Oscillation of Nonlinear Differential Equations with Advanced Arguments

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Abstract

This paper is concerned with the oscillation of all solutions of the *n-th* order delay differential equation $x^{(n)}(t)+P(t)\,f(x(\tau(t)))=0,\quad n\geq 2$. The necessary and sufficient conditions for oscillatory solutions are obtained and other conditions for nonoscillatory solution to converge to zero are established.

1. Introduction

Consider the nonlinear differential equation of order n with advanced argument of the type:-

$$x^{(n)}(t)+P(t)\,f(x(\tau(t)))=0,\qquad n\geq 2\quad ,t\geq t_0 \tag{1.1}$$

Where the continuous function $P:[t_0,\infty)\to R$ is allowed to oscillate, while the continuous functions $f:R\to R$, $\tau:[t_0,\infty)\to R$ satisfies the following conditions :

H1 : $\tau(t)$ is continuous nondecreasing

$$\tau(t)>t,\quad \lim_{t\to\infty}\tau(t)=\infty.$$

H2 : f is nondecreasing such that

$$u\,f(u)>0\quad for\quad u\neq 0.$$

By a solution of eq. (1.1) we mean a function $x\in C^1([t_0,\infty),R)$ which satisfies eq. (1.1) for all $t\geq t_0$ where $\sup\{|x(t)|:t\in[t_x,\infty)\}>0$. A solution is said to be non-oscillatory if it is eventually of constant sign otherwise is said oscillatory , eq. (1.1) is said oscillatory if all of its solutions are oscillatory. Many literatures studied the oscillation of eq.(1.1) of first and second order , and a few investigated the higher order. One may see the monographs due to Berezansky [1] Stavrovlakis [2] Ladde [3],Rath and Padhy [4], Kulenovic [5], Kiguradze,[6].

2. Mean Results

In this section we give some theorems describe the oscillatory behavior of the solutions of equation (1.1).

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Definition:- [see Kiguradze,[6]]

Let $u(t) \in C^n([0,\infty), R)$ be of constant sign and let $u^{(n)}(t)$ be also of constant sign and not equivalent to zero in any interval $[T,\infty)$, $T \geq 0$ and $u(t)\, u^{(n)}(t) \leq 0$ then there exists $t_0 \geq 0$ such that $u^{(i)}(t)$, $i = 1,2,\ldots,n-1$ are of constant sign on $[t_0,\infty)$ and there exists an integer $k \in \{1,3,5,\ldots,n-1\}$ when n is even, or exists an integer $k \in \{0,2,4\ldots,n-1\}$ when n is odd such that

$$u(t)u^{(i)}(t) > 0 \quad for \quad 0 \leq i \leq k \quad on \ [t_0,\infty)$$
$$(-1)^{n+i-1}u(t)u^{(i)}(t) > 0, \quad k+1 \leq i \leq n-1, t \geq t_0$$

(2.1)

Such $u(t)$ is said to be of degree k

The following Lemma improve Lemma 6, Mohamad H. [7]

Lemma 1. Suppose that

$$\delta\{u^{(n)}(t) + \delta P(t)\ f(u(\tau(t)))\}\operatorname{sgn} u(t) \leq 0, \qquad (2.2)$$

Where $\delta = \pm 1$, P, f, τ satisfies H1-H2 , $P(t) \geq 0$ and

$$\int_T^\infty t^2 P(t)\, dt = \infty \qquad (2.3)$$

Then the following statements are true:

1- Let $\delta = 1$, if n even then every possible non-oscillatory solution of (2.2) are of degree n-1. if n odd then every possible non- oscillatory solution of (2.2) are either of degree 0 or degree n-1.

2- Let $\delta = -1$, if n even then every possible non oscillatory solution of (2.2) are either of degree 0 or degree n. if n odd then every possible non oscillatory solution of (2.2) are of degree n.

<u>Proof</u> : See [7].

Other literatures study the case when the coefficient is constant see [8]-[10], while Olach [11] study eq.(1.1) and gives some sufficient conditions for oscillation. Now we give the first result in this paper.

<u>Theorem</u> (1) :*Suppose that* $P(t) \geq 0$, *and*

$$\int^\infty t^{n-1} P(t)\, dt = \infty\,, \qquad (2.4)$$

Then every bounded solution of (1.1) is oscillatory if n even, and every bounded solution are either oscillatory or $\lim_{t\to\infty} x(t) = 0$ *if n is odd.*

<u>Proof</u>: Let $x(t)$ is bounded nonoscillatory solution of eq. (1.1) on $[t_0,\infty)$ .Without loss of generality let

$x(t)>0$ on $[t_0,\infty)$, we have

$x^{(n)}(t) = -P(t)\, f(x(\tau(t))) \le 0$

Consider the equality

$$x^{(i)}(t) = \sum_{i=1}^{n-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} x^{(i)}(s) + \frac{(-1)^{n-j}}{(n-j-1)!} \int_t^s (u-t)^{n-j-1} x^{(n)}(u)\, du$$

Where $s \ge t \ge t_0$. By using eq. (1.1) the last equality leads to

$$x^{(j)}(t) = \sum_{i=j}^{n-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} x^{(i)}(s) + \frac{(-1)^{n-j+1}}{(n-j-1)!} \int_t^s (u-t)^{n-j-1} P(u)\, f(x(\tau(u)))\, du \qquad (2.5)$$

let n be even, since $x(t)$ is positive bounded and $x^{(n)}(t) \le 0$ then $x(t)$ must be of degree 1 , let j=1 then the last equality reduce to :

$$x'(t) \ge \frac{1}{(n-2)!} \int_t^\infty (u-t)^{n-2} P(u)\, f(x(\tau(u)))\, du \ge 0$$

Integrate the last inequality from t_1 to t where $t \ge t_1 \ge t_0$ we obtain

$$x(t) \ge \frac{1}{(n-1)!} \int_{t_1}^t (u-t_1)^{n-1} P(u)\, f(x(\tau(u)))\, du \qquad (2.6)$$

Since $x(t)$ is non-decreasing and bounded, then $\lim_{t\to\infty} x(t) = c > 0,\quad x(t) \le c$ so we can find t_2 large enough such that.

$\frac{c}{2} \le x(\tau(t)) \le c$ and

$f(x(\tau(t))) \ge f(\frac{c}{2}) = c_1, \quad t \ge t_2 \ge t_1$

, where c_1 is positive constant, as $t \to \infty$ we get from inequality (2.6)

$$c \ge \frac{c_1}{(n-1)!} \int_{t_2}^\infty (u-t_2)^{n-1} P(u)\, du \ ,$$

which is a contradiction with (2.4).

Now let n be odd, since $x^{(n)}(t) \le 0$ then $x(t)$ must be of degree 0 which implies that $x(t)$ is non-increasing, from eq.(2.5) with j = 0, we get :

$$x(t_1) \ge \frac{1}{(n-1)!} \int_{t_1}^t (u-t_1)^{n-1} P(u)\, f(x(\tau(u)))\, du\ , \quad t \ge t_1 \ge t_0$$

Since $x'(t) \le 0$, and $x(t)$ is bounded, then $\lim_{t\to\infty} x(t) = c \ge 0, \quad x(t) \ge c$.

if $c \ne 0$ then $c > 0$, we can find $t_2 \ge t_1$ large enough such that $x(\tau(t)) \ge c$, and

$f(x(\tau(t))) \ge f(c) = c_1$ for $t \ge t_2$ then

$$x(t_1) \ge \frac{c_1}{(n-1)!} \int_{t_2}^t (u-t_1)^{n-1} P(u)\, du$$

as $t \to \infty$ we get a contradiction , so either $c = 0$ or $x(t)$ is oscillatory.

<u>Theorem (2)</u> : *Suppose that* $P(t) \le 0$, *and*

$$\int^\infty t^{n-1} |P(t)| dt = \infty\ ,$$

(2,4)′

If n even then every bounded solution of (1.1) are either oscillatory or

$\lim_{t\to\infty} x(t) = 0$ *, and if n odd then every bounded solution of (1.1) are oscillatory.*

<u>*Proof*</u>: Let $x(t)$ is non oscillatory bounded solution of eq. (1.1) , $t \ge t_0$

Without loss of generality let $x(t)$>0 on $[t_0, \infty)$, then

$$x^{(n)}(t) = -P(t)\, f(x(\tau(t))) \ge 0$$

Let n be even , since $x(t)$ is bounded and $x^{(n)}(t) \ge 0$, then $x(t)$ must be of degree 0, which implies that $x(t)$ is non-increasing and bounded hence (2.5) reduce to

$$x(t) \ge \frac{1}{(n-1)!}\int_t^s (u-t)^{n-1} \,|\, P(u) \,|\, f(x(\tau(u)))\,du \ge 0$$

$$x(t_1) \ge \frac{1}{(n-1)!}\int_{t_1}^s (u-t_1)^{n-1} \,|\, P(u) \,|\, f(x(\tau(u)))\,du$$

$$\ge 0, \qquad t \ge t_1 \qquad (2.7)$$

Let $\lim_{t\to\infty} x(t) = c \ge 0, \quad x(t) \ge c$, if $c \ne 0$ then $c > 0$, we can find $t_2 \ge t_1$ large enough such that $x(\tau(t)) \ge c$, and

$f(x(\tau(t))) \ge f(c) = c_1 > 0$ for $t \ge t_2$ then (2.7) implies to

$$x(t_1) \ge \frac{c_1}{(n-1)!}\int_{t_2}^t (u-t_1)^{n-1} \,|\, P(u) \,|\, du$$

as $t \to \infty$ we get a contradiction , so either c=0 or $x(t)$ is oscillatory.

Let n be odd, since $x^{(n)}(t) \ge 0$ then $x(t)$ must be of degree 1 which implies that $x(t)$ is non decreasing and bounded, from eq.(2.5) with j = 1, we get :

$$x'(t) \ge \frac{1}{(n-2)!}\int_t^\infty (u-t)^{n-2} \,|\, P(u) \,|\, f(x(\tau(u)))\,du$$

Integrate the last inequality from t_1 to t where $t \ge t_1 \ge t_0$ we obtain

$$x(t_1) \ge \frac{1}{(n-1)!}\int_{t_1}^s (u-t_1)^{n-1} \,|\, P(u) \,|\, f(x(\tau(u)))\,du$$

$$\ge 0, \qquad t \ge t_0 \qquad (2.8)$$

$\lim_{t\to\infty} x(t) = c > 0, \quad x(t) \le c$ so we can find t_2 large enough such that.

$\frac{c}{2} \le x(\tau(t)) \le c$ and

$f(x(\tau(t))) \ge f(\frac{c}{2}) = c_1, \qquad t \ge t_2 \ge t_1$

, where c_1 is positive constant, and as $t \to \infty$ we get from inequality (2.8) a contradiction.

<u>***Theorem (3)***</u>

Suppose that $P(t) \ge 0$*, and (2.3) holds then if n is even every solutions of (1.1) are oscillatory and if n is odd every solutions of (1.1) are either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

<u>*Proof*</u>**:** Let $x(t)$ be non oscillatory solution of (1.1) and say $x(t) > 0,\ t \ge t_0$

$x^{(n)}(t) = -P(t)\, f(x(\tau(t))) \le 0,$ then

$x^{(i)}(t)$ are monotone $i = 0,1,\ldots,n-1$

If n even, then by Lemma 1 every non oscillatory solution of (1.1) are of degree $n-1$, integrate eq.(1.1) from s to t $s \le t$ we get

$$x^{(n-1)}(t) - x^{(n-1)}(s) = \int_s^t P(\xi)\, f(x(\tau(\xi)))\, d\xi, \quad s < t$$

$$x^{(n-1)}(s) \ge \int_s^t P(\xi)\, f(x(\tau(\xi)))\, d\xi, \qquad (2.9)$$

consider the integral equality

$$\int_{t_0}^t \xi^2 x^{(n)}(\xi)\, d\xi = t^2 x^{(n-1)}(t) - t_0^2 x^{(n-1)}(t_0) - t\, x^{(n-2)}(t) + t_0 x^{(n-2)}(t_0) + x^{(n-3)}(t) - x^{(n-3)}(t_0)$$

$$f(x(\tau(t_0))) \int_{t_0}^t \xi^2 P(\xi)\, d\xi \le t_0^2 x^{(n-1)}(t_0) - t^2 x^{(n-1)}(t) + t\, x^{(n-2)}(t) - t_0 x^{(n-2)}(t_0) - x^{(n-3)}(t) + x^{(n-3)}(t_0)$$

as $t \to \infty$ and apply (2.3) it follows that

$$\lim_{t\to\infty} \{t\, x^{(n-2)}(t) - t^2 x^{(n-1)}(t) - x^{(n-3)}(t)\} = \infty$$

there is $t_1 \ge t_0$ such that

$$t\, x^{(n-2)}(t) - t^2 x^{(n-1)}(t) - x^{(n-3)}(t) \ge 0, \quad t \ge t_1$$

which implies that

$$x^{(n-2)}(t) \ge t\, x^{(n-1)}(t), \quad t \ge t_1$$

$$\int_{t_1}^t \{s\, x^{(n-2)}(s) - s^2 x^{(n-1)}(s) - x^{(n-3)}(s)\} ds = t_1^2 x^{(n-2)}(t_1) - t^2 x^{(n-2)}(t) + 3t\, x^{(n-3)}(t) - 3t_1 x^{(n-3)}(t_1) - 4x^{(n-4)}(t) + 4x^{(n-3)}(t_1)$$

as $t \to \infty$ we get

$$\lim_{t\to\infty} [3t\, x^{(n-3)}(t) - t^2 x^{(n-1)}(t) - 4x^{(n-4)}(t)] = \infty$$

there is $t_2 \ge t_1$ such that

$$3t\, x^{(n-2)}(t) - t^2 x^{(n-1)}(t) - 4x^{(n-3)}(t) \ge 0, \; t \ge t_2$$

which implies that

$3x^{(n-2)}(t) \ge t\, x^{(n-1)}(t), \quad t \ge t_2$, follow in this procures we get there is $t_{n-2} \ge t_{n-3}$ such that

$$(2n-3)\, t\, x(t) - t^2 x'(t) - (n-1)^2 \int_{t_{n-2}}^t x(s)\, ds \ge 0, \quad t \ge t_{n-2}$$

which implies that

$$(2n-3)\, x(t) \ge t\, x'(t), \quad t \ge t_{n-2},$$

then $x(t) \ge \dfrac{t^{n-1}}{\prod_{i=1}^{n-1}(2i-1)} x^{(n-1)}(t)$,

from the last inequality and (2.9) we obtain

$$\begin{aligned} x(s) &\ge \frac{s^{n-1}}{\prod_{i=1}^{n-1}(2i-1)} x^{(n-1)}(s) \\ &\ge \frac{s^{n-1}}{\prod_{i=1}^{n-1}(2i-1)} \int_s^t P(\xi)\, f(x(\tau(\xi)))\, d\xi, \\ &\ge \frac{s^{n-1} f(x(\tau(s)))}{\prod_{i=1}^{n-1}(2i-1)} \int_s^t P(\xi)\, d\xi \\ &\ge \frac{f(x(\tau(s)))}{\prod_{i=1}^{n-1}(2i-1)} \int_s^t \xi^{n-1} P(\xi)\, d\xi, \end{aligned}$$

$$x(t_{n-2}) \ge \frac{f(x(\tau(t_{n-2})))}{\prod_{i=1}^{n-1}(2i-1)} \int_{t_{n-2}}^t \xi^{n-1} P(\xi)\, d\xi,$$

as $t \to \infty$ and according to (2.3) we get a contradiction

Let n be odd, by Lemma 1 every non-oscillatory solution of (1.1) are either of degree 0 or of degree n-1.

Suppose $x(t)$ is of degree 0, then $x(t)$ is positive decreasing and so it is bounded and by Theorem 1 either $x(t)$ oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Suppose $x(t)$ is of degree n-1, in this case the proof is similar when n even.

Theorem (4)

Assume that $P(t) \le 0$, *and* $\frac{f(u)}{u} \ge M > 0$

$$\int_T^\infty t^2 \,|\, P(t) \,|\, dt = \infty \,, \qquad (2.3)'$$

$$\limsup_{t\to\infty} \frac{M}{(n-1)!} \int_{\tau^{-1}(t)}^{t} (t-s)^{n-1} \,|\, P(s) \,|\, ds > 1, \qquad (2.10)$$

If n is even then every solution of (1.1) are either oscillatory or $\lim_{t\to\infty} x(t) = 0$,

If n is odd then every solution of (1.1) are oscillatory.

Proof : Let $x(t)$ be non-oscillatory solution of (1.1) and assume that

$x(t) > 0$, $t \ge t_0$ hence

$x^{(n)}(t) = -P(t)\, f(x(\tau(t))) \le 0,$ then

$x^{(i)}(t)$ are monotone $i = 0,1,\ldots,n-1$

Let n be even, eq.(1.1) can be written as

$$x^{(n)}(t) - |\, P(t) \,|\, f(x(\tau(t))) = 0,$$

so by Lemma 1 the only possible non-oscillatory solution of (1.1) are either of degree 0 or of degree n. Let $x(t)$ be of degree 0 ,then $x(t)$ is positive non-increasing so it is bounded then by Theorem 1 $x(t)$ is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Let $x(t)$ be of degree n, using the equality where $\xi < t$

$$x(t) = \sum_{i=0}^{n-1} \frac{(t-\xi)^i}{i!} x^{(i)}(\xi) + \frac{1}{(n-1)!} \int_\xi^t (t-s)^{n-1} x^{(n)}(s)\, ds \,,$$

$$x(t) \ge \frac{1}{(n-1)!} \int_\xi^t (t-s)^{n-1} \,|\, P(s) \,|\, f(x(\tau(s)))\, ds \,,$$

Let $\tau(\xi) \ge t,\quad t \ge \xi \ge t_1$, hence

$$x(\tau(\xi)) \ge \frac{1}{(n-1)!} \int_\xi^t (t-s)^{n-1} \,|\, P(s) \,|\, f(x(\tau(s)))\, ds$$

$$1 \ge \frac{1}{(n-1)!\, x(\tau(\xi))} \int_\xi^t (t-s)^{n-1} \,|\, P(s) \,|\, f(x(\tau(s)))\, ds$$

$$\ge \frac{1}{(n-1)!} \int_{\tau^{-1}(t)}^{t} (t-s)^{n-1} \,|\, P(s) \,|\, \frac{f(x(\tau(s)))}{x(\tau(s))}\, ds$$

$$\ge \frac{M}{(n-1)!} \int_\xi^t (t-s)^{n-1} \,|\, P(s) \,|\, ds,$$

This contradicts (2.10).

Let n be odd, by Lemma 1 $x(t)$ must be of degree n, the prove is similar to the case when n is even.

3. Remarks and Examples :

In this section we give some remarks and examples to illustrate the obtained results given in section 2

Remark1. If we use the condition $\frac{f(u)}{u} \ge M > 0$ then the nondecreasing property of $f(u)$ needed not be necessary as we can see in Theorem 4.

Remark2. We can use the condition (2.10) with $\frac{f(u)}{u} \ge M > 0$ to excluded the non-decreasing property of $f(u)$ in Theorem 1-Theorem 3.

Remark3. The conclusion of Theorem 4 remains true if we replace (2,10) by the condition

$$\liminf_{t\to\infty} \int_t^{\tau(t)} \frac{(\tau(s)-h(s))^{n-1}}{(n-1)!} |q(s)| ds > \frac{1}{e}$$

where h(t) is continuous function such that $\tau(t) \ge h(t) \ge t$.

Example1. Consider the delay differential equation:

$$x'''(t) + \frac{6}{(2t^2-1)^2} f(x(\tau(t))) = 0, \qquad t \ge 1$$

(*E.1*) with

$$p(t) = \frac{6}{(2t^2-1)^2}, \quad f(x(\tau(t))) = 1 - \frac{8t^2}{2t^2-1},$$

satisfies H1,H2 and all the conditions of Theorem 1, so all the solutions of

equation (*E.1*) are either oscillatory or tends to zero as $t \to \infty$ for instance $x(t) = \ln(\frac{2t-1}{2t+1})$ is such solution.

Example 2. Consider the delay differential equation:

$$x''(t) + (a - \sin t) f(x(\tau(t))) = 0, t \ge t_0$$

E.2

with

$$a \ge 1, \quad P(t) = 1 - \sin t, \quad f(x(\tau(t))) = -e^{-t}$$

it is easy to see that all conditions of Theorem 1 or Theorem 3 are hold so all solutions of equation (*E.2*) are either oscillatory or $\lim_{t\to\infty} x(t) = 0$, for instance $x(t) = -e^{-t}(a + \frac{\sin t - \cos t}{4})$ is such oscillatory solution of (*E.2*).

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تذبذب المعادلات التفاضلية غير الخطية ذات المعاملات التقدمية

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الخلاصة:

في هذا البحث قمنا باستخراج شروط كافية وضرورية لحلول المعادلات التفاضلية التباطئية ذات الرتب n من النوع $n \geq 2$, $x^{(n)}(t) + P(t)\, f(x(\tau(t))) = 0,$.

كذلك اعطيت بعض الشروط الكافية للحلول غير المتذبذبة كي تكون متقاربة الى الصفر. كما واعطيت في البحث بعض الامثلة لتوضيح النتائج المستخرجة.