Some Results on Reduced Rings

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Abstract:

The main purpose of this paper is to study some results concerning reduced ring with another concepts as semiprime ring ,prime ring,essential ideal ,derivations and homomorphism ,we give some results a bout that.

Key words: Reduced Ring ,Semiprime Ring ,Prime Ring ,Essential Ideal ,Derivations and Homomorphism.

1- Introduction

Many authors study behaviour and interrelations between various types of rings with reduced ring, they gave many results a bout that .By using the concept of annihilator and reduced ring, Fraser and Nicholson [1] showed that a ring R is a reduced p.p.-ring if and only if is a(left and right**)** p.p.-ring in which every idempotent is central ,where R is left p.p-ring ,in brevity ,an 1.p.p-ring, if every principal left ideal of R ,regarded as a left R –module ,is projective. Dually ,we may define the right p.p.-rings (r.p.p.-rings),we call a ring R a p.p.-ring if R is both on l.p.p.-ring and r.p.p-ring.Reduced rings with the maximum condition on annihilator were first studied by Cornish and Stewart [2] ,Xiaojiang and Shum[3] proved that ,let R be a p.p.-ring and E(R) the set of all idempotents of R ,then R is reduced ,Muhittin and Nazim [4] proved that let R be a reduced ring,then R is right nonsingular ,we say R is a right nonsingular ring if $Z(R_R)$=o ,let M be a right module over a ring R , an element m$\in$M is said to be a singular element of M if the right ideal r_R(m)is essential in R_R ,the set of all singular elements of M is denote by Z(M),we say that M_R is a singular (resp. nonsingular)module if Z(M)=M (resp.Z(M)=o)Kosan [5] proved that ,let R[x] be a right IN-ring,then R is a right IN- ring in R is a reduced ring ,where a ring R is called a right Ikeda – Nakayama (for short IN-ring) if the left annihilator of the intersection of any two right ideals is the sum of the left annihilators .The objective of this paper is to study behaviour of reduced ring with types of noncommutative ring (prime and semiprime ring) and additive mapping (a derivations)we give some results a bout that.

2- Preliminaries

Throughout R will represent an associative ring with center Z(R),R is said reduced if there are no nilpotent elements not equal to zero , is said to be prime if x Ry=o for x,y $\in$R implies x=o or y=o ,and semiprime if xRx=o with x$\in$R implies x=o . An ideal I is said to essential ideal if whenever I∩J=o, where J is an ideal of R implies that J=o ,and nilpotent proveded that I^n=o,n is nilpotencey index of I a positive integer. If I is a non-empty subset of R ,then the centralizer of I in R ,denoted by C_R(I),is defined by: C_R(I)={a$\in$R/ax=xa for all x$\in$I} .If a$\in C_R$(I) we say that a centralizes I. An a dditive mapping d:R→R is called a derivation if d(xy)=d(x)y+xd(y)holds for all x,y$\in$R,and an inner derivation if there exists a$\in$R such that d(x) =[a,x] for all x$\in$R.Also is called skew-centralizing on subset I of R (resp. skew-commuting on subset I of R)if d(x)x+xd(x) $\in$Z(R)holds for all

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x∈I(resp.d(x)x+xd(x)=o holds for all x∈I),and d acts as a homomorphism on I if d(xy)=d(x)d(y)holds for all x,y ∈I.

We write [x,y] for xy-yx and make extensive use of basic commutator identities [xy,z]=x[y,z]+[x,z]y and [x,yz]=y[x,z]+[x,y]z .

To a chieve our purposes ,we mention the following results .

Proposition1[6]

The following properties of a reduced ring R are equivalent

(a) xR∩yR=o implies xy=o for all x,y∈R.

(b) I∩J=o implies I J=o for all right ideals I,J of R.

(c)Qm is strongly regular ,Qm is the maximal ring of quotiets.

Proposition 2 [7 :Proposition 1.1.6]

The following statement are equivalent ,for any ring R

(a) R is prime .

(b) If I and J are ideals of R and I J =o ,then either I=o or J=o.

Lemma 3

Every reduced ring R has no divisors of zero.

Proof: Let x∈R ,suppose that x^n=o for all x∈R. for some n positive integer.

Since R is reduced ring ,then x=o .Replacing x by xy in the relation x^n=o ,we obtain $(xy)^n$=o for all x,y∈R.Since R is reduced ring then xy=o for all x,y ∈R.But x=o ,therefore,R has no zero divisore.

Remark 4[6]

If Qm is a strongly regular ring ,then R must be reduced.

3-Reduced Ring as Prime or Semiprime Rings .

In this section we will discuss interrelations between prime and semiprime rings with reduced rings.

Theorem 3.1

Let R be a reduced ring .If R satisfies one of the following conditions

(i) xR∩yR=o for all a,b∈R.

(ii)I∩J=o for all a non-zero right ideals I,J of R.Then Ris semiprime ring.

Proof: (i)For all x,y∈R,we have xR∩yR=o. Since R is reduced ring ,then by Proposition 1(a),we obtain xy=o.By Lemma 3,we get either x=o or y=o.Replacing y by rx, we get xrx=o for all x,r∈R.

Then ,we have xRx=o ,with x∈ R,x=o, therefore, R is semiprime ring.Similarly for y.

(ii) For all a non-zero right ideals I,J of R ,we have I∩J=o .Since R is reduced ring ,by Proposition 1(b) ,we obtain IJ=o.

Let x∈I and y∈J,then xy=o for all x∈I. By same method in part (i) we get ,R is semiprime ring.

Theorem3.2

Let R be a ring and I is a right nilpotent ideal with nilpotency index 2 and J is a right ideal of R such that I ∩J=o. Then R is reduced ring.

Proof: We have I∩J=o ,let x∈I and I∩J implies x∈J,we obtain I⊂J.Similarly ,when y∈J and I∩J,we obtain J⊂I,then I=J. Left-multiplying this relation by I,we get

Thus ,we have

I∩J =IJ=o .Since I∩J =M,M is ideal of R.

Let x∈I and y∈J ,then xy=o for all x∈I,y∈J.But xy∈M,then xy=o=z for all z∈M.Now,it is easy we obtain R is semiprime ring.

Theorem 3.3

Let R be a ring and I is a right essential ideal of R ,then R is reduced ring.

Prrof: We have I is a right essential ideal of R ,then there exists J is a right ideal of R such that I ∩J=o implies J=o .Left- multiplying the relation J=o by I

,we obtain I J =o .Then by Proposition 1(b and c) and Remark 4,R is reduced ring.

Corollary3.4

Let R be a reduced ring and I is a right essential ideal of R ,then R is prime ring.

Proof: Since I is a right essential ideal of R ,then there exists J is a right ideal of R such that I∩J=o, by Proposition 1(b) ,we get IJ=o . But I is right essential ideal ,then we obtain J=o.By Proposition 2(a and b) ,we get R is prime ring.

By same method in Corollary 3.4 and Theorem 3.1(ii),we can prove the following theorem .

Theorem 3.5

Let R be a reduced ring and I is a right essential ideal of R ,then R is semiprime ring.

Theorem3.6

Let R be a prime ring , I and J are a right ideals of R such that IJ=o. Then R is reduced ring .

Proof: We have IJ=o for all right ideals of R .

By Proposition 2(b) ,we get,either I=o or J=o .

When I=o ,by intersection J with I ,we obtain I∩J =o for all right ideals. And we have IJ=o .By Proposition 1(b and c)and Remark 4 , we obtain,R is reduced ring. Similary when J=o.

Proposition 3.7

If R is reduced ring and I is one-sided ideal of R ,x is nilpotent element of R ,$x \in I$ then R is semiprime ring.

Proof: Since x is nilpotent element and $x \in I$ i.e.

x^n =o (for some n positive integer). Since R is reduced ring ,then x=o. Left –multplying by xr ,we obtain xRx=o ,with $x \in R$.It is easy we obtain ,R is semiprime ring.

4-Reduced Ring with Derivations.

In this section we obtain necessary and sufficient conditions for a derivation d of reduced ring with char.R≠2 to become skew-centralizing and skew-commuting .Also we study effect of d acts as skew –centralizing and skew-commutiong on subset of reduced rings with char.R≠ 2.

Theorem4.1

Let R be a reduced ring with char .R≠2, and I is a subset of R .

If R admits a derivation d to satisfy

(a)d acts as a skew-commuting on I ,then d(I)=o.

(b)d acts as a skew-centralizing on I ,then d(I)centralizes I.

Proof: (a) Since d is skew-commuting ,then

$d(x)x+xd(x)=o$ for all $x \in I$ (1)

Left –multiplying (1) by x,we obtain

$xd(x)x+x^2d(x)=o$ for all $x \in I$. (2)

From (1) ,we get $d(x^2)=o$ for all $x \in$ (3)

In (3) replacing x by x+y,we obtain

$d(x^2)+d(xy)+d(yx)+d(y^2)=o$ for all $x,y \in I$.

A ccording to (3) ,a bove equation become

$d(xy)+d(yx)=o$ for all $x,y \in I$.Then

$d(x)y+xd(y)+d(y)x+yd(x)=o$ for all $x,y \in I$.

Replacing y by x^2 and according to(3),

gives

$d(x)x^2+x^2d(x)=o$ for all $x \in I$ (4)

Then

$x^2d(x)=-d(x)x^2$ for all $x \in I$ (5)

By substituting (4) in (2) ,we get

$xd(x)x-d(x)x^2=o$ for all $x \in I$ .Then

$[x,d(x)]x=o$ for all $x \in I$.Then by Lemma 3,we obtain either x=o

or $[x,d(x)]=o$ for all $x \in I$.

If x=o for all $x \in I$ .Then d(x)=o for all $x \in I$, i.e. d(I)=o.

We have ,when $[x,d(x)]=o$ for all $x \in I$.Then

$d(x)x-xd(x)=o$ for all $x \in I$. (6)

From (6) and(1) ,we obtain

$2d(x)x=o$ for all $x\in I$.Since char .R$\neq$2,then
$d(x)x=o$ for all $x\in I$ .Then by Lemma3,we get
$d(x)=o$ for all $x\in I$.Then $d(I)=o$.
(b) We will now discuss, when d acts as a skew- centralizing on I.Then we have $d(x)x+xd(x)\in Z(R)$ for all $x\in I$.
$d(x^2)\in Z(R)$for all $x\in I$. i.e.
$[d(x^2),r]=o$ for all $x\in I,r\in R$. (7)
Also ,by replacing r by x in(7) ,we obtain
$(d(x))x+xd(x))x=x(d(x)x+xd(x))$ for all $x\in I$. (8)
Then $d(x)x^2+xd(x)x= xd(x)x+x^2d(x)$ for all $x\in I$ .Then
$d(x)x^2-x^2 d(x)=o$ for all $x\in I$.Then
$[d(x),x^2]=o$ for all $x\in I$. (9)
In(7) ,replacing x by x+y ,we obtain
$[d(x^2)+d(xy)+d(ys)+d(y^2),r]=o$ for all $x,y\in I$,$r\in R$.
According to(7),we obtain
$[d(x)y+xd(y)+d(y)x+yd(x),r]=o$ for all $x,y\in I,r\in R$.
Replacing y by x^2,we obtain
$[d(x)x^2+xd(x^2)+d(x^2)x+x^2d(x),r]=o$ for all $x\in I,r\in R$.
According to(7) and (9) ,we get
$[x^2d(x)+xd(x^2)+xd(x^2)+x^2d(x),r]=o$ for all$x\in I,r\in R$. Then
$2[x^2d(x)+xd(x^2),r]=o$ for all $x\in I,r\in R$.
Since char. R$\neq$2,then we obtain
$[x^2d(x)+xd(x^2),r]=o$ for all $x\in I,r\in R$.Then
$[x(xd(x)+d(x^2)),r]=o$ for all $x\in I$,$r\in R$.Then
$x[xd(x)+d(x^2),r]+[x,r](xd(x)+d(x^2))=o$ for all $x\in I$,$r\in R$.
According to (7) ,a bove equation become
$x[xd(x),r]+[x,r](xd(x)+d(x^2))=o$ for all $x\in I,r\in R$. Replacing r by x ,we obtain
$x[xd(x),x]=o$ for all $x\in I$.Then
$x^2[d(x),x]=o$ for all $x\in I$ .Then by Lemma3,we get either $x^2=o$ or
$[d(x),x]=o$ for all $x\in I$.
If $x^2=o$. Since R is reduced ring ,then $x=o$, for all $x\in I$.
Right multiplying by d(x) ,we obtain
$xd(x)=o$ for all $x\in I$. (10)
Again left – multiplying by d(x) ,we obtain
$d(x)x=o$ for all $x\in I$ (11)
Subtracting (10) and(11) ,we get
$[d(x),x]=o$ for all $x\in I$.Thus ,d(I)centralizes of I .We get same result when $[d(x),x]=o$ for all $x\in I$.

Theorem 4.2

Let R be a reduced ring with char. R$\neq$2 and I a subset of R, then aderivation d is skew-centralizing and skew-commuting on I.If R admits d to satisfy
(a) d acts as a homomorphism on I.
(b)d acts as an anti-homomorphism on I.

Proof:(a) d acts as a homomorphism on I.We have d is a derivation ,then
$d(xy)=d(x)y+xd(y)$ for all $x,y\in I$.Then
$[d(xy),r]=[d(x)y,r]+[xd(y),r]$ for all $x,y\in I$,$r\in R$.Since d acts as a homomorphism ,then
$[d(x)d(y),r]=[d(x)y,r]+[xd(y),r]$ for all $x,y\in I,r\in R$.Replacing r by d(y),we obtain
$[d(x),d(y)]d(y)=[d(x)y,d(y)]+[xd(y),d(y)]$ for all $x,y\in I$ .Then
$[d(x),d(y)]d(y)=d(x)[y,d(y)]+[d(x),d(y)]y+[x,d(y)]d(y)$ for all $x,y\in I$.
Replacing y by x,we obtain
$d(x)[x,d(x)]+[x,d(x)]d(x)=o$ for all $x\in I$.
Then $[d(x)^2,x]=o$ for all $x\in I$. Then $d(x)^2\in Z(R)$ for all $x\in I$.Since d acts as a homomorphism ,then $d(x^2)\in Z(R)$ for all $x\in I$,i.e. $d(x)x+xd(x)\in Z(R)$ for all $x\in I$,then d is skew- centralizing on I.
We will ,now discuss when d is skew-commuting on I.
By same method in first part ,we obtain $d(x)^2\in Z(R)$ for all $x\in I$.
Also ,we have $d(xy)=d(x)y+xd(y)$ for all $x,y\in I$.Replacing x by x^2,we obtain
$d(x^2y)=d(x^2)y+x^2d(y)$for all $x,y\in I$.
Since d acts as a homomorphism ,then
$d(x^2)d(y)=d(x^2)y+x^2d(y)$ for all $x,y\in I$. Then

$d(x^2)(d(y)-y) = x^2 d(y)$ for all $x,y \in I$. (12)

Then $[d(x^2)(d(y)-y),r]=[x^2 d(y),r]$ for all $x,y \in I, r \in R$.

$d(x^2)[d(y)-y,r]+[d(x^2),r](d(y)-y)-x^2[d(y),r]-[x^2,r]d(y)=o$

for all $x,y \in I, r \in R$ .Since d acts as a homomorphism and $d(x)^2 \in Z(R)$for all $x \in I$, above equation become

$d(x)^2[d(y)-y,r]-x^2[d(y),r]-[x^2,r]d(y)=o$

for all $x,y \in I, r \in R$. Replacing r by y ,we obtain

$d(x)^2[d(y)-y,y]-x^2[d(y),y]-[x^2,y]d(y)=o$

for all $x,y \in I$. Then

$d(x)^2[d(y),y]-x^2[d(y),y]-[x^2,y]d(y)=o$

for all $x,y \in I$.Then

$(d(x)^2-x^2)[d(y),y]-[x^2,y]d(y)=o$ for all $x,y \in I$. Replacing y by x,we obtain $(d(x)^2-x^2)[d(x),x]=o$ for all $x \in I$. Since R is reduced ring ,then by Lemma3,we obtain either $d(x)^2-x^2=o$ or $[d(x),x]=o$ for all $x \in I$.We have when $d(x)^2-x^2=o$ for all $x \in I$.Then $d(x)^2=x^2$ for all $x \in I$. Substituting the relation $d(x)^2=x^2$, in (12) ,we obtain

$d(x)^2(d(y)-y)=d(x)^2 d(y)$ for all $x,y \in I$ .Thus $- d(x)^2 y=o$ for all $x,y \in I$.

Since R is reduced ring then by Lemma3,we have either $d(x)^2=o$ for all $x \in I$ or $y=o$ for all $y \in I$. If $d(x)^2=o$ for all $x \in I$.

Since d acts as a homomorphism ,then $d(x^2)=o$ for all $x \in I$ .Then

$d(x)x + xd(x)=o$ for all $x \in I$.Then d is skew –commuting on I.

If $y=o$ for all $y \in I$. Replacing y by x,we obtain $x=o$ for all $x \in I$.

Left –multiplying by d(x) ,we obtain

$d(x)x=o$ for all $x \in I$. (13)

Again right – multiplying by d(x) ,we obtain

$xd(x)=o$ for all $x \in I$. (14)

From (13) and (14) ,we get $d(x)x+xd(x)=o$ for all $x \in I$ .Then

d is skew –commuting on I .We will ,now discuss when $[d(x),x]=o$ for all $x \in I$.Since d is a derivation, we have

$d(x^2)=d(x)x+xd(x)$ for all $x \in I$. (15)

Also ,we have

$d(x)x-xd(x)=o$ for all $x \in I$. (16)

From (15) and (16),we obtain $d(x^2)=2d(x)x$ for all $x \in I$.Since d acts as a homomorphism ,we have $d(x)^2 = 2d(x)x$ for all $x \in I$.Then

$d(x)(d(x)-2x)=$for all $x \in I$.Since R is reduced ring ,then by Lemma3,we obtain either $d(x)=o$ or $d(x)-2x=o$ for all $x \in I$.

If $d(x)=o$ for all $x \in I$.Then it is easy to get d is skew –commuting on I.

When $d(x)=2x$ for all $x \in I$. (17)

Substiuting (17) in (15) ,we obtain

$d(x^2)=4x^2$ for all $x \in I$. (18)

Substittuting (18) in (12) ,we obtain

$4d(x^2)(d(y)-y)=d(x^2)d(y)$ for all $x,y \in I$. Then

$4d(x^2)d(y)-4d(x^2)y-d(x^2)d(y)=o$ for all $x,y \in I$.

Then since d acts as a homomorphism ,we obtain

$4d(x^2y) -4d(x^2)y-d(x^2y)=o$ for all $x,y \in I$.Then

$3d(x^2y)-4d(x^2)y=o$ for all $x,y \in I$. Replacing y by x^2 ,we obtain

$3d(x^4)-4d(x^2)x^2=o$ for all $x \in I$. According to(18) ,we obtain $3d(x^4)-d(x^2)d(x^2)=o$ for all $x \in I$.Since d is acts as a homomorphism , we obtain

$3d(x^4)-d(x^4)=o$ for all $x \in I$. Then

$2d(x^4)=o$ for all $x \in I$.Since char $.R \neq 2$,then

$d(x^4)=o$ for all $x \in I$.Since d is acts as a homomorphism ,we obtain $d(x^2)^2=o$ for all $x \in I$. Since R is reduced ring ,then

$d(x^2)=o$ for all $x \in I$.Then

$d(x)x+xd(x)=o$ for all $x \in I$ .Thus, we obtain d is skew-commuting on I.The proof of (b) is similar .

Then by Theorem 4.1 and Theorem 4.2 ,we obtain the following corollary.

Corollary 4.3

Let R be a reduced ring with char $.R \neq 2$ and I a subset of R then a derivation $d(I)=o$ and d(I) centralizes I .If R admits d to satisfy

(a) d acts as a homomorphism on I.

(b) d acts as an anti- homomorphism on I.

Remark 4.4

In Theorem 4.1 and Theorem 4.2 ,we can not exclude the condition char.R≠2,as it is shown in the following example.

Example 4.5

Let R be the ring of all 2×2matrices over a field F with char.R=2,let a=$\begin{pmatrix} o & o \\ o & 1 \end{pmatrix}$,I ={$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$}/a,b∈F}be a subset of R.

Let d be the inner derivation given by:

d(x) =x$\begin{pmatrix} o & o \\ o & 1 \end{pmatrix}$-$\begin{pmatrix} o & o \\ o & 1 \end{pmatrix}$x ,when x∈I

,then x=$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$therefore

,d(x)=$\begin{pmatrix} o & b \\ -b & o \end{pmatrix}$. Then

d(x)

x+xd(x)=$\begin{pmatrix} o & b \\ -b & o \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}$+$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} o & b \\ -b & o \end{pmatrix}$

=$\begin{pmatrix} b^2 & ba \\ -ba & -b^2 \end{pmatrix}$+$\begin{pmatrix} -b^2 & ab \\ -ba & b^2 \end{pmatrix}$

=$\begin{pmatrix} o & 2ba \\ -2ba & o \end{pmatrix}$.Since char. R=2,then

=$\begin{pmatrix} o & o \\ o & o \end{pmatrix}$.Then d is skew- centralizing and skew- commuting on I ,i.e.d(I)=o and d(I) centralizes I.

Also when we have d acts as homomorphism (resp. acts as an anti- homomorphism).

d(x)d(x) =d$\begin{pmatrix} a & b \\ b & a \end{pmatrix}d\begin{pmatrix} a & b \\ b & a \end{pmatrix}$

=d($\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}$)

=d$\begin{pmatrix} a^2+b^2 & ab+ba \\ ba+ab & b^2+a^2 \end{pmatrix}$

=$\begin{pmatrix} o & 2ab \\ -2ab & o \end{pmatrix}$.Since char .R=2,we obtain

=$\begin{pmatrix} o & o \\ o & o \end{pmatrix}$.Thus d is skew –cenralizing and skew- commuting on I.

References:

1. Fraser ,J.A.and Nicholson. W.K. 1989.Reduced p.p.- rings, Mathematical Japonica 34.(5):715-725.
2. Cornish ,W.H.and Stewart .P.N.1974.Rings with no nilpotent elements and with the maximum condition on annihilators .Canadian Mathematical Bulletin 17:35-38.
3. Xiaojiang ,G.and Shum.K.P.2006,On p.p.-ring which are reduced .International Journal of Mathematics and Mathematical Sciences.2006.Article ID34694;1-5.
4. Muhittin,B.and Nazim.A.2006.On reduced and semicomm- utative modules .Turk.J.Math.30:285-291.
5. Kosan,M.T.2006.The armendariz module and its application to the ikeda-nakayama module .International Journal of Mathematics and Mathematical Sciences.2006.Article ID 35238:1-7.
6. Stenstrom ,B.1971.Rings and modules of quotients,Lecture Notes in Mathematics 237.Springer-Verlag.NewYork.
7. Herstein,I.N.1968.Non commutative rings .Univ.of Chicago, the Mathematical Association of America.

بعض النتائج حول الحلقات الاختزالية

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الملخص :

الغرض الرئيسي من هذا البحث هودراسة بعض النتائج بخصوص الحلقات الاختزالية مع مفاهيم اخرى كالحلقات شبه الاولية والحلقات الاولية والمثالي الاساسي والاشتقاقات والتشاكلات وقد اعطينا بعض النتائج حول ذلك.