \times C_n) = \begin{cases} n \text{ if } k = 1 \text{ and } n \text{ is odd}; \\ \frac{n}{2} \text{ if } k = 1 \text{ and } n \text{ is even}; \\ 1 \text{ if } k = 2. \end{cases}$

Proof: The following cases for k should be considered:

Case 1: $k = 1$. Let us consider the following sub-cases for n:

Case 1.a: n *is odd*. Therefore, $P_2 \times C_n$ is isomorphic to a cycle C_{2n}. This means:

- $C_1(P_2 \times C_n) = 1$ since $P_2 \times C_n$ is connected.
- $CT_1(P_2 \times C_n) = n$ by Observation 3.

Case 1.b: n *is even*. Therefore, $P_2 \times C_n$ is isomorphic to the sum of two cycles $C'_n + C''_n$ defined as:

$$C'_n = \left\{(1,2i+1),(2,2j): 0 \leq i \leq \frac{n}{2} - 1;\ 1 \leq j \leq \frac{n}{2}\right\};$$

$$C''_n = \left\{(1,2j),(2,2i+1): 0 \leq i \leq \frac{n}{2} - 1;\ 1 \leq j \leq \frac{n}{2}\right\}.$$

This means $S_0 = \{x, y: x \in C'_n \text{ and } y \in C''_n\}$ which makes $C_1(P_2 \times C_n) = 2$. Since the conversion processes run separately on C'_n and C''_n, then $CT_1(P_2 \times C_n) = CT_1(C'_n) = CT_1(C''_n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$.

Case 2: $k = 2$. In a similar way to Case 1 both sub-cases for n are considered as follows:

Case 2.a: n *is odd***.** Since $P_2 \times C_n$ is isomorphic to a cycle C_{2n}, then:

- $C_2(P_2 \times C_n) = C_2(C_{2n}) = n$ by Theorem 3.
- $CT_2(P_2 \times C_n) = CT_2(C_{2n}) = 1$ by Observation 1 since C_{2n} is 2-regular.

Case 2.b. n *is even***.** Since $P_2 \times C_n$ is isomorphic to the sum of two cycles $C'_n + C''_n$, then:

- $C_2(P_2 \times C_n) = 2C_2(C_n) = 2(\frac{n}{2}) = n$ by Theorem 3.
- $CT_2(P_2 \times C_n) = CT_2(C'_n) = CT_2(C''_n) = 1$ by Observation 1 because C'_n, C''_n are 2-regular.

From the previous cases and sub-cases, the requested is proven. □

Proposition 4: For $n \geq 3$:

i. $C_2(P_3 \times C_n) = n$.

ii. $CT_2(P_3 \times C_n) = 1$.

Proof: It is obvious that $V(P_3 \times C_n)$ can be divided based on vertex degree into two subsets:

$$Q_1 = \{v \in V; deg(v) = 2\} = \{(1,j),(3,j): 1 \leq j \leq n\};$$

$$Q_2 = \{v \in V; deg(v) = 4\} = \{(2,j): 1 \leq j \leq n\}.$$

For $1 \leq j \leq n$ let us define some special sets on $P_3 \times C_n$ as:

$$W_j = \{(1,j-1),(1,j+1),(2,j),(3,j-1)\};$$

$$X_j = \{(1,j-1),(1,j+1),(2,j),(3,j+1)\};$$

$$Y_j = \{(1,j-1),(2,j),(3,j-1),(3,j+1)\};$$

$$Z_j = \{(1,j+1),(2,j),(3,j-1),(3,j+1)\}.$$

Each version of any of these sets cannot be converted at any step if it does not contain at least one vertex of S_0 because it consists of:

- three vertices of degree 1 that are adjacent to one vertex of the same set.
- One vertex of degree 4 that is adjacent to three vertices of the same set.

Fig 1 shows that W_3, X_3, Y_3 and Z_3 are 2-immune on $P_3 \times C_5$. (In Fig 1 the vertices of W_3 are denoted by $\{w_1, w_2, w_3, w_4\}$ and the same notation style is used for X_3, Y_3 and Z_3). Since for all $w \in W$, $|N(w) - W| < 2 = k$; then these sets are 2-immune

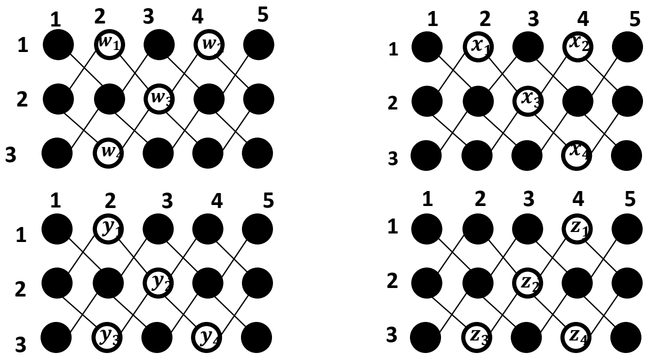

Figure 1. 2-immune sets W_3, X_3, Y_3 and Z_3 on $P_3 \times C_5$.

Let us now try to distribute only three vertices from S_0 on the four columns CO_4, CO_5, CO_6, CO_7 of $P_3 \times C_{10}$ without leaving any unconverted version of $W_j, X_j, Y_j, Z_j: j \in \{5,6\}$. The following cases are considered:

Case 1: $(2,5), (2,6) \notin S_0$. Let $(1,4), (3,4), (1,7) \in S_0$. This would leave $Y_6 \cap S_0 = \emptyset$ and since Y_6 is 2-immune, then and the process fails. Without loss of generality, a 2-immune set will be left if neither $(2,4), (2,5)$ is included in S_0.

Case 2: Only one of $(2,5), (2,6)$ belongs to S_0. Let us assume that $(2,5) \in S_0$ taking into consideration that without loss of generality, the same argument applies if $(2,6) \in S_0$. Since $(2,5) \in S_0$, this leaves two converted vertices to be distributed in a way that does not leave any of W_6, X_6, Y_6, Z_6 unconverted. This is achievable if the two converted vertices were two vertices from $\{(1,5), (1,7), (3,5), (3,7)\}$.

Let us discuss the possibilities of the two chosen converted vertices in regards to the following sets:

$B_1 = \{(1,5), (2,4)\}, B_2 = \{(2,4), (3,5)\}, B_3 = \{(1,6), (2,7)\}, B_4 = \{(2,7), (3,6)\}$.

Case 2.a: The two converted vertices are $(1,5), (3,5)$. This would prevent leaving any of W_6, X_6, Y_6, Z_6 unconverted. However, it would also leave B_3, B_4 fully unconverted, this means both $(1,8), (3,8)$ need to be included in S_0 to avoid having unconverted W_7, X_7, Y_7, Z_7.Therefore, 5 vertices from the 5 columns $CO_j: 4 \leq j \leq 8$ must be included in S_0 or else the process automatically fails.

Case 2.b: Only one of the two chosen converted vertices belongs to $\{(1,5), (3,5)\}$, if it is $(1,5)$, then B_2, B_3, B_4 are all left unconverted therefore in addition to $(1,8), (3,8)$ this means one vertex from $\{(1,3), (3,3)\}$ must be included in S_0 to avoid leaving Y_4 unconverted. Therefore, 6 vertices from the 6 columns $CO_j: 3 \leq j \leq 8$ should be included in S_0 or else the process automatically fails. Without loss of generality, the same result is obtained if $(3,5) \in S_0$.

Case 2.c: In order to prevent leaving any of B_1, B_2, B_3, B_4 entirely unconverted, the two chosen converted vertices should be $(2,4), (2,7)$. However, that would leave W_6, X_6, Y_6, Z_6 unconverted and the process automatically fails.

From all the cases and subcases and without loss of generality it can be concluded that the $n-2$ columns $CO_j: 2 \leq j \leq n-1$ must include n vertices of S_0, and since $(2,1)$ is adjacent to each of $(1,n), (3,n)$ while $(2,n)$ is adjacent to each of $(1,1), (3,1)$, then the same argument applies to CO_1 and CO_n. Therefore:

$$C_2(P_3 \times C_n) \geq n \qquad 2$$

Let the seed set be $S_0 = \{(2,j): 1 \leq j \leq n\}$ which is of cardinality n. The process goes as follows:

$t = 0: S_0 = \{(2,j): 1 \leq j \leq n\}$.

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$t = 1: S_1 = S_0 \cup \{(1,j),(3,j): 1 \leq j \leq n\} = V(P_3 \times C_n)$. This means S_0 is an I2CS on $P_3 \times C_n$ and $C_2(P_3 \times C_n) \leq n$. From 2 it can be concluded that $C_2(P_3 \times C_n) = n$ for $n \geq 10$. However, since $(2,1)$ is adjacent to each of $(1,n),(3,n)$ while $(2,n)$ is adjacent to each of $(1,1),(3,1)$, then the same argument applies for all values of $n \geq 3$. Therefore: $C_2(P_3 \times C_n) = n$ for $n \geq 10$, which means $CT_2(P_3 \times C_n) \leq 1$, but since $S_0 \neq V(P_3 \times C_n)$ then $CT_2(P_3 \times C_n) \geq 1$ which means $CT_2(P_3 \times C_n) = 1$ for $n \geq 3$.

From all the above the requested is concluded.□

Proposition 5: For $n \geq 3$:

i. $C_3(P_3 \times C_n) = 2n$.
ii. $CT_3(P_3 \times C_n) = 1$.

Proof: It is obvious that all vertices of Q_1 are 3-immune therefore they must be included in S_0 which means $C_3(P_3 \times C_n) \geq |Q_1| = 2n$. Let $S_0 = Q_1 = \{(1,j),(3,j): 1 \leq j \leq n\}$ be the seed set, then $S_1 = S_0 \cup Q_1 = V(P_3 \times C_n)$. This means S_0 is an I3CS on $P_3 \times C_n$ and $C_3(P_3 \times C_n) \leq 2n$. Therefore, $C_3(P_3 \times C_n) = 2n$.

It can also be concluded that $CT_3(P_3 \times C_n) \leq 1$ and since $S_0 \neq V(P_3 \times C_n)$, then $CT_3(P_3 \times C_n) \geq 1$ which means $CT_3(P_3 \times C_n) = 1$. From all the above the requested is proven.□

Proposition 6: For $n \geq 3$:

i. $C_3(P_4 \times C_n) = \begin{cases} 2n+1 \; if \; n \; is \; odd; \\ 2n+2 \; if \; n \; is \; even. \end{cases}$

ii. $CT_3(P_4 \times C_n) = \begin{cases} n \; if \; n \; is \; odd; \\ \frac{n}{2} \; if \; n \; is \; even. \end{cases}$

Proof: Since $V(P_3 \times C_n)$ can be divided based on vertex degree into two subsets:

$$Q_1 = \{v \in V; \deg(v) = 2\} = \{(1,j),(4,j): 1 \leq j \leq n\};$$

$$Q_2 = \{v \in V; \deg(v) = 4\} = \{(2,j),(3,j): 1 \leq j \leq n\}.$$

it is obvious that all vertices of Q_1 are 3-immune which means $Q_1 \subseteq S_0$. Let $S_0 = Q_1$ be the seed set, then $S_1 = S_0 \neq V(P_4 \times C_n)$ and the process fails. This means:

$$C_3(P_4 \times C_n) > 2n \qquad 3$$

Now let us consider the two following cases for n:

Case 1: n is odd. Let the seed set be $S_0 = Q_1 \cup \{(2,1)\}$ which is of cardinality $2n+1$. The process goes as follows:

$S_1 = S_0 \cup \{(3,2),(3,n)\}$; $S_2 = S_1 \cup \{(2,3),(2,n-1)\}$; $S_3 = S_2 \cup \{(3,4),(2,n-2)\}$;

For $2 \leq t \leq n-1$ and t is even: $S_t = S_{t-1} \cup \{(2,t+1),(2,n-t+1)\}$;

For $3 \leq t \leq n-2$ and t is odd: $S_t = S_{t-1} \cup \{(3,t+1),(3,n-t+1)\}$;

The process ends at $t = n$ for which $S_n = S_{n-1} \cup \{(3,1)\} = V(P_4 \times C_n)$. Therefore, $S_0 = Q_1 \cup \{(2,1)\}$ is I3CS which means if n is odd:

$$C_3(P_4 \times C_n) \leq 2n+1 \qquad 4$$

From Eqs 3 and 4; $C_3(P_4 \times C_n) = 2n+1$ if n is odd. Without loss of generality and due to symmetry, the same argument applies for any $S_0 = Q_1 \cup \{x: x \in R_2 \cup R_3\}$ and the same results is obtained. Therefore, $CT_3(P_4 \times C_n) = n$ if n is odd.

Case 2: n is even. Due to (3); $C_3(P_4 \times C_n) > 2n$. It is obvious that $R_2 \cup R_3 = M_1 \cup M_2$:

$$M_1 = \{(2,2i+1),(3,2j): 0 \leq i \leq \frac{n}{2}-1;\; 1 \leq j \leq \frac{n}{2}\};$$

$$M_2 = \{(2,2j),(3,2i+1): 0 \leq i \leq \frac{n}{2}-1;\; 1 \leq j \leq \frac{n}{2}\}.$$

It is noticeable that each one of G_{M_1}, G_{M_2} is a cycle of order n. It can also be noticed that $G_{R_2 \cup R_3} = G_{M_1 \cup M_2}$ is not connected because no vertex of M_1 is adjacent to any vertex of M_2 and vice versa. This means $Q_1 \cup \{x: x \in R_2 \cup R_3\}$ is not an I3CS on $P_4 \times C_n$ because the conversion will not reach any vertex of M_2 if $x \in M_1$ and vice versa, therefore one more vertex must be added to S_0 so it becomes:

$S_0 = Q_1 \cup \{x, y: x \in M_1 \text{ and } y \in M_2\}$. Let us choose $x = (2,1), y = (3,1)$. The process goes as:

$S_0 = Q_1 \cup \{(2,1), (3,1)\}$; $S_1 = S_0 \cup \{(2,2), (3,2), (2,n), (3,n)\}$;

For $2 \le t \le \frac{n}{2} - 1$: $S_t = S_{t-1} \cup \{(2, t+1), (2, n-t+1), (3, t+1), (3, n-t+1)\}$;

The process ends at $t = \frac{n}{2}$ for which $S_{\frac{n}{2}} = S_{\frac{n}{2}-1} \cup \{(2, \frac{n}{2}+1), (3, \frac{n}{2}+1)\} = V(P_4 \times C_n)$.

Therefore, $S_0 = Q_1 \cup \{(2,1)\}$ is I3CS which means $C_3(P_4 \times C_n) \le 2n + 2$ if n is even. This means that $C_3(P_4 \times C_n) = 2n + 2$ if n is even. Without loss of generality and due to symmetry, the same argument applies for any $S_0 = Q_1 \cup \{x, y: x \in M_1 \text{ and } y \in M_2\}$ and the same results are obtained. Therefore, $CT_3(P_4 \times C_n) = \frac{n}{2}$ if n is even. From all the previous cases the requeated is proven.□

Proposition 7: For $n \ge 3$:

i. $C_3(P_5 \times C_n) = \begin{cases} \frac{5n}{2} \; if \; n \equiv 0 (mod\ 4); \\ \frac{5n+1}{2} \; if \; n \equiv 1,3 (mod\ 4); \\ \frac{5n}{2} + 1 \; if \; n \equiv 2 (mod\ 4). \end{cases}$

ii. $CT_3(P_5 \times C_n) = 2$.

Proof: In a similar way to previous cases, $V(P_3 \times C_n)$ can be divided based on vertex degree into:

$$Q_1 = \{v \in V; \deg(v) = 2\} = \{(1,j), (5,j): 1 \le j \le n\};$$

$$Q_2 = \{v \in V; \deg(v) = 4\} = \{(2,j), (3,j), (4,j): 1 \le j \le n\}.$$

Since $k = 3$, all vertices of Q_1 must be included in S_0. Let us define the sets $U_j: 2 \le j \le n-3$ as $U_j = \{(3,j), (2,j+1), (4,j+1), (3,j+2)\}$. It can be noticed that for any j then U_j is 3-immune because each vertex of $u \in U_j$ is of degree 4 and is adjacent to two vertices of U_j. Fig 2 shows that U_3 is 3-immune on $P_5 \times C_5$ which means if $U_3 \cap S_0 = \emptyset$ then the process fails even when $S_0 = V - U_3$.

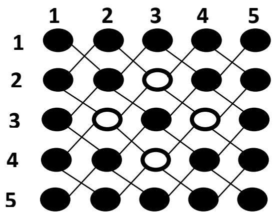

Figure 2. 3-immune U_3 on $P_5 \times C_5$.

This means every set $\{(2,j), (3,j), (4,j), (2,j+1), (3,j+1), (4,j+1), (2,j+2), (3,j+2), (4,j+2): 2 \le j \le n-2\}$ must contain at least one vertex of S_0, otherwise at least one version of $U_j \cap S_0 = \emptyset$ will be left on $P_5 \times C_n$ while as shown in Fig 3, every set $U_j' = \{(2,j), (3,j), (4,j), (2,j+1), (3,j+1), (4,j+1), (2,j+2), (3,j+2), (4,j+2), (2,j+3), (3,j+3), (4,j+3): 2 \le j \le n-4\}$ must contain at least two vertices of S_0, otherwise at least one version of $U_j \cap S_0 = \emptyset$ will be left on $P_5 \times C_n$ and the process will fail.

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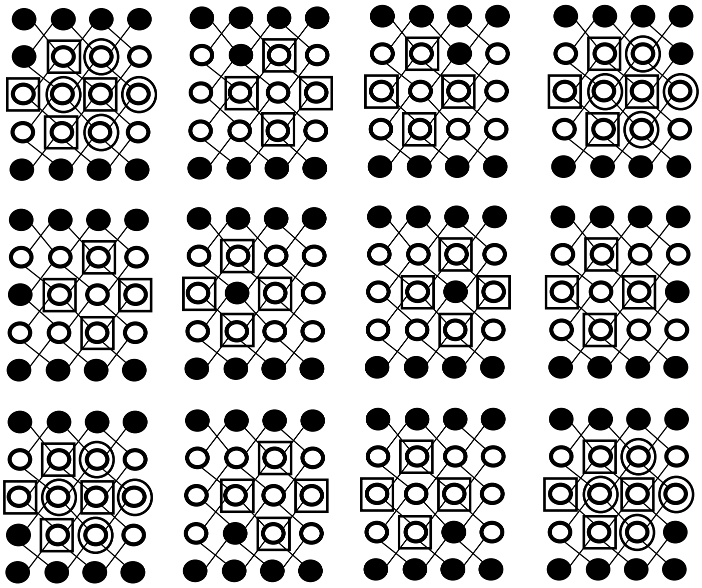

Figure 3. the bold squares and circles represent 3-immune sets on an arbitrary U_j' when $|U_j' \cap S_0| = 1$.

Since each vertex of $(2,1),(3,1),(4,1)$ is adjacent to two verices of CO_n while each of $(1,1),(5,1)$ is adjacent to one vertex of CO_n (and vice versa), then the same argument studied above applies to CO_1, CO_{n-1}, CO_n. This means every $U_j': 1 \le j \le n$ must contain at least two vertices of S_0. The following cases for n are considered:

Case 1: $n \equiv 0(mod\ 4)$. As it was found out, in addition to Q_1, at least $2(\frac{n}{4}) = \frac{n}{2}$ vertices of Q_2 must be included in S_0 to avoid leaving any version of U_j for which $U_j \cap S_0 = \emptyset$. This means for $n \equiv 0(mod\ 4)$:

$$C_3(P_5 \times C_n) \ge 2n + \frac{n}{2} = \frac{5n}{2}. \qquad 5$$

Let the seed set be $S_0 = Q_1 \cup \{(3,4l+1),(3,4l+2): 0 \le l \le \frac{n}{4} - 1\}$ which is of cardinality $\frac{5n}{2}$. The process goes as follows:

$t = 0: S_0 = Q_1 \cup \{(3,4l+1),(3,4l+2): 0 \le l \le \frac{n}{4} - 1\}$;

$t = 1: S_1 = S_0 \cup \{(2,l),(4,l): 1 \le l \le n\}$.

$t = 2: S_2 = S_1 \cup \{(3,4l+3),(3,4l+4): 0 \le l \le \frac{n}{4} - 1\} = V(P_5 \times C_n)$ which means S_0 is an I3CS on $P_5 \times C_n$. Therefore, $C_3(P_5 \times C_n) \le \frac{5n}{2}$. From (5) it is obtained that $C_3(P_5 \times C_n) = \frac{5n}{2}$ if $n \equiv 0(mod\ 4)$.

It can also be concluded that $CT_3(P_5 \times C_n) \le 2$. Looking at Fig 2, it is noticeable that converting all vertices of U_j' in one step starting from two converted vertices is impossible. This means $CT_3(P_5 \times C_n) = 2$ if $n \equiv 0(mod\ 4)$.

Let us define S_0 for the remaining cases. However, the process in these cases goes similarly to Case 1 (with the same number of steps):

Case 2: $n \equiv 1(mod\ 4)$.

$S_0 = Q_1 \cup \{(3,4l+1),(3,4l+2): 0 \le l \le \lfloor\frac{n}{4}\rfloor - 1\} \cup \{(3,n)\}$ of cardinality $\frac{5n+1}{4}$.

Case 3: $n \equiv 2(mod\ 4)$.

$S_0 = Q_1 \cup \{(3,4l+1),(3,4l+2): 0 \le l \le \lfloor\frac{n}{4}\rfloor - 1\} \cup \{(3,n-1),(3,n)\}$ of cardinality $\frac{5n}{2} + 1$.

Case 4: $n \equiv 3(mod\ 4)$.

$S_0 = Q_1 \cup \{(3,4l+1),(3,4l+2): 0 \le l \le \lfloor\frac{n}{4}\rfloor - 1\} \cup \{(3,n-2),(3,n-1)\}$ of cardinality $\frac{5n+1}{4}$.

From all the previous cases the requested is concluded.□

Proposition 8: For $m, n \geq 3$;

i. $C_4(P_m \times C_n) = \begin{cases} nm - \max\{(n-2)\left\lceil\frac{m-2}{2}\right\rceil, (m-2)\left\lceil\frac{n-2}{2}\right\rceil\} \text{ if } m \text{ or } n \text{ is odd;} \\ \frac{mn+2m+2n-4}{2} \text{ if } m \text{ and } n \text{ are even.} \end{cases}$

ii. $CT_4(P_m \times C_n) = 1$.

Proof: Since $k = 4$, all vertices of $Q_1 = R_1 \cup R_m$ must be included in S_0. Otherwise, the process automatically fails. Since every $u \in Q_2 = V(P_m \times C_n) - Q_1$ is of degree 4, there cannot be two adjacent unconverted vertices of Q_2 at $t = 0$ or else neither one of these two vertices will satisfy the conversion rule at any step of the process, therefore the process fails. To avoid that, $Q_2 - S_0$ must be independent. In order to make S_0 as small as possible, $Q_2 - S_0$ must be as large as possible, thus $Q_2 - S_0$ must be the largest independent set of the graph G_{Q_2} which is induced by Q_2 on $P_m \times C_n$, which means $|Q_2 - S_0| = \alpha(G_{Q_2})$. It is noticeable that G_{Q_2} represents a $P_{m-2} \times C_{n-2}$ graph. Therefore, due to Theorem 2 it can be concluded that $\alpha(G_{Q_2}) = \alpha(P_{m-2} \times C_{n-2}) = \max\{\alpha(P_{m-2})|C_{n-2}|, |P_{m-2}|\alpha(C_{n-2})\}$ and the smallest seed set S_0 on $P_m \times C_n$ that contains Q_1 and guarantees not leaving two adjacent unconverted vertices from Q_2 is of cardinality:

$|S_0| = |Q_1| + |Q_2| - \alpha(P_{m-2} \times C_{n-2})$. Due to Theorem 1 this means:

$C_4(P_m \times P_n) = nm - max\{(n-2)\left\lceil\frac{m-2}{2}\right\rceil, (m-2)\left\lceil\frac{n-2}{2}\right\rceil\}$. However, in case m, n are even, then $\left\lceil\frac{m-2}{2}\right\rceil = \frac{m-2}{2}$ and $\left\lceil\frac{n-2}{2}\right\rceil = \frac{n-2}{2}$, then $max\{(n-2)\left\lceil\frac{m-2}{2}\right\rceil, (m-2)\left\lceil\frac{n-2}{2}\right\rceil\} = \frac{(m-2)(n-2)}{2}$ which means $C_4(P_m \times C_n) = nm - \frac{(m-2)(n-2)}{2} = \frac{mn+2m+2n-4}{2}$ and thus proving the requested in (i).

Since $k = 4 = \Delta(G)$ and by Proposition 2; $CT_4(P_m \times C_n) = 1$ for $m, n \geq 3$.□

Proposition 9: For $n \geq 2$; $CT_1(L_n) = \left\lceil\frac{n+1}{2}\right\rceil$.

Proof: It is known that $C_1(L_n) = 1$ since L_n is connected. Let us consider the following cases for n:

Case 1. n is odd.

Let the seed set be $S_0^{(0)} = \{(1, \frac{n+1}{2})\}$; the process goes as follows:

$S_1^{(0)} = S_0^{(0)} \cup \{(1, \frac{n-1}{2}), (1, \frac{n+3}{2}), (2, \frac{n+1}{2})\}$; $S_2^{(0)} = S_1^{(0)} \cup \{(1, \frac{n-3}{2}), (1, \frac{n+5}{2}), (2, \frac{n-1}{2}), (2, \frac{n+3}{2})\}$;

For $3 \leq t \leq \frac{n-1}{2}$: $S_t^{(0)} = S_{t-1}^{(0)} \cup \{(1, \frac{n-2t+1}{2}), (1, \frac{n+2t+1}{2}), (2, \frac{n-2t+3}{2}), (2, \frac{n+2t-1}{2})\}$

$= \{(1, l), (2, r): \frac{n-2t+1}{2} \leq l \leq \frac{n+2t+1}{2}; \frac{n-2t+3}{2} \leq r \leq \frac{n+2t-1}{2}\}$.

$S_{\frac{n-1}{2}}^{(0)} = \{(1, l), (2, r): 1 \leq l \leq n;\ 2 \leq r \leq n-1\}$.

The process at $t \leq \frac{n+1}{2}$ for which $S_{\frac{n+1}{2}}^{(0)} = S_{\frac{n-1}{2}}^{(0)} \cup \{(2,1), (2, n)\} = V(L_n)$. Due to symmetry purposes, the same result is obtained if $S_0^{(0)} = \{(2, \frac{n+1}{2})\}$. Now Let us study the process if a different vertex from the upper row is chosen as the seed set as follows:

For any $1 \leq i \leq \frac{n-3}{2}$: $S_0^{(i)} = \{(1, \frac{n+1}{2} - i) = (1, \frac{n-2i+1}{2})\}$;

$S_1^{(i)} = S_0^{(i)} \cup \{(1, \frac{n-2i-1}{2}), (1, \frac{n-2i+3}{2}), (2, \frac{n-2i+1}{2})\}$;

$S_2^{(i)} = S_1^{(i)} \cup \{(1, \frac{n-2i-3}{2}), (1, \frac{n-2i+5}{2}), (2, \frac{n-2i-1}{2}), (2, \frac{n-2i+3}{2})\}$;

For $3 \leq t \leq \frac{n-2i-1}{2}$: $S_t^{(i)} = S_{t-1}^{(i)} \cup \{(1, \frac{n-2i-2t+1}{2}), (1, \frac{n-2i+2t+1}{2}), (2, \frac{n-2i-2t+3}{2}), (2, \frac{n-2i+2t-1}{2})\}$

$= \{(1, l), (2, r): \frac{n-2i-2t+1}{2} \leq l \leq \frac{n-2i+2t+1}{2}; \frac{n-2i-2t+3}{2} \leq r \leq \frac{n-2i+2t-1}{2}\}$;

$S_{\frac{n-2i+1}{2}}^{(i)} = S_{\frac{n-2i-1}{2}}^{(i)} \cup \{(1, n-2i+1), (2,1), (2, n-2i)\}$;

For $\frac{n-2i+3}{2} \leq t \leq \frac{n+2i-1}{2}$: $S_t^{(i)} = S_{t-1}^{(i)} \cup \left\{(1, \frac{n-2i+2t+1}{2}), (2, \frac{n-2i+2t-1}{2})\right\}$

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$= \{(1,l),(2,r): 1 \leq l \leq \frac{n-2i+2t+1}{2};\ 1 \leq r \leq \frac{n-2i+2t-1}{2}\}$;

$S^{(i)}_{\frac{n+2i-1}{2}} = \{(1,l),(2,r): 1 \leq l \leq n;\ 1 \leq r \leq n-1\}$;

The process ends at $t = \frac{n+2i+1}{2}$ for which $S^{(i)}_{\frac{n+2i+1}{2}} = S^{(i)}_{\frac{n+2i-1}{2}} \cup \{(2,n)\} = V(L_n)$.

It is easy to notice that for $i = \frac{n-1}{2}$ then $S_0^{\left(\frac{n-1}{2}\right)} = \{(1,1)\}$ and in this distinct case the process goes as follows:

$S_0^{\left(\frac{n-1}{2}\right)} = \{(1,1)\};\ S_1^{\left(\frac{n-1}{2}\right)} = \{(1,2),(2,1)\}$;

For $2 \leq t \leq n-1$: $S_t^{\left(\frac{n-1}{2}\right)} = S_{t-1}^{\left(\frac{n-1}{2}\right)} \cup \{(1,t+1),(2,t)\} = \{(1,l),(2,r): 1 \leq l \leq t+1; 1 \leq r \leq t\}$;

The process ends at $t = n$ for which $S_n^{\left(\frac{n-1}{2}\right)} = S_{n-1}^{\left(\frac{n-1}{2}\right)} \cup \{(2,n)\} = V(L_n)$.

It is concluded that for any $1 \leq i \leq \frac{n-1}{2}$ the process ends at $= \frac{n+2i+1}{2} > \frac{n+1}{2}$. This means the lowest value of t_{final} is obtained when $i = 0$ and due to symmetry, the same result is obtained if $S_0^{(i)} = \{(1, \frac{n+1}{2} + i): 1 \leq i \leq \frac{n-1}{2}\}$. It is also obvious that the same study applies similarly if $S_0^{(i)} \subset R_2$. All of the above leads to the conclusion that for $n \geq 3$; $CT_1(L_n) = \frac{n+1}{2} = \left\lceil \frac{n+1}{2} \right\rceil$ if n id odd.

Case 2. n is even. By following the same argument of Case 1, for $n \geq 2$; $CT_1(L_n)$ is obtained by choosing $S_0 = \{x; x \in \{(1,\frac{n}{2}).(1,\frac{n}{2}+1).(2,\frac{n}{2}),(2,\frac{n}{2}+1)\}\}$, for which the process ends at $t = \frac{n}{2} + 1 = \left\lceil \frac{n+1}{2} \right\rceil$.

From Case 1 and Case 2; $CT_1(L_n) = \left\lceil \frac{n+1}{2} \right\rceil$ for $n \geq 2$.□

Proposition 10: For $n \geq 2$; $CT_2(L_n) = \begin{cases} \frac{n-1}{2} \ if\ n\ is\ odd; \\ n-1\ if\ n\ is\ even. \end{cases}$

Proof: For $1 \leq j \leq n-1$; Let $W_j = CO_j \cup CO_{j+1}$. It was implied in a previous paper by the authors that $C_2(L_n) = \left\lceil \frac{n+1}{2} \right\rceil$ for $n \geq 2$. This conclusion was obtained due to the following sets being 2-immune: $R_1; R_2; CO_1; CO_n; \{W_j:\ 1 \leq j \leq n-1\}$ because for any $u \in U$ when $U \in \{R_1, R_2, CO_1, CO_n, W_j\}$; $|N(u) - U| < 2$. This means when creating S_0 the following cases for n must be considered:

Case 1: n is odd.

To avoid leaving any unconvertible 2-immune sets one vertex from each odd indexed column must be included in S_0. However, the process then ends at $t = 0$. Therefore, it is necessary to include one more vertex in S_0. Let $S_0^{(0)} = \left\{(1,2l+1): 0 \leq l \leq \frac{n-1}{2}\right\} \cup \{(2,\frac{n+1}{2})\}$ which is of cardinality $\frac{n+1}{2}$. By tracking the process in a similar way to Proposition 9, the process goes as:

$$S_0^{(0)} = \{(1,2l+1): 0 \leq l \leq \frac{n-1}{2}\} \cup \{(2,\frac{n+1}{2})\}\ ;\ S_1 = \{(1,l): 1 \leq l \leq n\} \cup \{(2,\frac{n-1}{2}),(2,\frac{n+3}{2})\};$$

For $2 \leq t \leq \frac{n-1}{2}$: $S_t = S_{t-1} \cup \left\{(2,\frac{n+2t-1}{2}),(2,\frac{n+2t+1}{2})\right\} = \{(1,l),(2,r): 1 \leq l \leq n;\ \frac{n+2t-1}{2} \leq r \leq \frac{n+2t+1}{2}\}$;

The process ends successfully at $t^{(0)}_{final} = \frac{n-1}{2}$ and similarly to Proposition 9; $t^{(i)}_{final} = \frac{n+2i-1}{2}$ for any $S_0^{(i)} = \left\{(1,2l+1): 0 \leq l \leq \frac{n-1}{2}\right\} \cup \{(2,\frac{n+1}{2} \mp i)\}: 1 \leq i \leq \frac{n-1}{2}$. This means $t^{(0)}_{final} = \frac{n-1}{2} = min\{t^{(i)}_{final}\}$ for $0 \leq i \leq \frac{n-1}{2}$. Due to symmetry the same result is obtained in the case of alternating between R_1 and R_2 when creating $S_0^{(0)}$. From all the above; $CT_2(L_n) = \frac{n-1}{2}$ for $n \geq 3$ if n is odd.

Case 2: n is even**.**

To avoid leaving any unconvertible 2-immune set on L_n when distributing the $\left\lceil\frac{n+1}{2}\right\rceil$ seed vertices, let us consider the following cases (options):

Case 2.a: Include one vertex from each even indexed column and one vertex of CO_1 in S_0.

Case 2.b: Include one vertex from each odd indexed column and one vertex of CO_n in S_0.

In both subcases there cannot be any other vertex located freely (unlike Case 1). For Case 2.a, the process goes as:

$$S_0 = \{(1,2l+1): 0 \le l \le \frac{n}{2} - 1\} \cup \{(2,1)\};\ S_1 = \{(1,l): 1 \le l \le n\} \cup \{(2,1),(2,2)\};$$

For $2 \le t \le n-1$: $S_t = S_{t-1} \cup \{(2,t+1)\} = \{(1,l),(2,r): 1 \le l \le n;\ 1 \le r \le t+1\}$;

The process ends successfully at $t = n-1$. It is obvious that the same result is obtained in Case 2.b. Therefore, for $n \ge 2$; $CT_2(L_n) = n-1$ if n is even.

From Case 1 and Case 2 the requested is proven.□

Observation 4: Due to Proposition 2 and since $\Delta(L_n) = 3$; $CT_3(L_n) = 1$.

Conclusion

In this paper a new invariant called the irreversible k-threshold conversion time (denoted by $CT_k(G)$ was defined. This invariant retrieves the minimum number of steps (t) that the minimum IkCS needs in order to convert V(G) entirely. $CT_k(G)$ was also studied on some simple graphs, then both $C_k(P_m \times C_n)$ and $CT_k(P_m \times C_n)$ were determined for some values of k, m, n. Finally, $C_k(G)$ of the Ladder graph L_n for $n \ge 2$ was determined.

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Authors' Declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Tishreen University, Syria.

Authors' Contribution Statement

R. Sh. designed the study and found the irreversible k-threshold conversion number of the tensor product of a path and a cycle. A. K. introduced the irreversible k-threshold conversion time definition and studied it for $P_m \times C_n$ and L_n. S. M. applied the new definition to some simple graphs and organized the manuscript. The submitted version of the manuscript was checked and approved by all authors.

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معامل جديد متعلق بعمليات التحول غير العكوس ذو عتبة الانتشار k في البيان

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الخلاصة

تشكل عملية انتشار التحول غير العكوس ذو العتبة k في بيان ما $G = (V, E)$ نوعاً خاصاً من عمليات الانتشار البيانية والتي تهتم بشكل خاص بدراسة انتشار تغير في حالة رؤوس للبيان G انطلاقاً من مجموعة اختيارية من رؤوسه حيث يتحقق انتشار التحول إلى الرؤوس المجاورة وفقاً لقاعدة تحول محددة مسبقاً. إن عملية انتشار التحول غير العكوس ذو العتبة k في البيان G هي عملية تكرارية تدرس انتشار تغير أحادي الاتجاه (من الحالة 0 إلى الحالة 1) على $V(G)$. تبدأ العملية باختيار مجموعة $S_0 \subseteq V$، ومن أجل كل خطوة $t(t = 1,2, \dots,)$ فإن S_t تنتج عن S_{t-1} بإضافة جميع الرؤوس التي تجاور k رأسا على الأقل من S_{t-1} إلى S_{t-1}. تدعى S_0 بذرة عملية التحول غير العكوس ذو العتبة k وإذا تحقق أن $S_t = V(G)$ من أجل قيمة ما $t \geq 0$، عندئذ تسمى S_0 مجموعة تحول غير عكوس ذو عتبة انتشار k (IkCS) للبيان G. عدد التحول غير العكوس ذو عتبة الانتشار k للبيان G (يرمز له $C_k(G)$) هو عدد عناصر أصغر IkCS للبيان G. في هذه الورقة البحثية نقوم بتعريف معامل جديد يسمى زمن التحول غير العكوس ذو عتبة الانتشار k (نرمز له بـ $CT_k(G)$) والذي يقيس أقل عدد ممكن من الخطوات (t) التي تحتاجها مجموعة IkCS أصغرية لنشر التحول إلى كافة رؤوس البيان. ونقوم بدراسة $CT_k(G)$ لبعض البيانات الخاصة البسيطة مثل المسارات والحلقات والبيان النجمي، كما نقوم أيضاً بإيجاد $C_k(G)$ و $CT_k(G)$ للجداء المباشر لمسار P_m وحلقة C_n (والذي يرمز له بالرمز $P_m \times C_n$) وذلك من أجل بعض قيم k و m و n. كذلك نوجد CT_k للبيان السلمي L_n من أجل $k = 1,2,3$ وأي قيمة عشوائية لـ $n \geq 2$.

الكلمات المفتاحية: عمليات التحول البيانية، عدد التحول ذو عتبة الانتشار k، زمن التحول ذو عتبة الانتشار k، مجموعة البذرة، الجداء المباشر لبيانين، البيان السلمي.