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Hopf and Zero-Hopf Bifurcation of the Four-Dimensional Lotka-Volterra Systems

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Abstract

In this work, the four-dimensional Lotka-Volterra model (4DLV) involving four species in a constant environment is considered. The objective of this investigation is to study the local bifurcations occurring in the system. This system has at most sixteen possible equilibrium points. One of the equilibrium points is considered in order to investigate the periodic solutions that bifurcate from the Hopf and the zero-Hopf equilibrium points, respectively. It has been proven that, five families of sufficient conditions exist on the parameters of the system in which the Jacobian matrix at equilibrium point has a pair of purely imaginary $\pm i\omega$, $\omega > 0$ and two non-positive eigenvalues. Moreover, eight families of sufficient conditions exist on the parameters in which the Jacobian matrix at the equilibrium point has a pair of purely imaginary eigenvalues $\pm i\omega$ and at least one of the other eigenvalues is zero. Next, this investigation reveals that certain four-dimensional Lotka-Volterra subsystems exhibit one periodic solution bifurcating from the Hopf equilibrium point and three periodic solutions bifurcating from the zero-Hopf equilibrium point respectively. The averaging method in any order for computing periodic solutions consists of providing sufficient conditions for the existence of periodic solutions in polynomial differential systems by studying the equilibrium points of their associated averaged systems. Then, the main tool utilized is the first-order averaging method to compute periodic solutions that bifurcate from the Hopf and zero-Hopf singular points of the four-dimensional Lotka-Volterra system under certain conditions. Finally, the obtained theoretical results are supported and verified by numerical examples.

Keywords: Averaging theory, Lotka-Volterra system, Periodic solutions, Quadratic polynomial differential system, Zero-Hopf bifurcation.

Introduction

In biological populations theory, Alfred Lotka and Vito Volterra separately proposed the Lotka-Volterra systems in 1925 and 1926, respectively. These systems describe the interaction of n species. There are n first-order differential equations in them.

$$\frac{dx_i(t)}{dt} = x_i(t)\left(b_i + \sum_{j=1}^{n} a_{ij}\, x_j(t)\right), i = 1, \dots, n, \quad 1$$

where $b_i > 0$ is the growth rate of the $i\,th$ population, $a_{i,j}/b_i$ are interaction coefficients that determine how much the $j\,th$ species affects the $i\,th$ population's growth rate, and $x_i(t)$ represents population's size at time t. Numerous mathematical models in fields like as physics, ecology, economics, etc. have used this framework as their foundation [1].

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This system provides a simplified model of prey-predator interactions.

Our consideration is started with the following 4DLV system [2]:

$$\begin{cases}\dot{x} = x(a_1 + a_{11}x + a_{12}y + a_{13}z + a_{14}w),\\ \dot{y} = y(b_1 + b_{11}x + b_{12}y + b_{13}z + b_{14}w),\\ \dot{z} = z(c_1 + c_{11}x + c_{12}y + c_{13}z + c_{14}w),\\ \dot{w} = w(d_1 + d_{11}x + d_{12}y + d_{13}z + d_{14}w),\end{cases} \quad 2$$

as a model for the competition of the four biological species (x, y, z and w are written instead of $x(t), y(t), z(t)$ and $w(t)$, respectively). In this model, each of variables x, y, z and w describes the number of individuals of species (so that $x, y, z, w > 0$), also $a_1, b_1, c_1, d_1, a_{1j}, b_{1j}, c_{1j}, d_{1j}$ for $j = 1,2,3,4$ are real parameters which are the interaction coefficients, and the a_1, b_1, c_1, d_1 are parameters that depend on the environment. For instance, if one of a_1, b_1, c_1, d_1 is greater than zero, it means that this species is able to increase with food from the environment, while if one of a_1, b_1, c_1, d_1 is smaller than zero, it means that this species cannot survive when left alone in the environment. One can also have some of a_1, b_1, c_1, d_1 be zero, which means that the population stays constant if the species do not interact.

The Hopf bifurcation in a differential system happens when a complex conjugate pair of eigenvalues of a Jacobian matrix at a singular point becomes purely imaginary eigenvalues. As a result, a Hopf bifurcation can arise only in systems of dimension two or higher. The crucial aspect is that Hopf bifurcation is concerned with the birth or death of a periodic solution as it emanates from or shrinks onto a singular point, the focus. Hopf bifurcation has been vastly investigated by many researchers [3-5]. Hopf bifurcation, as is well known, produces periodic solutions, which are characteristic oscillatory behaviors of many nonlinear systems [6-8]. Here, the first object is studying the Hopf bifurcation in system (2). A singular point of system (2) having eigenvalues $\pm i\omega$ and nonpositive real numbers α, β with nonpositive $\alpha, \beta \in R$ such that $\alpha\beta \neq 0$ is called the Hopf equilibrium. The second object is studying a zero-Hopf bifurcation in system (2). So, a singular point of system (2) having eigenvalues $\pm i\omega$ and other eigenvalues which are zero is called the zero-Hopf singular point. Such a kind of bifurcation is described in [9,10]. It is known that Lotka–Volterra systems can demonstrate zero-Hopf singular points [10].

A closed orbit Γ_ϵ of system (2) bifurcates from the singular point if it tends to it as $\epsilon \to 0$. The study will explore the existence of periodic orbits of system (2) in the neighborhood of the singular point when ϵ equal to zero. The zero-Hopf and Hopf bifurcations are the bifurcations in a neighborhood of an isolated singular point. It is common for a small-amplitude periodic orbit to either appear or disappear around this kind of singular point if the stability type of this point changes when it varies near zero. Such a kind of bifurcation is described in [11]. Then a natural question is: how many periodic orbits can bifurcate from system (4) with a zero-Hopf or the Hopf singular point inside the class of all Lotka-Volterra systems (2). Therefore, the existence of the number of periodic orbits which can bifurcate from a zero-Hopf singular point as well as the Hopf singular point when this is perturbed inside the class of all systems (4) is studied, see Theorems 1 and 2 which are the main results in this investigation.

There are a few investigations on system (2); Kowgier [2] showed that how survival probabilities of four populations alter with an assumption that they arrived an equilibrium level given by the same member of individuals. Farhan et al.[12] investigated the stability of the four-dimensional Lotka-Volterra model. Wang and Xiao [13] explore the method of occurrence of chaotic behavior and studied numerically that a periodic orbit by Hopf bifurcation can undergo successive period doubling cascade of system (2).

Computing the periodic solutions may be done in a variety of methods, such as using the bifurcation theory, Melnikov integral, and Poincare return map, the Center manifold theorem and Normal forms. To find the number of the periodic solutions which bifurcate from a zero- Hopf and the Hopf point, the first-order averaging technique is used. As you can see in [14, 15], one of the most crucial methods for analyzing periodic orbits for second- and higher-order differential systems is the averaging technique. The publications written by Djedid et al. [16] have further information on the averaging theory.

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The focus is on the existence of periodic solutions for the differential system (2) that bifurcate from a single singular point with eigenvalues $\pm i\omega$ and two zeros.

Hopf and Zero-Hopf Singular Points

System (2) has sixteen $E_i, i = 1, \dots, 16$ singular points which are calculated in [9]. Only the following equilibria E_i can have a pair of purely conjugate complex eigenvalues

$E_1 = \left(0, -\frac{b_1c_{13}-b_{13}c_1}{b_{12}c_{13}-b_{13}c_{12}}, \frac{b_1c_{12}-b_{12}c_1}{b_{12}c_{13}-b_{13}c_{12}}, 0\right)$, such that $b_{12}c_{13} - b_{13}c_{12} \neq 0$,

$E_2 = (-\frac{a_1c_{13}-a_{13}c_1}{a_{11}c_{13}-a_{13}c_{11}}, 0, \frac{a_1c_{11}-a_{11}c_1}{a_{11}c_{13}-a_{13}c_{11}}, 0)$, such that $a_{11}c_{13} - a_{13}c_{11} \neq 0$,

$E_3 = (-\frac{a_1b_{12}-a_{12}b_1}{a_{11}b_{12}-a_{12}b_{11}}, \frac{a_1b_{11}-a_{11}b_1}{a_{11}b_{12}-a_{12}b_{11}}, 0, 0)$, such that $a_{11}b_{12} - a_{12}b_{11} \neq 0$,

$E_4 = (0, 0, -\frac{c_1d_{14}-c_{14}d_1}{c_{13}d_{14}-c_{14}d_{13}}, \frac{c_1d_{13}-c_{13}d_1}{c_{13}d_{14}-c_{14}d_{13}})$, such that $c_{13}d_{14} - c_{14}d_{13} \neq 0$,

$E_5 = (0, -\frac{b_1d_{14}-b_{14}d_1}{b_{12}d_{14}-b_{14}d_{12}}, 0, \frac{b_1d_{12}-b_{12}d_1}{b_{12}d_{14}-b_{14}d_{12}})$, such that $b_{12}d_{14} - b_{14}d_{12} \neq 0$,

$E_6 = (-\frac{a_1d_{14}-a_{14}d_1}{a_{11}d_{14}-a_{14}d_{11}}, 0, 0, \frac{a_1d_{11}-a_{11}d_1}{a_{11}d_{14}-a_{14}d_{11}})$, such that $a_{11}d_{14} - a_{14}d_{11} \neq 0$,

$E_7 = \frac{1}{\eta_1}(-(a_1b_{12}c_{13} - a_1b_{13}c_{12} - a_{12}b_1c_{13} + a_{12}b_{13}c_1 + a_{13}b_1c_{12} - a_{13}b_{12}c_1), (a_1b_{11}c_{13} - a_1b_{13}c_{11} - a_{11}b_1c_{13} + a_{11}b_{13}c_1 + a_{13}b_1c_{11} - a_{13}b_{11}c_1), -(a_1b_{11}c_{12} - a_1b_{12}c_{11} - a_{11}b_1c_{12} + a_{11}b_{12}c_1 + a_{12}b_1c_{11} - a_{12}b_{11}c_1), 0)$,

$E_8 = (0, -\frac{1}{\eta_2}(b_1c_{13}d_{14} - b_1c_{14}d_{13} - b_{13}c_1d_{14} + b_{13}c_{14}d_1 + b_{14}c_1d_{13} - b_{14}c_{13}d_1)$,

$\frac{1}{\eta_2}(b_1c_{12}d_{14} - b_1c_{14}d_{12} - b_{12}c_1d_{14} + b_{12}c_{14}d_1 + b_{14}c_1d_{12} - b_{14}c_{12}d_1)$,

$-\frac{1}{\eta_2}(b_1c_{12}d_{13} - b_1c_{13}d_{12} - b_{12}c_1d_{13} + b_{12}c_{13}d_1 + b_{13}c_1d_{12} - b_{13}c_{12}d_1))$,

$E_9 = (-\frac{1}{\eta_3}(a_1c_{13}d_{14} - a_1c_{14}d_{13} - a_{13}c_1d_{14} + a_{13}c_{14}d_1 + a_{14}c_1d_{13} - a_{14}c_{13}d_1), 0,$

$\frac{1}{\eta_3}(a_1c_{11}d_{14} - a_1c_{14}d_{11} - a_{11}c_1d_{14} + a_{11}c_{14}d_1 + a_{14}c_1d_{11} - a_{14}c_{11}d_1)$,

$-\frac{1}{\eta_3}(a_1c_{11}d_{13} - a_1c_{13}d_{11} - a_{11}c_1d_{13} + a_{11}c_{13}d_1 + a_{13}c_1d_{11} - a_{13}c_{11}d_1))$,

$E_{10} = (-\frac{1}{\eta_4}(a_1b_{12}d_{14} - a_1b_{14}d_{12} - a_{12}b_1d_{14} + a_{12}b_{14}d_1 + a_{14}b_1d_{12} - a_{14}b_{12}d_1)$,

$\frac{1}{\eta_4}(a_1b_{11}d_{14} - a_1b_{14}d_{11} - a_{11}b_1d_{14} + a_{11}b_{14}d_1 + a_{14}b_1d_{11} - a_{14}b_{11}d_1), 0,$

$-\frac{1}{\eta_4}(a_1b_{11}d_{12} - a_1b_{12}d_{11} - a_{11}b_1d_{12} + a_{11}b_{12}d_1 + a_{12}b_1d_{11} - a_{12}b_{11}d_1))$,

where

$\eta_1 = \left|\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ b_{11} & b_{12} & b_{13} \\ c_{11} & c_{12} & c_{13} \end{pmatrix}\right|$, $\eta_2 = \left|\begin{pmatrix} b_{12} & b_{13} & b_{14} \\ c_{12} & c_{13} & c_{14} \\ d_{12} & d_{13} & d_{14} \end{pmatrix}\right|$,

$\eta_3 = \left|\begin{pmatrix} a_{11} & a_{13} & a_{14} \\ c_{11} & c_{13} & c_{14} \\ d_{11} & d_{13} & d_{14} \end{pmatrix}\right|$, $\eta_4 = \left|\begin{pmatrix} a_{11} & a_{12} & a_{14} \\ b_{11} & b_{12} & b_{14} \\ d_{11} & d_{12} & d_{14} \end{pmatrix}\right|$, such that $\eta_1\eta_2\eta_3\eta_4 \neq 0$

After computations using the Maple computer software, the print-out of the four non-zero coordinates of the singular point E_{11}, which takes up nearly two pages, has been excluded. Farhan et al.[4], they analyzed the local stability of all possible singular points and showed that eight of them are unstable, while the rest of them are locally asymptotically stable under certain conditions.

Without losing generality, only the singular point E_2 is considered in this study. The following proposition is to establish conditions under which system (2)'s singular point has a pair of purely conjugate complex eigenvalues.

Proposition 1: There are only five families $\mathcal{H}_i, i = 1, \cdots, 5$ of conditions on the parameters of system (2)

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in which the Jacobian matrix at the singular point E_2 has a pair of purely conjugate complex eigenvalues $\mp i\omega$ and non-zero eigenvalues $\alpha,\beta\ \in\mathbb{R}$. That is

$$\mathcal{H}_1=\{a_1=0, b_{13}=\frac{a_{13}c_1(\beta-b_1)}{\omega^2}, c_{11}=\frac{a_{11}(\omega^2+c_1^2)}{\omega^2}, c_{13}=a_{13}, d_1=\beta+c_1, d_{11}=\frac{c_1^2a_{11}}{\omega^2}, d_{13}=0\}.$$

$$\mathcal{H}_2=\{a_1=-\frac{\omega^2}{c_1}, a_{11}=0, b_1=\frac{\beta a_{13}c_1-\omega^2b_{13}}{a_{13}c_1}, c_{13}=0, d_1=\frac{\beta c_{11}+c_1d_{11}}{c_{11}}, d_{13}=0\}.$$

$$\mathcal{H}_3=\{a_1=\frac{\omega^2a_{11}-\omega^2c_{11}+c_1^2a_{11}}{c_1c_{11}}, a_{13}=\frac{c_{13}a_{11}^2(\omega^2+c_1^2)-c_{11}\omega^2(2a_{11}-c_{11})}{a_{11}c_1^2c_{11}}, b_1=\frac{\beta a_{11}c_{13}-\beta c_{11}c_{13}+a_{11}b_{13}c_1}{c_{13}(a_{11}-c_{11})},$$
$$d_1=\beta, d_{11}=0, d_{13}=0\}.$$

$$\mathcal{H}_4=\{a_1=\frac{a_{11}(a_{11}-c_{11})(\beta-d_1)^2+\omega^2d_{11}^2}{d_{11}c_{11}(\beta-d_1)}, a_{13}=\frac{c_{13}((\beta-d_1)^2a_{11}^2+\omega^2d_{11}^2)}{a_{11}(\beta-d_1)^2c_{11}}, b_1=\frac{a_{11}b_{13}(\beta-d_1)+\beta c_{13}d_{11}}{c_{13}d_{11}},$$
$$c_1=\frac{(a_{11}-c_{11})(\beta-d_1)}{d_{11}}, d_{13}=0\}.$$

$$\mathcal{H}_5=\{b_{13}=\frac{(a_{13}-c_{13})(b_1-\beta)}{a_1}, c_1=\frac{(c_{13}-a_{13})\omega^2+a_1^2c_{13}}{a_{13}a_1}, c_{11}=\frac{a_{11}\left(c_{13}^2(\omega^2+a_1^2)+\omega^2a_{13}(a_{13}-2c_{13})\right)}{a_1^2a_{13}c_{13}},$$
$$d_1=\frac{\beta a_1a_{13}-\omega^2a_{13}+\omega^2c_{13}+a_1^2c_{13}}{a_{13}a_1}, d_{11}=\frac{a_{11}(a_{13}-c_{13})(\omega^2a_{13}-\omega^2c_{13}-a_1^2c_{13})}{a_1^2a_{13}c_{13}}, d_{13}=0\}.$$

Proposition 2: There are only eight families $\mathrm{F}_j,\ j=1,\cdots,8$ of conditions on the parameters of system (2) in which the Jacobian matrix at the singular point E_2 has a pair of purely conjugate complex $\mp i\omega$, and $\alpha,\beta\in\mathbb{R}$ eigenvalues with $\alpha=0$ or $\beta=0$. That is

$$\mathrm{F}_1=\{a_{13}=\frac{c_{13}(\omega^2+a_1^2)}{\omega^2}, b_{11}=\frac{(\omega^2b_{13}-a_1b_1c_{13})c_{11}}{\omega^2c_{13}}, c_1=0, c_{11}=a_{11}\}.$$

$$\mathrm{F}_2=\{a_1=-\frac{\omega^2}{c_1}, a_{11}=0, b_{11}=\frac{(\omega^2b_{13}+a_{13}b_1c_1)c_{11}}{c_1^2a_{13}}, c_{13}=0\}.$$

$$\mathrm{F}_3=\{a_{11}=0, c_1=-\frac{\omega^2}{a_1}, c_{13}=0, d_{13}=\frac{(\omega^2d_{11}+c_{11}d_1a_1)a_{13}}{c_{11}a_1^2}\}.$$

$$\mathrm{F}_4=\{a_{11}=c_{11}, a_{13}=\frac{c_{13}(\omega^2+a_1^2)}{\omega^2}, c_1=0, d_{13}=\frac{(\omega^2d_{11}+c_{11}d_1a_1)c_{13}}{\omega^2c_{11}}\}.$$

$$\mathrm{F}_5=\{a_1=\frac{\omega^2a_{11}-\omega^2c_{11}+c_1^2a_{11}}{c_1c_{11}}, a_{13}=\frac{c_{13}(\omega^2a_{11}^2-2\omega^2a_{11}c_{11}+\omega^2c_{11}^2+a_{11}^2c_1^2)}{a_{11}c_1^2c_{11}}, b_{11}=\frac{a_{11}b_{13}c_1-a_{11}b_1c_{13}+b_1c_{11}c_{13}}{c_1c_{13}}\}.$$

$$\mathrm{F}_6=\{a_1=\frac{\omega^2a_{11}-\omega^2c_{11}+c_1^2a_{11}}{c_1c_{11}}, a_{13}=\frac{c_{13}(\omega^2a_{11}^2-2\omega^2a_{11}c_{11}+\omega^2c_{11}^2+a_{11}^2c_1^2)}{a_{11}c_1^2c_{11}}, d_{13}=\frac{c_{13}(a_{11}d_1+c_1d_{11}-c_{11}d_1)}{a_{11}c_1}\}.$$

$$\mathrm{F}_7=\{a_1=\frac{b_1(a_{13}-c_{13})}{b_{13}}, c_1=-\frac{\omega^2b_{13}^2-a_{13}b_1^2c_{13}+b_1^2c_{13}^2}{b_{13}b_1a_{13}}, c_{11}=\frac{(\omega^2b_{13}^2+b_1^2c_{13}^2)a_{11}}{a_{13}b_1^2c_{13}}, d_{11}=-\frac{a_{11}b_{13}d_1}{b_1c_{13}}, d_{13}=0\}.$$

$$\mathrm{F}_8=\{b_{13}=a_{13}-c_{13}, c_1=-\frac{\omega^2a_{13}-\omega^2c_{13}-a_1^2c_{13}}{a_{13}a_1}, c_{11}=\frac{a_{11}(\omega^2a_{13}^2-2\omega^2a_{13}c_{13}+\omega^2c_{13}^2+a_1^2c_{13}^2)}{a_1^2a_{13}c_{13}}, d_{11}=$$
$$-\frac{a_{11}d_1(a_{13}-c_{13})}{a_1c_{13}}, d_{13}=0\}.$$

The proof of Propositions 1 and 2 are given in the next Section. Because of this, the singular point E_2 under each family $\mathcal{H}_i, i=1,\cdots,5$ and $\mathrm{F}_j, j=1,\cdots,8$ of conditions in Propositions 1 and 2, respectively, is the Hopf point.

Furthermore, without loss of generality, in order to find periodic solutions bifurcating from the Hopf point of system (2), the family $\mathcal{H}_1$ of conditions in Proposition 1 is examined and the first-order averaging method is applied. The most important result of this investigation, which is the following theorem, explains this.

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Theorem 1: Consider system (2) with the family $\mathcal{H}_1$ of conditions in Proposition 1. Let $d_{13} = \epsilon\mu_1$, with a sufficiently small parameter $\epsilon > 0$. If

$$\mu_1\, f \neq 0,$$

where f is a polynomial of parameters in system (2). Then, system (2) has the Hopf bifurcation at the singular point which localizes at the singular point E_2, and when $\epsilon = 0$, a periodic solution is produced at this singular point.

Theorem 1 is proved in the next section.

Now, this study investigates a zero-Hopf bifurcation of system (2), focusing on the family F_1 of conditions in Proposition 2. The Jacobian matrix of system (2) at the singular point E_2 with the family F_1 of conditions in Proposition 2 has the following eigenvalues

$$\lambda_{1,2} = \pm\omega, \quad \lambda_3 = \beta = 0, \quad \lambda_4 = \alpha = -\frac{\omega^2 a_{11} d_{13} - \omega^2 c_{13} d_{11} - a_1 a_{11} c_{13} d_1}{a_1 a_{11} c_{13}}.$$

In order to have two zero and two purely conjugate complex eigenvalues, assume that

$$d_1 = \frac{(a_{11} d_{13} - c_{13} d_{11})\omega^2}{a_1 a_{11} c_{13}}. \qquad 3$$

Determining the following theorem as a result.

Theorem 2: Consider system (2) with the family F_1 of conditions in Proposition 2 and Eq. 3. Let $c_1 = \epsilon\, \mu_1$, with $\epsilon > 0$, a sufficiently small parameter. If

$$\mu_1\, f_2 \neq 0,$$

where f_2 is a polynomial of parameters in system (2). Then, at the singular point, system (2) exhibits the zero-Hopf bifurcation, which localizes at E_2, and when $\epsilon = 0$, at most three closed orbits bifurcate at this singular point.

Theorem 2 is proved in the next section using the averaging method of the first order.

To investigate the closed orbits that bifurcate from the zero-Hopf and Hopf singular points of system (2) by employing the first-order averaging technique of the first-order (see The Main Tool), a small parameter will be introduced to create a new independent variable in a periodic system.

The rest of this research is structured as follows: In the next section, Theorems 1 and 2 are proven, and a first-order averaging method is described in The Main Tool for computing periodic solutions bifurcating from zero-Hopf and Hopf singular points.

Proof of the main results

In this section, firstly, propositions 1 and 2 will be proved. Theorems 1 and 2 will then be demonstrated after that.

1. Proof of Proposition 1

Proof: The characteristic polynomial at E_2 of system (2) is

$$\phi(\lambda) = \lambda^4 + S_1\lambda^3 + S_2\lambda^2 + S_3\lambda + S_4,$$

where S_i for $i = 1, \cdots, 4$ are given in the (Supplemental Material section). Suppose that the system (2) has two purely conjugate complex eigenvalues at E_2. Then, $\phi(\lambda)$ must have the following form

$$\phi(\lambda) = (\lambda - \alpha)(\lambda - \beta)(\lambda^2 + \omega^2) = \lambda^4 - (\alpha + \beta)\lambda^3 + (\alpha\beta + \omega^2)\lambda^2 - \omega^2(\alpha + \beta)\lambda + \alpha\beta\omega^2,$$

with $\omega > 0$, and for any nonpositive real numbers $\alpha, \beta \in \mathbb{R}$. The proof is directly made by comparing the coefficients in both $\phi(\lambda)$. After calculations with the Maple computer program, the families $\mathcal{H}_i, i = 1, \cdots, 5$ of conditions given in Proposition 1 are followed.

The Jacobian matrix of system (2) under the family conditions $\mathcal{H}_1$ at the singular point E_2 has the eigenvalues $\lambda_{1,2} = \mp i\omega, \lambda_3 = \beta$ and $\lambda_4 = \frac{a_{11} c_1 \beta - \omega^2 b_{11}}{a_{11} c_1} = \alpha$, with $a_{11} c_1 \neq 0$. This means that E_2 is the Hopf point.

2. Proof of Proposition 2

Proof: The proof is omitted as it is similar to the proof of Proposition 1, only by imposing the characteristic polynomial of the Jacobian matrix at E_2 of system (2) as

$$\phi(\lambda) = \lambda(\lambda - \alpha)(\lambda^2 + \omega^2) = \lambda^4 + S_1\lambda^3 + S_2\lambda^2 + S_3\lambda + S_4.$$

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After transforming equilibrium E_2 to the origin, system (2) becomes

$$\begin{cases} \dot{x} = (x + x_0)\begin{pmatrix} a_1 + a_{11}(x + x_0) + \\ a_{12}y + a_{13}(z + z_0) + a_{14}w \end{pmatrix}, \\ \dot{y} = y\begin{pmatrix} b_1 + b_{11}(x + x_0) + \\ b_{12}y + b_{13}(z + z_0) + b_{14}w \end{pmatrix}, \\ \dot{z} = (z + z_0)\begin{pmatrix} c_1 + c_{11}(x + x_0) + \\ c_{12}y + c_{13}(z + z_0) + c_{14}w \end{pmatrix}, \\ \dot{w} = w\begin{pmatrix} d_1 + d_{11}(x + x_0) + d_{12}y + \\ d_{13}(z + z_0) + d_{14}w \end{pmatrix}, \end{cases} \quad 4$$

where $x_0 = -\frac{a_1c_{13} - a_{13}c_1}{a_{11}c_{13} - a_{13}c_{11}}$, $z_0 = \frac{a_1c_{11} - a_{11}c_1}{a_{11}c_{13} - a_{13}c_{11}}$.

3. Proof of Theorem 1

Proof: Let the perturbation $d_{13} = \epsilon\,\mu_1$ be satisfied. Then, after transforming the interior singular point E_2 to the origin and re-scaling the variables, $(x, y, z, w) = (\epsilon x, \epsilon y, \epsilon z, \epsilon w)$, system (2) is written as follows

$$\begin{cases} \dot{x} = \frac{1}{a_{11}c_1}((\epsilon a_{11}c_1x - \omega^2)(a_{11}x + a_{12}y + a_{13}z + a_{14}w)), \\ \dot{y} = \frac{y}{a_{11}c_1\omega^2}((c_1a_{11}(\epsilon a_{13}c_1(\beta - b_1)z + \omega^2(\epsilon(b_{11}x + b_{12}y + b_{14}w) + \beta)) - b_{11}\omega^4)), \\ \dot{z} = \frac{1}{a_{13}c_1\omega^2}((\epsilon c_1a_{13}z + \omega^2)(c_1^2a_{11}x + \omega^2(a_{11}x + c_{12}y + a_{13}z + c_{14}w))), \\ \dot{w} = \frac{w}{a_{13}c_1\omega^2}((c_1a_{13}(\epsilon c_1^2a_{11}x + \omega^2(\mu_1z\epsilon^2 + \epsilon(d_{14}w + d_{12}y) + \beta)) + \epsilon\omega^4\mu_1)). \end{cases} \quad 5$$

Now, the linear part of system (5) at the origin may be represented in its real Jordan normal form, that is, when $\epsilon = 0$, the linear part at the origin of the system (5) will be transformed

$$J = \begin{pmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}.$$

This is accomplished by using the linear change of the coordinates $(x, y, z, w) = P.(X, Y, Z, W)$, and P with its inverse P^{-1} are

$$P = \begin{pmatrix} \frac{a_{11}\omega^2}{c_1} & \frac{\nu_5}{\nu_3} & \frac{a_{13}\omega^2}{c_1} & \frac{\nu_1}{c_1{}^3(\beta^2+\omega^2)} \\ -a_{11}\omega^2 & -\frac{c_1\nu_4}{\nu_3} & 0 & -\frac{\nu_2}{c_1{}^3(\beta^2+\omega^2)} \\ 0 & -\frac{a_{11}\omega^2}{\nu_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{c_1}{a_{11}a_{13}\omega^5} \end{pmatrix},$$

$$P^{-1} = \begin{pmatrix} 0 & 0 & \frac{c_1\nu_4}{a_{11}^2\,\omega^4} & \frac{\nu_2a_{13}\omega^3}{c_1{}^4(\beta^2 + \omega^2)} \\ 0 & 0 & -\frac{\nu_3}{\omega^2a_{11}} & 0 \\ \frac{c_1}{\omega^2a_{13}} & -\frac{1}{\omega^2a_{13}} & \frac{-c_1(\nu_4 - \nu_5)}{a_{11}a_{13}\omega^4} & \frac{-a_{11}\omega^3(c_1\nu_1 + \nu_2)}{c_1{}^4(\beta^2 + \omega^2)} \\ 0 & 0 & 0 & \frac{a_{11}a_{13}\omega^5}{c_1} \end{pmatrix}$$

where

$\nu_1 = \omega^4(\beta a_{14} - \beta c_{14} + a_{14}c_1)$, $\nu_2 = \omega^3(a_{14}c_1\beta - a_{14}\omega^2 + c_{14}\omega^2)$,

$\nu_3 = c_1(a_{11}{}^2c_1{}^2\beta^2 - 2a_{11}b_{11}c_1\beta\omega^2 + b_{11}{}^2\omega^4 + \omega^2a_{11}{}^2c_1{}^2)$,

$\nu_4 = a_{11}\omega^3(a_{11}a_{12}c_1\beta - a_{11}a_{12}\omega^2 + a_{11}c_{12}\omega^2 + a_{12}b_{11}\omega^2)$,

$\nu_5 = a_{11}\omega^4(a_{11}a_{12}c_1\beta - a_{11}c_{12}c_1\beta - a_{12}b_{11}\omega^2 + b_{11}c_{12}\omega^2 + a_{11}a_{12}c_1{}^2)$.

Then, in order to analyze the closed orbits of this system when ϵ is too small, the new variables (X, Y, Z, W) is then created by transforming system (5). To do this, the class of the cylindrical coordinates described below is employed:

$$(X, Y, Z, W) = (r\cos(\theta), r\sin(\theta), Z, W).$$

Introducing also θ as the new independent variable. Thus, system (5) becomes

$$\left(\frac{dr}{d\theta}, \frac{dZ}{d\theta}, \frac{dW}{d\theta}\right)^T = \epsilon(f_{11}, f_{21}, f_{31})^T + O(\epsilon^2) \quad 6$$

Using the notations from Theorem 3 (see The Main Tool for the state of Theorem 3), there is $t = \theta, T = 2\pi, x = (r, Z, W)$, and

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$$f_1(\theta,x)=f_1(\theta,r,Z,W)=\begin{pmatrix}f_{11}(\theta,r,Z,W)\\ f_{21}(\theta,r,Z,W)\\ f_{31}(\theta,r,Z,W)\end{pmatrix}.$$

The averaged function is given by

$$F_1(x)=F_1(r,Z,W)=\begin{pmatrix}F_{11}(r,Z,W)\\ F_{21}(r,Z,W)\\ F_{31}(r,Z,W)\end{pmatrix},$$

$$F_{11}(r,Z,W)=\frac{1}{2\pi}\int_0^{2\pi}f_{11}(\theta,r,Z,W)d\theta =\frac{r}{2c_1\nu_1\omega^3}(\nu_2 Z+\nu_5 W),$$

$$F_{21}(r,Z,W)=\frac{1}{2\pi}\int_0^{2\pi}f_{21}(\theta,r,Z,W)d\theta = -\frac{Z}{2\nu_1 c_1^2\omega^5}(c_1\nu_1\nu_3 Z+\nu_4\omega^5 W),$$

7

$$F_{31}(r,Z,W)=\frac{1}{2\pi}\int_0^{2\pi}f_{31}(\theta,r,Z,W)d\theta = -\frac{(a_{13}c_1\nu_1\nu_6 Z+a_{13}\nu_0\omega^5 W-2\mu_1\nu_1\omega^6)\,W}{2a_{13}c_1\nu_1\omega^5}$$

where ν_i for $i=0,\cdots,6$ are given in the (Supplemental Material section). Therefore, the system $F_{11}(r,Z,W)=F_{21}(r,Z,W)=F_{31}(r,Z,W)=0$ has three solutions $s_1=(0,0,r), s_2=(0,0,W_0)$ and $s_3=(0,Z_0,W_0)$. After calculations with the Maple computer program, the print-out of singular points s_1, s_2 and s_3 occupies more than one page. Only the solution $s_3=(0,Z_0,W_0)$ can be here considered. The others are not good solutions.

In addition, the determinant of the Jacobian $\frac{\partial(F_{11},F_{21},F_{31})}{\partial(r,Z,W)}$ at the solution s_3 takes non-zero value. Existing the solution s_3 and $\det(\frac{\partial F_1}{\partial x})|_{(0,Z_0,W_0)}=\mu_1 f$, such that $\mu_1 f\neq 0$. As a result, Theorem 3 (see The Main Tool for the state of Theorem 3) states that system (6) has a periodic solution when $\epsilon>0$.

An illustration, by following example, is shown to demonstrate that system (2) has a limit cycle that bifurcates from the Hopf bifurcation. The following system is under consideration:

$$\begin{cases}\dot{x}=x(-2x-y-2z+w),\\ \dot{y}=y(-2x-y-6z-4),\\ \dot{z}=z(-4x-y-2z-w+1),\\ \dot{w}=w(-2x+y+\epsilon z-2w),\end{cases} \quad (8)$$

with $\epsilon>0$. The singular point $(\frac{1}{2},0,\frac{-1}{2},0)$ of system (8) is the Hopf point with eigenvalues $-1,-2$ and $\mp i$. The averaged functions of Eq.7 of Theorem 1 are

$$F_{11}(r,Z,W)=\frac{3r}{2}(Z+W), F_{21}=-\frac{Z}{32}(112Z+27W+16), F_{31}=-\frac{W}{16}(128Z+15W).$$

System $F_{11}=F_{21}=F_{31}=0$ has the following three solutions

$s_1=(0,0,r)$, $s_2=(0,-\frac{1}{7},0)$ and $s_3=(0,\frac{5}{37},-\frac{128}{111})$.

s_1 and s_2 are not good solutions. In addition, the following is the value of the Jacobian matrix at s_3

$$det(\frac{\partial(F_{11},F_{21},F_{31})}{\partial(r,Z,W)})|s_3=-\frac{1130}{1369}.$$

which is different from zero. By Theorem 1 using the averaging method, system (8) has a periodic solution. By MATLAB simulation, each periodic solution in the projection space is plotted see Fig. 1.

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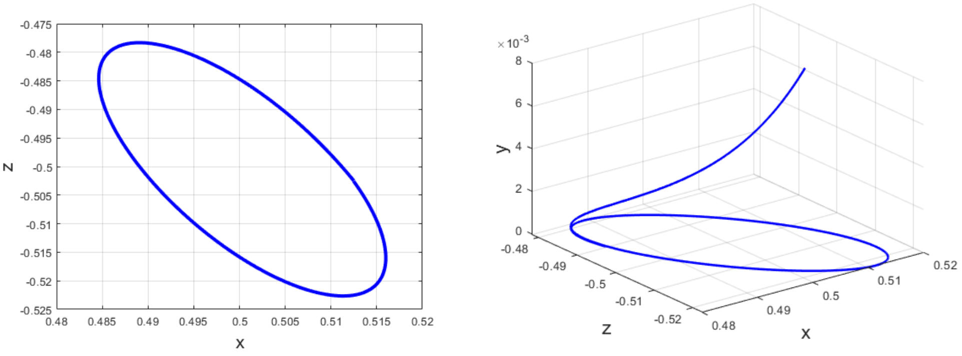

(a) (b)

Figure 1. in (2D and 3D): MATLAB simulation showing the (x, z) −space and (x, y, z) −space plot of the Lotka-Volterra model (11) depicting limit cycle for $X(0) = (25\epsilon + 0.5, 15\epsilon, -5\epsilon - 0.5, 1.6\epsilon)$, trange= $0{:}\,0.01{:}\,9$ and $\epsilon = 0.0005$.

4. Proof of Theorem 2

Proof: Let system (2) satisfy the perturbation $c_1 = \epsilon\,\mu_1$ and Eq. 3. Then, after re-scaling the variables $(x, y, z, w) = (\epsilon x, \epsilon y, \epsilon z, \epsilon w)$, and translating the interior singular point E_2 to the origin, system (2) is expressed as follows:

$$\begin{cases} \dot{x} = \frac{1}{a_{11}a_1^2\omega^2}\left(\left((a_1 - \epsilon\mu_1)\omega^2 + \epsilon a_1^2(a_{11}x - \mu_1)\right)(\omega^2(a_{11}x + a_{12}y + c_{13}z + a_{14}w) + c_{13}a_1^2 z)\right), \\ \dot{y} = \frac{1}{a_1 c_{13}\omega^2}\left(\epsilon y\left(a_1(\omega^2 b_{13} - a_1 b_1 c_{13})(a_{11}x - \mu_1) + \omega^2 c_{13}\left((b_{12}y + b_{13}z + b_{14}w)a_1 + \mu_1 b_1\right)\right)\right), \\ \dot{z} = \frac{1}{c_{13}a_1^2}\left((c_{14}w + a_{11}x + c_{12}y + c_{13}z)(\epsilon c_{13}a_1^2 z + (\epsilon\mu_1 - a_1)\omega^2(\omega^2))\right) \\ \dot{w} = \frac{1}{a_{11}c_{13}a_1^2}(\epsilon w(c_{13}(a_{11}a_1^2(d_{11}x + d_{12}y + d_{13}z + d_{14}w) - d_{11}\mu_1(\omega^2 + a_1^2)) + a_{11}d_{13}\mu_1\omega^2)). \end{cases}$$

9

Now that $\epsilon = 0$, the real Jordan normal form of the linear component at the origin of system (9) may be expressed, the linear part at the origin of system (9) will be transformed into its real Jordan normal form, as follows

$$\begin{pmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is accomplished by using the linear change of the coordinates $(x, y, z, w) = P.(X, Y, Z, W)$, where P with its inverse P^{-1} are

$$P = \begin{pmatrix} -\frac{c_{13}}{a_{11}} & \frac{a_1 c_{13}}{a_{11}\omega} & \frac{p_1}{a_{11}{a_1}^2} & \frac{p_3}{a_{11}{a_1}^2} \\ 0 & 0 & 1 & 1 \\ 1 & 0 & -\frac{p_2}{c_{13}{a_1}^2} & -\frac{p_4}{{a_1}^2 c_{13}} \\ 0 & 0 & 1 & 2 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 0 & \frac{2p_2 - p_4}{c_{13}{a_1}^2} & 1 & -\frac{p_2 - p_4}{c_{13}{a_1}^2} \\ \frac{a_{11}\omega}{c_{13}a_1} & \frac{-\omega(2p_1 - 2p_2 + p_4 - p_3)}{{a_1}^3 c_{13}} & \frac{\omega}{a_1} & \frac{\omega(p_1 - p_2 + p_4 - p_3)}{{a_1}^3 c_{13}} \\ 1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

where

$$p_1 = (a_{12} + a_{14} - c_{14} - c_{12}) - {a_1}^2(c_{12} + c_{14}), \quad p_2 = \omega^2(a_{12} + a_{14} - c_{12} - c_{14}),$$

$$p_3 = (a_{12} + 2a_{14} - 2c_{14} - c_{12}) - {a_1}^2(c_{12} + 2c_{14}), \quad p_4 = \omega^2(a_{12} + 2a_{14} - c_{12} - 2c_{14}).$$

Then, in order to analyze the closed orbits of this system when ϵ is too small, the new variables (X, Y, Z, W) is then created by transforming system (9). To do this, the class of the cylindrical coordinates described below is employed:

$$(X, Y, Z, W) = (r\cos(\theta), r\sin(\theta), Z, W).$$

By introducing also θ as a new independent variable, as demonstrated in Theorem 1, the averaged function is given

$$F_1(x) = F_1(r, Z, W) = \begin{pmatrix} F_{11}(r, Z, W) \\ F_{21}(r, Z, W) \\ F_{31}(r, Z, W) \end{pmatrix},$$

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$$F_{11}(r,Z,W) = \frac{r}{2a_{11}c_{13}a_1^2\omega^3}(u_1Z + u_2W - \mu_1 a_{11}c_{13}a_1^2\omega^2),$$
$$F_{21}(r,Z,W) = \frac{1}{a_1^2c_{13}\omega^3a_{11}}(Z(u_5Z - \mu_1u_3) + W(u_6Z + 2u_7W - 2\mu_1u_4)), \qquad 10$$
$$F_{31}(r,Z,W) = \frac{1}{a_1^2c_{13}\omega^3a_{11}}(Z(\mu_1u_4 - u_9Z) - W(u_{10}Z + u_{11}W - \mu_1u_8)),$$

where u_i for $i = 1,\cdots,8$ are given in the (Supplemental Material section).

Therefore, the system $F_{11}(r,Z,W) = F_{21}(r,Z,W) = F_{31}(r,Z,W) = 0$ has the following four solutions

$$s_1 = (0,0,0), s_2 = (0,\frac{2\mu_1\rho_2}{\rho_1},-\frac{\mu_1\rho_2}{\rho_1}), s_3 = (0,-\frac{\mu_1\rho_3}{\rho_4},\frac{\mu_1\rho_3}{\rho_4}), s_4 = (0,\frac{\mu_1\rho_7}{\rho_6},-\frac{\mu_1\rho_5}{\rho_6}),$$

where ρ_i for $i = 1,\cdots,7$ are given in the (Supplemental Material section). After calculations, s_1 is not a good solution. Only the solutions s_2, s_3 and s_4 are considered. In addition, the determinant of the Jacobian matrix $\frac{\partial(F_{11},F_{21},F_{31})}{\partial(r,Z,W)}$ at the solutions s_2, s_3 and s_4 takes non-zero value. Existing the solutions s_2, s_3 and s_4 and $\det\left(\frac{\partial F_1}{\partial x}\right)|_{s_2,s_3,s_4} = \mu_1 f_2$, such that $\mu_1 f_2 \neq 0$. Thus, from Theorem 3 (see also The Main Tool for Theorem 3), gives that system (2) under the generic conditions has three periodic solutions bifurcations from zero-Hopf singular point for $\epsilon > 0$.

An illustration, by following example, is shown to demonstrate that system (2) has three limit cycles that bifurcates from the zero-Hopf bifurcation. The following system is under consideration

$$\begin{cases} \dot{x} = x(-2x - y - 2z + w - 1), \\ \dot{y} = y(-2y - 2z + w - 2), \\ \dot{z} = z(-2x - 2y - z - w + 2\epsilon), \\ \dot{w} = w(-2x - y - z - 2w), \end{cases} \qquad 11$$

with $\epsilon > 0$. The singular point $\left(\frac{1}{2},0,-1,0\right)$ of system (11) is the zero-Hopf point. Following the steps of first-order averaging theory, the following averaged functions are obtained:

$$\begin{cases} F_{11}(r,Z,W) = -r(8Z + 6W + 1), \\ F_{21}(r,Z,W) = 2Z(29Z + 11) + 3W(29Z + 11W + 6), \\ F_{31}(r,Z,W) = -8Z(8Z + 3) - 2W(19W + 49Z + 10). \end{cases} \qquad 12$$

Then, system (12) has the following four solutions

$$s_1 = (0,0,0),\ s_2 = (0,-1,1),\ s_3 = (0,3,-4),\ s_4 = (0,-\frac{4}{7},\frac{2}{7}).$$

s_1 is not a good solution. The other solutions s_2, s_3 and s_4 give us

$$\det\left(\frac{\partial F_1}{\partial x}\right)\Big|_{s_2} = 4 \neq 0,\ \det\left(\frac{\partial F_1}{\partial x}\right)\Big|_{s_3} = -20 \neq 0,\ \text{and}\ \det\left(\frac{\partial F_1}{\partial x}\right)\Big|_{s_4} = -\frac{520}{49} \neq 0.$$

In addition, the determinant of the Jacobian matrix $\frac{\partial(F_{11},F_{21},F_{31})}{\partial(r,Z,W)}$ at each solution s_2, s_3 and s_4 takes non-zero. Using the averaging method, system (11) has three periodic solutions. By MATLAB simulation, each periodic solution in the projection space is plotted see Figs. 2-4. Furthermore, all periodic solutions together in one figure are plotted (see Fig. 5).

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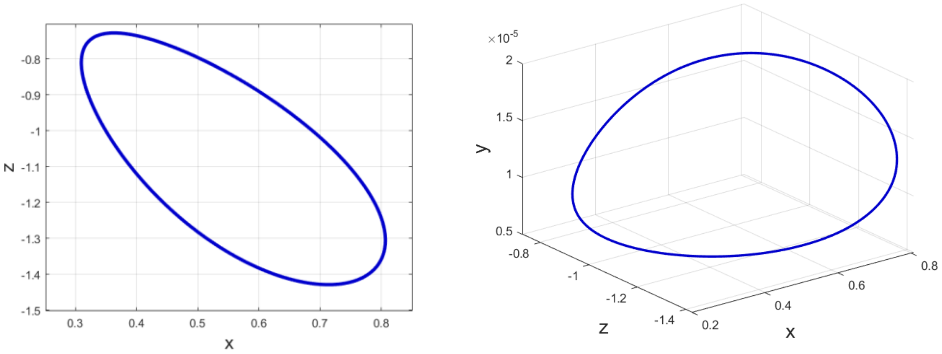
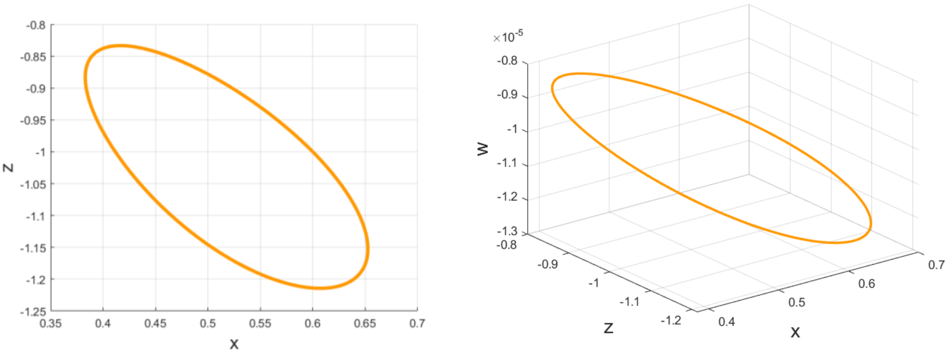


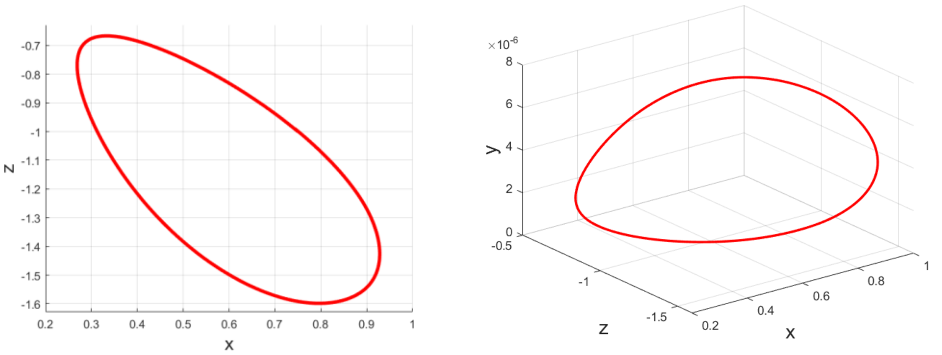
(a) (b)

Figure 2. in (2D and 3D): MATLAB simulation showing the (x, z) −space and (x, y, z) −space plot of the Lotka-Volterra model (11) depicting limit cycle for $X(0) = (0.35(\epsilon + 1), \epsilon, -\epsilon - 1, 0)$, trange= $0: 0.01: 6.4$ and $\epsilon = 0.000005$.



(a) (b)

Figure 3. in (2D and 3D): MATLAB simulation showing the (x, z) −space and (x, z, w) −space plot of the Lotka-Volterra model (11) depicting limit cycle for $X(0) = (0.6(10\epsilon + 1), 0, -6\epsilon - 1, -2\epsilon)$, trange= $0: 0.01: 6.4$ and $\epsilon = 0.000005$.



(a) (b)

Figure 4. in (2D and 3D): MATLAB simulation showing the (x, z) −space and (x, y, z) −space plot of the Lotka-Volterra model (11) depicting limit cycle for $X(0) = (0.75(1.42\epsilon + 1),\ 1.43\epsilon, -1.71\epsilon - 1, -0.57\epsilon)$, trange= $0: 0.01: 6.4$ and $\epsilon = 0.000005$.

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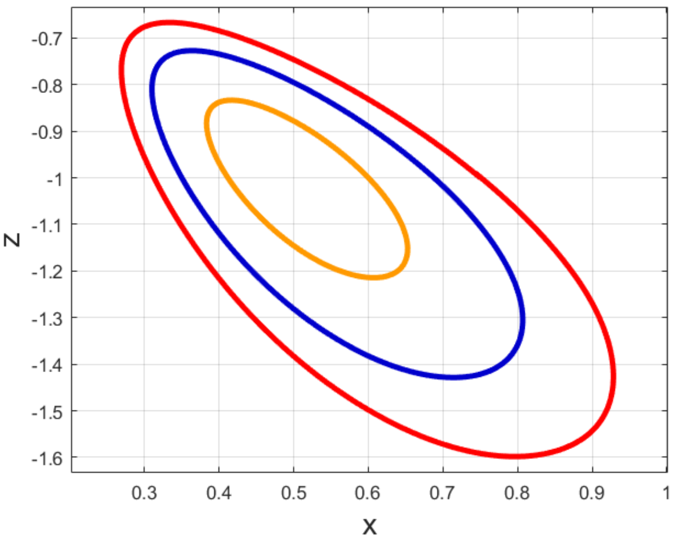

Figure 5. in (2D): MATLAB simulation showing the (x, z) −space

The Main Tool

First-order averaging method

This section presents the results of the averaging theory (For a general introduction to this method, one can see [17-19]), which is necessary to prove our results (Theorems 1 and 2). Consider the differential equation below

$$\dot{x} = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x,\varepsilon), \quad x(0) = x_0 \qquad 13$$

and

$$\dot{y} = \varepsilon g(y), \quad y(0) = x_0, \qquad 14$$

with x, y and x_0 where an open subset Q of $\mathbb{R}^n, t \in [0,\infty), \varepsilon \in (0, \varepsilon_0]$. Setting

$$g(y) = \frac{1}{T}\int_0^T F_1(t,y)dt.$$

and suppose that F_1 and F_2 are periodic with respect to period T in the variable t. Additionally, denoting all first derivatives of g by $D_x g$ and all second derivatives of g by $D_{xx} g$, respectively. For a proof of the next result, see [20].

Theorem 3: Suppose that $y(t) \in Q$ for $t \in [0,1/\varepsilon]$ and that $F_1, D_x F_1, D_{xx} F_1$ and $D_x F_2$ are continuous and bounded by that is not dependent on ε in $[0,\infty) \times Q \times (0, \varepsilon_0]$. Then the following statements hold:

1. $x(t) - y(t) = O(\varepsilon)$, is satisfied for $t \in [0,1/\varepsilon]$ as $\varepsilon \to 0$.

2. If system (21) has a singular point $E \neq 0$ such that

$$det D_y g(E) \neq 0,$$

then, a T-periodic solution $x(t,\varepsilon)$ of system (13) exists, which is close to E and such that $x(0,\varepsilon) - E = O(\varepsilon)$ as $\varepsilon \to 0$

Supplemental Material

$$S_1 = \frac{1}{a_{11}c_{13} - a_{13}c_{11}}\Big(\big((a_1 - b_1 + c_1 - d_1)c_{13} + c_1(d_{13} - a_{13} + b_{13})\big)a_{11} + a_1 c_{13}(d_{11} + b_{11} - c_{11}) - c_{11}(d_{13} + b_{13})a_1 - \big((d_{11} + b_{11})c_1 - c_{11}(b_1 + d_1)\big)a_{13}\Big),$$

$$S_2 = \frac{1}{(a_{11}c_{13} - a_{13}c_{11})^2}\Big(\Big(\big(((c_1 - b_1 - d_1)a_1 - (b_1 + d_1)c_1 + b_1 d_1)c_{13}^2 + c_1\big((d_{13} + b_{13})a_1 \quad + c_1(d_{13} - a_{13} + b_{13}) + (b_1 + d_1)a_{13} - d_{13}b_1 - d_1 b_{13}\big)c_{13} - c_1^2\big((d_{13} + b_{13})a_{13} - d_{13}b_{13}\ \big)\big)a_{11}^2 + \big(a_1(a_1(d_{11} + b_{11} - c_{11}) + (d_{11} + b_{11})c_1 + c_{11}(b_1 + d_1) -$$

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$d_{11}b_1 - d_1b_{11})c_{13}^2 \quad -\big(c_{11}(d_{13}+b_{13})a_1^2 +$
$\big((2\big((d_{11}+b_{11})a_{13} + (d_{13}+b_{13})c_{11}\big) - d_{11}b_{13} -$
$d_{13}b_{11})c_1 - ((b_1 \quad +d_1)a_{13} + d_{13}b_1 +$
$d_1b_{13})c_{11})a_1 + a_{13}\big((d_{11}+b_{11})c_1^2 - (c_{11}(b_1 +$
$d_1) + d_{11}b_1 + d_1b_{11}) \quad c_1 + 2b_1d_1c_{11})\big)c_{13} +$
$c_1\big(c_{11}((d_{13}+b_{13})a_{13} - 2d_{13}b_{13})a_1 +$
$a_{13}\big(((d_{11}+b_{11}+c_{11})a_{13} \quad -d_{11}b_{13} -$
$d_{13}b_{11})c_1 - ((b_1+d_1)a_{13} - d_{13}b_1 -$
$d_1b_{13})c_{11})\big)\big)a_{11} - ((d_{11}+b_{11})c_{11} -$
$d_{11}b_{11})a_1^2c_{13}^2 + \big((a_{13}c_{11} + (d_{13}+b_{13})c_{11} -$
$d_{11}b_{13} - d_{13}b_{11})c_{11}a_1 + a_{13}\big(((d_{11}+b_{11})c_{11}$
$-2d_{11}b_{11})c_1 - ((b_1+d_1)c_{11} - d_{11}b_1 -$
$d_1b_{11})c_{11})\big)a_1c_{13} + a_1^2b_{13}c_{11}^2d_{13} +$
$a_{13}c_{11}((d_{13}b_{11} \quad -a_{13}c_{11} + d_{11}b_{13})c_1 -$
$c_{11}(d_{13}b_1 + d_1b_{13}))a_1 - a_{13}^2(c_{11}b_1 -$
$c_1b_{11})(c_1d_{11} - c_{11}d_1)\big),$

$S_3 = \frac{1}{(a_{11}c_{13}-a_{13}c_{11})^3}\Big(\Big(\big(((b_1+d_1)c_1 - b_1d_1)a_1 -$
$b_1c_1d_1\big)c_{13}^3 - c_1\big(((d_{13}+b_{13})c_1 -$
$d_{13}b_1 - d_1b_{13})a_1 + ((b_1+d_1)a_{13} - d_{13}b_1 -$
$d_1b_{13})c_1 - b_1d_1a_{13})c_{13}^2 - c_1^2(a_1d_{13}b_{13} -$
$(a_{13}(d_{13} \quad +b_{13}) - d_{13}b_{13})c_1 + a_{13}\,(d_{13}b_1 +$
$d_1b_{13}))c_{13} + d_{13}c_1^3a_{13}b_{13})a_{11}^3 + \Big(\Big(\big((c_{11}(2d_{13} +$
$2b_{13}) - \quad d_{11}b_{13} - d_{13}b_{11})c_1 - c_{11}(d_{13}b_1 +$
$d_1b_{13}))a_1^2 + \big((2(d_{11}+b_{11})a_{13} - d_{11}b_{13} -$
$d_{13}b_{11})c_1^2 \quad -((c_{11}(b_1+d_1) + 2d_{11}b_1 +$
$2d_1b_{11})a_{13} + 2c_{11}(d_{13}b_1 + d_1b_{13}))c_1 +$
$2b_1d_1a_{13}c_{11})a_1 - \quad a_{13}c_1((d_{11}b_1 + d_1b_{11})c_1 -$
$2b_1d_1c_{11})\big)\,c_{13}^2 - a_1\big(((d_{11}+b_{11})c_1 + c_{11}(b_1 +$
$d_1) - d_{11}b_1 - \quad d_1b_{11})a_1 - (d_{11}b_1 + d_1b_{11})c_1 -$
$b_1d_1c_{11})c_{13}^3 + (2a_1^2d_{13}b_{13}c_{11} - c_1((c_{11}(d_{13} +$
$b_{13}) - 2d_{13}b_{11} - 2d_{11}b_{13})a_{13} - 3d_{13}b_{13}c_{11})a_1 -$
$a_{13}\big(((d_{11}+b_{11})a_{13} - d_{11}b_{13} - d_{13}b_{11})c_1^2 -$
$((2c_{11}(b_1+d_1) + d_{11}b_1 + d_1b_{11})a_{13} -$
$c_{11}(d_{13}b_1 + d_1b_{13}))c_1 + 2b_1d_1a_{13}c_{11})\big)\,c_1c_{13} -$
$a_{13}\big(2a_1d_{13}b_{13}c_{11} + a_{13}((c_{11}(d_{13}+b_{13}) +$
$d_{11}b_{13} + d_{13}b_{11})c_1c_{11}(d_{13}b_1 + d_1b_{13}))\big)\,c_1^2\big)$
$a_{11}^2 + \big(a_1^2\,\big(((d_{11}+b_{11})c_{11} - d_{11}b_{11})a_1 -$
$d_{11}b_{11}c_1 - c_{11}(d_{11}b_1 + d_1b_{11})\big)\,c_{13}^3 - a_1(c_{11}($
$c_{11}(d_{13}+b_{13}) - d_{11}b_{13} - d_{13}b_{11})a_1^2 +$
$\big(\big(((d_{11}+b_{11})c_{11} - 3d_{11}b_{11})a_{13} - 2c_{11}(d_{13}b_{11}$
$+d_{11}b_{13}))c_1 - c_{11}((2c_{11}(b_1+d_1) - d_{11}b_1 -$
$d_1b_{11})a_{13} + c_{11}(d_{13}b_1 + d_1b_{13}))\big)\,a_1 - 2a_{13}$
$(-b_1d_1c_{11}^2 + b_{11}c_1^2d_{11})\big)\,c_{13}^2 - \big(a_1^3d_{13}b_{13}c_{11}^2 +$
$c_{11}\,\big(((c_{11}(d_{13}+b_{13}) + 2d_{13}b_{11} + 2d_{11}b_{13})$
$a_{13} + 3d_{13}b_{13}c_{11})c_1 - c_{11}a_{13}(d_{13}b_1 +$
$d_1b_{13}))a_1^2 + a_{13}\,\big(\big(((d_{11}+b_{11})c_{11} +$
$3d_{11}b_{11})a_{13} + \quad 2c_{11}(d_{13}b_{11} + d_{11}b_{13}))c_1^2 +$
$((c_{11}(b_1+d_1) - 2d_{11}b_1 - 2d_1b_{11})a_{13} -$
$2(d_{13}b_1 + d_1b_{13}) \quad c_{11})c_{11}c_1 + b_1d_1a_{13}c_{11}^2)a_1 -$
$c_1a_{13}^2(b_1c_{11} - b_{11}c_1)(c_1d_{11} - c_{11}d_1)\big)\,c_{13} +$
$a_{13}c_1(a_1^2b_{13}c_{11}^2 \quad d_{13} + a_{13}c_{11}((2c_{11}(d_{13}+b_{13}) +$
$d_{11}b_{13} + d_{13}b_{11})c_1 - c_{11}(d_{13}b_1 + d_1b_{13}))a_1 +$
$a_{13}^2(((\quad b_{11} + d_{11})c_{11} + d_{11}b_{11})c_1^2 - c_{11}(c_{11}(b_1 +$
$d_1) + d_{11}b_1 + d_1b_{11})c_1 + b_1d_1c_{11}^2))\big)\,a_{11} +$
$a_1c_{11}\big(a_1^2c_{13}^3d_{11}b_{11} - a_1\big(c_{11}((d_{11}+b_{11})a_{13} +$
$d_{11}b_{13} + d_{13}b_{11})a_1 - a_{13}(c_{11}(d_{11}b_1 + \quad d_1b_{11}) -$
$2d_{11}b_{11}c_1)\big)c_{13}^2 + \big(((d_{13}+b_{13})a_{13} +$
$d_{13}b_{13})c_{11}^2a_1^2 + \big(((2d_{11}+2b_{11})a_{13} + \quad d_{11}b_{13} +$
$d_{13}b_{11})c_1 - ((b_1+d_1)a_{13} + d_{13}b_1 +$
$d_1b_{13})c_{11})a_{13}c_{11}a_1 - a_{13}^2(b_1c_{11} - b_{11}c_1)$
$(c_1d_{11} - c_{11}d_1)\big)c_{13} - a_{13}^2c_{11}c_1\big(c_{11}(d_{13} +$
$b_{13})a_1 + ((d_{11}+b_{11})c_1 - c_{11}(b_1+d_1))a_{13}\big)\big)\Big),$

$$S_4 = \frac{(a_1c_{11} - a_{11}c_1)(a_1c_{13} - a_{13}c_1)}{(a_{11}c_{13} - a_{13}c_{11})^3}\left(\left(\begin{matrix} a_1(c_{11}d_{13} - c_{13}d_{11}) \\ -a_{11}(c_1d_{13} - c_{13}d_1) + (c_1d_{11} \end{matrix}\right.\right.$$
$$\left.\left. - c_{11}d_1)a_{13}\right)\left(\begin{matrix} a_1(b_{11}c_{13} - b_{13}c_{11}) \\ -a_{11}(b_1c_{13} - b_{13}c_1) + a_{13}(b_1c_{11} - b_{11}c_1) \end{matrix}\right)\right).$$

$u_1 = (c_{13}d_{11} - a_{11}d_{13})(c_{14} - a_{14})\omega^4 +$
$\big(\big((b_1(a_{12} - c_{12}) - a_1(c_{14} + c_{12}))c_{13} +$
$a_1b_{13}c_{12}\big)\,a_{11} \quad +c_{14}d_{11}a_1c_{13})a_1\omega^2 -$
$c_{12}c_{13}a_1^3b_1a_{11},$

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$u_2 = 2(c_{13}d_{11} - a_{11}d_{13})(c_{14} - a_{14})\omega^4 + \left(a_{11}\left(\left(b_1(a_{12} - c_{12}) - a_1(2c_{14} + c_{12})\right)c_{13} + a_1 b_{13}c_{12}\right) + 2c_{14}d_{11}a_1c_{13}\right)a_1\omega^2 - c_{12}c_{13}a_1^3b_1a_{11}$,

$u_3 = \omega^4(a_{11}d_{13} - c_{13}d_{11}) + \omega^2 a_1^2(2a_{11}b_{13} - c_{13}d_{11}) - 2a_1a_{11}b_1c_{13}(\omega^2 + a_1^2)$,

$u_4 = \omega^4(a_{11}d_{13} - c_{13}d_{11}) + \omega^2 a_1^2(a_{11}b_{13} - c_{13}d_{11}) - a_1a_{11}b_1c_{13}(\omega^2 + a_1^2)$,

$u_5 = (c_{13}d_{11} - a_{11}d_{13})(c_{12} + c_{14} - a_{12} - a_{14})\omega^4 + \left(\left(\left(\left((2b_{12} + 2b_{14} - d_{12} - d_{14})a_1 + 2b_1(c_{12} + c_{14} - a_{12} - a_{14})\right)c_{13} - 2a_1b_{13}(c_{14} + c_{12})\right)a_{11} + c_{13}d_{11}a_1(c_{14} + c_{12})\right)a_1\omega^2 + 2c_{13}a_1^3b_1a_{11}(c_{14} + c_{12})$,

$u_6 = (c_{13}d_{11} - a_{11}d_{13})(3c_{12} + 4c_{14} - 3a_{12} - 4a_{14})\omega^4 + \left(\left(\left(a_1(4b_{12} - 3d_{12} - 4d_{14} + 6b_{14}) + 2b_1(2c_{12} + 3c_{14} - 2a_{12} - 3a_{14})\right)c_{13} - 2a_1b_{13}(2c_{12} + 3c_{14})\right)a_{11} + a_1c_{13}d_{11}(3c_{12} + 4c_{14})\right)a_1\omega^2 + 2a_{11}c_{13}b_1(2c_{12} + 3c_{14})a_1^3$,

$u_7 = (c_{13}d_{11} - a_{11}d_{13})(c_{12} + 2c_{14} - a_{12} - 2a_{14})\omega^4 + a_1\left(\left(\left(\left((b_{12} + 2b_{14} - d_{12} - 2d_{14})a_1 + b_1(c_{12} + 2c_{14} - a_{12} - 2a_{14})\right)c_{13} - a_1b_{13}(2c_{14} + c_{12})\right)a_{11} + c_{13}d_{11}a_1(2c_{14} + c_{12})\right)\omega^2 + c_{13}a_1^3b_1a_{11}(2c_{14} + c_{12})$,

$u_8 = (2\omega^4a_{11}d_{13} - 2\omega^4c_{13}d_{11} + \omega^2a_1^2a_{11}b_{13} - 2\omega^2a_1^2c_{13}d_{11} - \omega^2a_1a_{11}b_1c_{13} - a_{11}a_1^3b_1c_{13})$,

$u_9 = (c_{13}d_{11} - a_{11}d_{13})(c_{14} + c_{12} - a_{12} - a_{14})\omega^4 + a_1\left(\left(\left(\left((b_{12} + b_{14} - d_{12} - d_{14})a_1 + b_1(c_{14} + c_{12} - a_{12} - a_{14})\right)c_{13} - a_1b_{13}(c_{14} + c_{12})\right)a_{11} + c_{13}d_{11}a_1(c_{14} + c_{12})\right)\omega^2 + c_{13}a_1^3b_1a_{11}(c_{14} + c_{12})$,

$u_{10} = (c_{13}d_{11} - a_{11}d_{13})(3c_{12} + 4c_{14} - 3a_{12} - 4a_{14})\omega^4 + \left(\left(\left(\left(a_1(2b_{12} + 3b_{14} - 3d_{12} - 4d_{14}) + b_1(2c_{12} + 3c_{14} - 2a_{12} - 3a_{14})\right)c_{13} - (2c_{12} + 3c_{14})b_{13}a_1\right)a_{11} + c_{13}d_{11}a_1(3c_{12} + 4c_{14})\right)a_1\omega^2 + b_1(2c_{12} + 3c_{14})c_{13}a_1^3a_{11}$,

$u_{11} = 2(c_{13}d_{11} - a_{11}d_{13})(2c_{14} + c_{12} - a_{12} - 2a_{14})\omega^4 + a_1\left(\left(\left(a_1(b_{12} + 2b_{14} - 2d_{12} - 4d_{14}) + b_1(2c_{14} + c_{12} - a_{12} - 2a_{14})\right)c_{13} - a_1b_{13}(2c_{14} + c_{12})\right)a_{11} + 2c_{13}d_{11}a_1(2c_{14} + c_{12})\right)\omega^2 + c_{13}a_1^3b_1a_{11}(2c_{14} + c_{12})$.

$\rho_1 = \left((a_1b_{12} - a_{12}b_1 + b_1c_{12})\omega^2 + a_1^2b_1c_{12}\right)c_{13} - \omega^2a_1b_{13}c_{12}$,

$\rho_2 = -b_1(\omega^2 + a_1^2)c_{13} + \omega^2a_1b_{13}$, $\rho_3 = -d_{11}(\omega^2 + a_1^2)c_{13} + \omega^2a_{11}d_{13}$,

$\rho_4 = \left(-d_{11}(a_{14} - c_{14})\omega^2 - a_1^2(a_{11}d_{14} - c_{14}d_{11})\right)c_{13} + \omega^2a_{11}d_{13}(a_{14} - c_{14})$,

$\rho_5 = (\omega^2 + a_1^2)\left(d_{11}(b_{12} + b_{14})\omega^2 + a_1b_1a_{11}(d_{12} + d_{14})\right)c_{13}^2 - \omega^2\left((d_{13}(b_{12} + b_{14})a_{11} + d_{11}b_{13}(a_{12} + a_{14}))\omega^2 + a_{11}a_1\left(b_{13}(d_{12} + d_{14})a_1 + d_{13}b_1(a_{12} + a_{14})\right)\right)c_{13} + \omega^4d_{13}a_{11}b_{13}(a_{12} + a_{14})$,

$\rho_6 = \left(-d_{11}(-a_{12}b_{14} + a_{14}b_{12} - b_{12}c_{14} + b_{14}c_{12})\omega^4 - a_1\left(\left((b_{12}d_{14} - b_{14}d_{12})a_1 + b_1(a_{14}d_{12} - a_{12}d_{14} + c_{12}d_{14} - c_{14}d_{12})\right)a_{11} + d_{11}a_1(-b_{12}c_{14} + b_{14}c_{12})\right)\omega^2 - a_1^3b_1a_{11}(c_{12}d_{14} - c_{14}d_{12})\right)c_{13}^2 + \omega^2\left((d_{13}(-a_{12}b_{14} + a_{14}b_{12} - b_{12}c_{14} + b_{14}c_{12})a_{11} + d_{11}b_{13}(a_{14}c_{12} - a_{12}c_{14}))\omega^2 + a_{11}a_1\left(b_{13}(c_{12}d_{14} - c_{14}d_{12})a_1 + d_{13}b_1(-a_{12}c_{14} + a_{14}c_{12})\right)\right)$

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$c_{13} - \omega^4 d_{13} a_{11} b_{13}(-a_{12}c_{14} + a_{14}c_{12}),$

$$\rho_7 = (\omega^2 + a_1^2)\begin{pmatrix} d_{11}(b_{12}+2b_{14})\omega^2 \\ +a_1 b_1 a_{11}(d_{12}+2d_{14}) \end{pmatrix} c_{13}^2 - \left(\begin{pmatrix} (d_{13}(b_{12}+2b_{14})a_{11} + d_{11}b_{13}(a_{12}+2a_{14}))\omega^2 \\ +a_1 a_{11}(b_{13}(d_{12}+2d_{14})a_1 + d_{13}b_1(a_{12}+2a_{14})) \end{pmatrix} \right)\omega^2 c_{13} + \omega^4 d_{13} a_{11} b_{13}(a_{12}+2a_{14}).$$

Results and Discussion

The Lotka-Volterra systems have been widely utilized for modelling a broad range of natural phenomena, including the time evolution of competing species in biology, chemical processes, and many other phenomena. In non-linear differential systems, it is highly intriguing and crucial to know whether a system has periodic solutions. In this investigation, the first result in this paper proved that there are only five families of conditions on the parameters of the four-dimensional Lotka-Volterra system in which the Jacobian matrix at one of its singular points has a pair of purely conjugate complex eigenvalues and non-zero eigenvalues (this was explained in Proposition 1). Moreover, that there are only eight families of conditions on the parameters of the four-dimensional Lotka-Volterra system were proved in which the Jacobian matrix at one of its singular points has a pair of purely conjugate complex and at least one of the other eigenvalues is zero (this was explained in Proposition 2). The second result in this paper is Theorem 1 and Theorem 2, which give the periodic solutions of the four-dimensional Lotka-Volterra system. Theorem 1 states that the four-dimensional Lotka-Volterra system exhibits Hopf bifurcation at one of its singular points and that a periodic solution is bifurcated at that singular point. Theorem 2 states that the zero-Hopf bifurcation occurs at one of its singular points and that at most three periodic solutions bifurcate at this singular point. At the same time, the first-order averaging method was applied to determine the number of periodic solutions around the zero-Hopf and Hopf equilibrium points. It is believed by us that there are still other interesting relevancies between the two bifurcations, and further exploration is needed.

Conclusion

In this paper, the 4DLV differential system was considered. This type of system has sixteen equilibrium points. The values of the parameters were described for which the zero-Hopf and a Hopf bifurcation occur at one of its equilibrium points for this system. There are five families of sufficient conditions for parameters of this type of system (see Proposition 1) for which one of its equilibrium points is the Hopf equilibrium point. Moreover, there are eight families of sufficient conditions for parameters of this type of system (see Proposition 2) for which one of its equilibrium points becomes a zero-Hopf equilibrium point. Consequently, the existence of periodic solutions in a certain class of the Lotka-Volterra systems is proved by the classical first-order averaging method (see the above result and discussion section). This method gives the periodic solutions of the 4DLV system. It follows from Theorem 3 that any sufficiently small system has a periodic solution with period near 2π. Clearly, this periodic solution tends to the origin of the coordinates when $\epsilon \to 0$. Therefore, this is the small amplitude periodic solution, which starts at the zero-Hopf or a Hopf equilibrium point.

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Authors' Declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Soran.

Authors' Contribution Statement

The first author S.A.M. contributed to analysis and the data collection. The second author N.H.H. supervised the findings of this work. Both authors discussed the results and contributed to the final manuscript.

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تشعب هوبف و صفري هوف لأنظمة لوتكا فولتيرا رباعية الأبعاد

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الخلاصة

في هذا العمل، تم النظر في نموذج لوتكا-فولتيرا رباعي الأبعاد (4DLV) الذي يتضمن أربعة أنواع في بيئة ثابتة. ويهدف هذا البحث الى دراسة التشعبات المحلية (local bifurcations) في النظام. يحتوي هذا النظام على ست عشرة نقطة من التوازن (equilibrium point) على الأكثر. تم أخذ إحدى نقاط التوازن في الاعتبار من أجل دراسة الحلول الدورية (periodic solutions) التي تتشعب من نقطتي توازن هوبف (Hopf) وصفر هوبف (zero-Hopf) على التوالي. توجد خمس عائلات ذات شروط كافية على معلمات النظام التي تحتوي فيها المصفوفة الجاكوبية عند نقطة التوازن على زوج من القيم الوهمية البحتة $\pm i\omega$ وقيمتين ذاتيتين (eigenvalues) غير موجبتين. إضافة الى ذلك توجد ثماني عائلات ذات شروط كافية على المعلمات التي تحتوي فيها المصفوفة الجاكوبية عند نقطة التوازن على زوج من القيم الذاتية الخيالية البحتة $\pm i\omega$ وعلى الأقل احدى من القيم الذاتية الأخرى تكون صفرا. بعد ذلك، يكشف هذا البحث أن بعض أنظمة Lotka-Volterra الفرعية رباعية الأبعاد تظهر حلاً دوريًا متشعبًا من نقطة توازن هوبف وثلاثة حلول دورية متشعبة من نقطة توازن هوبف صفرية بشكل استقبالي. تتكون طريقة حساب المتوسط بأي ترتيب لحساب الحلول الدورية من توفير الظروف الكافية لوجود الحلول الدورية في الأنظمة التفاضلية متعددة الحدود من خلال دراسة نقاط التوازن للأنظمة المتوسطة المرتبطة بها. ثم، والأداة الرئيسية المستخدمة هي نظرية المتوسط من الدرجة الأولى لحساب الحلول الدورية التي تتشعب من نقاط هوبف وصفر هوبف المفردة في 4DLVS. وفي النهاية تدعم النتائج النظرية التي تم الحصول عليها والتحقق منها من خلال الأمثلة العددية.

الكلمات المفتاحية: نظرية المتوسطات، نظام لوتكا فولتيرا، حلول دورية، نظام تربيعي متعدد الحدود، تشعب صفري هوبف.