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On (J - δ) Semi Homogeneous Systems of Differential Equations

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RESEARCH ARTICLE

On $(J - \delta)$ Semi Homogeneous Systems of Differential Equations

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ABSTRACT

The concept of homogeneity in differential equations can be generalized to systems of differential equations as shown in this work. The classification of three-dimensional differential equation systems is presented based on the definition of the Jacobean matrix and its determinant, where two systems of the homogeneous system are defined, called the J-semi-homogeneous system and the other δ-semi-homogeneous system, where the first definition is based on the Jacobian matrix, while the second is based on the determinant of the Jacobite matrix. Examples are given for both definitions, and the relationship between the two definitions will be studied. In addition to finding an equivalent for these two definitions, some results for these two definitions have also been proven.

Keywords: δ-semi-homogeneous system, Jacobean matrix, J-semi-homogeneous system, Semi-homogenous, System of differential equations

Introduction

Numerous branches of mathematics and science, such as robotics,[1,2] differential equations, and optimization, depend heavily on the Jacobian matrix. When assessing stability,[3] gradients, and transformations in various mathematical models and algorithms, give crucial information on the local behavior of functions.[4]

The Jacobian matrix is a matrix made up of a multivariable function's first-order partial derivatives[5] and helps us convert one coordinate system into another. The Jacobian matrix is employed when there are several variables or functions. For determining the derivatives of implicit or composite functions. The derivative at J for a row vector of a function $f_i : \mathcal{R}^3 \to \mathcal{R}, (i = 1, 2, 3)$ is defined as: $\frac{\partial(f_1, f_2, \ldots f_n)}{\partial(x_1, x_2, \ldots x_n)}$. Consequently, the following function's Jacobian matrix[6–8] is represented as follows:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{1}$$

In this paper, we discussed using the Jacobian matrix to classify homogeneous systems of differential equations. Many researchers focus their research on different equations[9] while others are concerned with studying difference equations. Al-Asadi and Huda defined A system of first- and higher-order semi-homogeneous difference equations, 2021. A self-semi-homogeneous system of difference equations was introduced and generalized by Al-Asadi and Abed,[10] and was defined as follows P-self-semi-homogenous. Also, new definitions

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will be introduced for J-(δ) semi-homogenous systems, which are defined as follows: If there is a nonzero real matrix V such that the following equations hold:

$$F(VX) = J(Q)VF(X), \; F(VX) = det\; J(Q)VF(X) \tag{2}$$

then a homogeneous system of differential equations is said to be J-(δ) semi-homogenous.

The concept of a J-adjoint (δ-adjoint)-semi-homogenous system of differential equations, is defined as follows: a homogenous system is called J-adjoint (δ-adjoint)-semi-homogeneous if there are two non-zero matrices V and C such that the following equations hold:

$$F(VX) = J(Q)CF(X), \; F(VX) = det\; J(Q)CF(X) \tag{3}$$

and some definitions are given with examples, while certain theorems are also proved.

This work studied types of J-semi-homogenous.

Consider the system.

$$F(X) = QX \tag{4}$$

where $F = \begin{pmatrix} f \\ g \\ h \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, F(X) = \begin{pmatrix} f \\ g \\ h \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix}$ and $Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$

Definition 1: It is referred to as a J-semi-homogeneous if there is a non-zero matrix V such that system (4) satisfies the following equation.

$$F(VX) = J(Q)^k V\, F(X).$$

Definition 2: If there is a non-zero matrix V such that system (4) satisfies the following equation, then it is referred to as a δ-semi-homogeneous.

$$F(VX) = det\; (J(Q))^k V\, F(X).$$

Where J is a Jacobian matrix and det is determined.

Remark: The next example shows that the Definitions 1 and 2 are independent for a matrix V, that is the J-semi-homogeneous may be not δ-semi-homogenous with the same matrix V, and vice versa, the following examples explain that:

Example 1: *Consider the system of differential equations*

$$f(x) = x + y + z$$
$$g(y) = x - y$$
$$h(z) = x + y - z$$

then its δ-semi-homogenous but not J-semi-homogeneous, to show that:

$$V = \begin{bmatrix} \frac{56}{299} & \frac{-27}{299} & \frac{-15}{299} \\ \frac{9}{205} & \frac{28}{205} & \frac{12}{205} \\ \frac{21}{205} & \frac{-3}{205} & \frac{28}{205} \end{bmatrix}$$

satisfies the following equation:

$$F(VX) = det\ J(Q)\,VF(X)$$

$$F\begin{bmatrix} \frac{56}{299} & \frac{-27}{299} & \frac{-15}{299} \\ \frac{9}{205} & \frac{28}{205} & \frac{12}{205} \\ \frac{21}{205} & \frac{-3}{205} & \frac{28}{205} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 4\begin{bmatrix} \frac{56}{299} & \frac{-27}{299} & \frac{-15}{299} \\ \frac{9}{205} & \frac{28}{205} & \frac{12}{205} \\ \frac{21}{205} & \frac{-3}{205} & \frac{28}{205} \end{bmatrix}\begin{bmatrix} x+y+z \\ x-y \\ x+y-z \end{bmatrix}$$

$$\begin{bmatrix} \frac{56}{299}x + \frac{272}{299}y + \frac{284}{299}z \\ \frac{196}{205}x - \frac{28}{205}y - \frac{12}{205}z \\ \frac{184}{205}x + \frac{208}{205}y - \frac{28}{205}z \end{bmatrix} = \begin{bmatrix} \frac{56}{299}x + \frac{272}{299}y + \frac{284}{299}z \\ \frac{196}{205}x - \frac{28}{205}y - \frac{12}{205}z \\ \frac{184}{205}x + \frac{208}{205}y - \frac{28}{205}z \end{bmatrix}$$

Therefore δ-semi-homogenous
When $J(Q) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

$$F(V(c)X) = J(Q)VF(X)$$

$$F\begin{bmatrix} \frac{56}{299} & \frac{-27}{299} & \frac{-15}{299} \\ \frac{9}{205} & \frac{28}{205} & \frac{12}{205} \\ \frac{21}{205} & \frac{-3}{205} & \frac{28}{205} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{56}{299} & \frac{-27}{299} & \frac{-15}{299} \\ \frac{9}{205} & \frac{28}{205} & \frac{12}{205} \\ \frac{21}{205} & \frac{-3}{205} & \frac{28}{205} \end{bmatrix}\begin{bmatrix} x+y+z \\ x-y \\ x+y-z \end{bmatrix}$$

$$\begin{bmatrix} \frac{56}{299}x + \frac{272}{299}y + \frac{284}{299}z \\ \frac{196}{205}x - \frac{28}{205}y - \frac{12}{205}z \\ \frac{184}{205}x + \frac{208}{205}y - \frac{28}{205}z \end{bmatrix} \neq \begin{bmatrix} \frac{6255}{12259}x + \frac{5479}{12259}y + \frac{2313}{12259}z \\ \frac{-11781}{61295}x + \frac{16033}{61295}y + \frac{15452}{61295}z \\ \frac{3767}{61295}x - \frac{3701}{61295}y + \frac{15751}{61295}z \end{bmatrix}$$

And it is not a J-semi-homogeneous.
Now we try to find an equivalent definition through the following theorem

Theorem 1: *The homogeneous system of differential Eq. (4) is J-semi homogeneous if and only if the matrix V equal to* $\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}$, *Where*

$$v_{11} = \frac{(q_{21}q_{11}v_{12} + q_{11}q_{31}v_{13} + q_{12}q_{11}v_{21} + q_{21}q_{12}v_{22} + q_{12}q_{31}v_{23} + q_{13}q_{11}v_{31} + q_{13}q_{21}v_{32} + q_{13}q_{31}v_{33})}{q_{11}(1-q_{11})}$$

$$v_{12} = \frac{q_{11}q_{12}v_{11} + q_{11}q_{32}v_{13} + q_{12}^2v_{21} + q_{12}q_{22}v_{22} + q_{12}q_{32}v_{23} + q_{13}q_{12}v_{31} + q_{13}q_{22}v_{32} + q_{13}q_{32}v_{33} - q_{12}}{q_{11} - q_{11}q_{22}}$$

$$v_{13} = \frac{q_{11}q_{13}v_{11} + q_{11}q_{23}v_{12} + q_{12}q_{13}v_{21} + q_{12}q_{23}v_{22} + q_{12}q_{33}v_{23} + q_{13}^2v_{31} + q_{13}q_{23}v_{32} + q_{13}q_{33}v_{33} - q_{13}}{q_{11} - q_{11}q_{33}}$$

$$v_{21} = \frac{q_{21}q_{11}v_{11} + q_{21}^2v_{12} + q_{21}q_{31}v_{13} + q_{22}q_{21}v_{22} + q_{22}q_{31}v_{23} + q_{23}q_{11}v_{31} + q_{23}q_{21}v_{32} + q_{23}q_{31}v_{33} - q_{21}}{q_{22}(1-q_{11})}$$

$$v_{22} = \frac{q_{21}q_{12}v_{11} + q_{21}q_{22}v_{12} + q_{21}q_{32}v_{13} + q_{12}q_{22}v_{21} + q_{22}q_{32}v_{23} + q_{23}q_{12}v_{31} + q_{23}q_{22}v_{32} + q_{23}q_{32}v_{33}}{q_{22} - q_{22}^2}$$

$$v_{23} = \frac{q_{21}q_{13}v_{11} + q_{21}q_{23}v_{12} + q_{21}q_{33}v_{13} + q_{22}q_{13}v_{21} + q_{22}q_{23}v_{22} + q_{23}q_{13}v_{31} + q_{23}^2v_{32} + q_{23}q_{33}v_{33} - q_{23}}{q_{22} - q_{22}q_{33}}$$

$$v_{31} = \frac{q_{31}q_{11}v_{11} + q_{31}q_{21}v_{12} + q_{31}^2v_{13} + q_{32}q_{11}v_{21} + q_{32}q_{21}v_{22} + q_{32}q_{31}v_{23} + q_{33}q_{21}v_{32} + q_{33}q_{31}v_{33} - q_{31}}{q_{33}(1-q_{11})}$$

$$v_{32} = \frac{q_{31}q_{12}v_{11} + q_{31}q_{22}v_{12} + q_{31}q_{32}v_{13} + q_{32}q_{12}v_{21} + q_{32}q_{22}v_{22} + q_{32}^2v_{23} + q_{33}q_{12}v_{31} + q_{33}q_{32}v_{33} - q_{32}}{q_{33}(1-q_{22})}$$

$$v_{33} = \frac{q_{31}q_{13}v_{11} + q_{31}q_{23}v_{12} + q_{31}q_{33}v_{13} + q_{32}q_{13}v_{21} + q_{32}q_{23}v_{22} + q_{32}q_{33}v_{23} + q_{33}q_{13}v_{31} + q_{33}q_{23}v_{32}}{q_{33}(1-q_{33})}$$

Since F is a homogeneous system, there exists a non-zero matrix (for example, let's say V) that allows Eq. (2) to be held.

$$F(VX) = J(Q)VF(X)$$

$$F\left(\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = J(Q)\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

$$\begin{pmatrix} f(v_{11}x + v_{12}y + v_{13}z) \\ g\left(v_{21}x + v_{22}y + v_{23}z\right) \\ h(v_{31}x + v_{32}y + v_{33}z) \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}\begin{pmatrix} v_{11}f(x) + v_{12}g(y) + v_{13}h(z) \\ v_{21}f(x) + v_{22}g(y) + v_{23}h(z) \\ v_{31}f(x) + v_{32}g(y) + v_{33}h(z) \end{pmatrix}$$

The left hand is

$$\begin{pmatrix} q_{11}v_{11}x + (q_{11}v_{12} + q_{12})\,y + (q_{11}v_{13} + q_{13})\,z \\ (q_{22}v_{21} + q_{21})\,x + q_{22}v_{22}y + (q_{22}v_{23} + q_{23})\,z \\ (q_{31} + q_{33}v_{31})\,x + (q_{32} + q_{33}v_{32})\,y + q_{33}v_{33}z \end{pmatrix}$$

Suppose that the result of the right hand can be denoted by $\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}$, *then we have*

$$\begin{aligned} H_1 = {} & q_{11}^2 v_{11}x + q_{11}v_{11}q_{12}y + q_{11}v_{11}q_{13}z + q_{11}v_{12}q_{21}x + q_{11}v_{12}q_{22}y + q_{11}q_{23}v_{12}z + q_{11}v_{13}q_{31}x + q_{11}v_{13}q_{32}y \\ & + q_{11}v_{13}q_{33}z + q_{12}v_{21}q_{11}x + q_{12}^2 v_{21}y + q_{12}v_{21}q_{13}z + q_{12}v_{22}q_{21}x + q_{12}v_{22}q_{22}\,y + q_{12}v_{22}q_{23}z + q_{12}v_{23}q_{31}x \\ & + q_{12}v_{23}q_{32}y + q_{12}v_{23}q_{33}z + q_{13}v_{31}q_{11}x + q_{13}v_{31}q_{12}y + q_{13}^2 v_{31}z + q_{13}v_{32}q_{21}x + q_{13}v_{32}q_{22}y + q_{13}v_{32}q_{23}z \\ & + q_{13}v_{33}q_{31}x + q_{13}v_{33}q_{22}y + q_{13}v_{33}q_{33}z \end{aligned}$$

$$\begin{aligned} H_2 = {} & q_{21}v_{11}q_{11}x + q_{12}v_{11}q_{21}y + q_{21}v_{11}q_{13}z + q_{21}^2 v_{12}x + q_{21}v_{12}q_{22}y + q_{21}q_{23}v_{12}z + q_{21}v_{13}q_{31}x + q_{21}v_{13}q_{32}y \\ & + q_{21}v_{13}q_{33}z + q_{22}v_{21}q_{11}x + q_{22}v_{21}q_{12}y + q_{22}v_{21}q_{13}z + q_{22}v_{22}q_{21}x + q_{22}^2 v_{22}y + q_{22}v_{22}q_{23}z + q_{22}v_{23}q_{31}x \\ & + q_{22}v_{23}q_{32}y + q_{22}v_{23}q_{33}z + q_{23}v_{31}q_{11}x + q_{23}v_{31}q_{12}y + q_{23}v_{31}q_{13}z + q_{23}v_{32}q_{21}x + q_{23}v_{32}q_{22}y + q_{23}^2 v_{32}z \\ & + q_{23}v_{33}q_{31}x + q_{23}v_{33}q_{22}y + q_{23}v_{33}q_{33}z \end{aligned}$$

$$\begin{aligned} H_3 = {} & q_{31}v_{11}q_{11}x + q_{12}v_{11}q_{31}y + q_{31}v_{11}q_{13}z + q_{21}v_{12}q_{21}x + q_{31}v_{12}q_{22}y + q_{31}q_{23}v_{12}z + q_{31}^2 v_{13}x + q_{31}v_{13}q_{32}y \\ & + q_{31}v_{13}q_{33}z + q_{32}v_{21}q_{11}x + q_{32}v_{21}q_{12}y + q_{32}v_{21}q_{13}z + q_{32}v_{22}q_{21}x + q_{32}v_{22}q_{22}y + q_{32}v_{22}q_{23}z + q_{32}v_{23}q_{31}x \\ & + q_{32}^2 v_{23}y + q_{32}v_{23}q_{33}z + q_{33}v_{31}q_{11}x + q_{33}v_{31}q_{12}y + q_{33}v_{31}q_{13}z + q_{33}v_{32}q_{21}x + q_{33}v_{32}q_{22}y + q_{33}v_{32}q_{23}z \\ & + q_{33}v_{33}q_{31}x + q_{33}v_{33}q_{32}y + q_{33}^2 v_{33}z \end{aligned}$$

$$q_{11}v_{11} = q_{11}^2 v_{11} + q_{21}q_{11}v_{12} + q_{11}q_{31}v_{13} + q_{12}q_{11}v_{21} + q_{21}q_{12}v_{22} + q_{12}q_{31}v_{23} + q_{13}q_{11}v_{31} + q_{13}q_{21}v_{32} + q_{13}q_{31}v_{33}$$

$$\begin{aligned} q_{11}v_{12} + q_{12} = {} & q_{11}q_{12}v_{11} + q_{11}q_{22}v_{12} + q_{11}q_{32}v_{13} + q_{12}^2 v_{21} + q_{12}q_{22}v_{22} + q_{12}q_{32}v_{23} + q_{13}q_{12}v_{31} + q_{13}q_{22}v_{32} \\ & + q_{13}q_{32}v_{33} \end{aligned}$$

$$\begin{aligned} q_{13} + q_{11}v_{13} = {} & q_{11}q_{13}v_{11} + q_{11}q_{23}v_{12} + q_{11}q_{33}v_{13} + q_{12}q_{13}v_{21} + q_{12}q_{23}v_{22} + q_{12}q_{33}v_{23} + q_{13}^2 v_{31} + q_{13}q_{23}v_{32} \\ & + q_{13}q_{33}v_{33} \end{aligned}$$

$$\begin{aligned} q_{21} + q_{22}v_{21} = {} & q_{21}q_{11}v_{11} + q_{21}^2 v_{12} + q_{21}q_{31}v_{13} + q_{22}q_{11}v_{21} + q_{22}q_{21}v_{22} + q_{22}q_{31}v_{23} + q_{23}q_{11}v_{31} + q_{23}q_{21}v_{32} \\ & + q_{23}q_{31}v_{33} \end{aligned}$$

$$q_{22}v_{22} = q_{21}q_{12}v_{11} + q_{21}q_{22}v_{12} + q_{21}q_{32}v_{13} + q_{12}q_{22}v_{21} + q_{22}^2 v_{22} + q_{22}q_{32}v_{23} + q_{23}q_{12}v_{31} + q_{23}q_{22}v_{32} + q_{23}q_{32}v_{33}$$

$$\begin{aligned} q_{22}v_{23} + q_{23} = {} & q_{21}q_{13}v_{11} + q_{21}q_{23}v_{12} + q_{21}q_{33}v_{13} + q_{22}q_{13}v_{21} + q_{22}q_{23}v_{22} + q_{22}q_{33}v_{23} + q_{23}q_{13}v_{31} + q_{23}^2 v_{32} \\ & + q_{23}q_{33}v_{33} \end{aligned}$$

$$\begin{aligned} q_{31} + q_{33}v_{31} = {} & q_{31}q_{11}v_{11} + q_{31}q_{21}v_{12} + q_{31}^2 v_{13} + q_{32}q_{11}v_{21} + q_{32}q_{21}v_{22} + q_{32}q_{31}v_{23} + q_{33}q_{11}v_{31} + q_{33}q_{21}v_{32} \\ & + q_{33}q_{31}v_{33} \end{aligned}$$

$$\begin{aligned} q_{33}v_{32} + q_{32} = {} & q_{31}q_{12}v_{11} + q_{31}q_{22}v_{12} + q_{31}q_{32}v_{13} + q_{32}q_{12}v_{21} + q_{32}q_{22}v_{22} + q_{32}^2 v_{23} + q_{33}q_{12}v_{31} + q_{33}q_{22}v_{32} \\ & + q_{33}q_{32}v_{33} \end{aligned}$$

$$\begin{aligned} q_{33}v_{33} = {} & q_{31}q_{13}v_{11} + q_{31}q_{23}v_{12} + q_{31}q_{33}v_{13} + q_{32}q_{13}v_{21} + q_{32}q_{23}v_{22} + q_{32}q_{33}v_{23} + q_{33}q_{13}v_{31} + q_{33}q_{23}v_{32} \\ & + q_{33}^2 v_{33} \end{aligned}$$

It is simple to show that matrix V equals:

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}$$

Conversely
Suppose that there is a matrix V equal to

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}$$

Such that

$$v_{11} = \frac{(q_{21}q_{11}v_{12} + q_{11}q_{31}v_{13} + q_{12}q_{11}v_{21} + q_{21}q_{12}v_{22} + q_{12}q_{31}v_{23} + q_{13}q_{11}v_{31} + q_{13}q_{21}v_{32} + q_{13}q_{31}v_{33})}{q_{11}(1-q_{11})}$$
$$v_{12} = \frac{q_{11}q_{12}v_{11} + q_{11}q_{32}v_{13} + q_{12}^2v_{21} + q_{12}q_{22}v_{22} + q_{12}q_{32}v_{23} + q_{13}q_{12}v_{31} + q_{13}q_{22}v_{32} + q_{13}q_{32}v_{33} - q_{12}}{q_{11} - q_{11}q_{22}}$$
$$v_{13} = \frac{q_{11}q_{13}v_{11} + q_{11}q_{23}v_{12} + q_{12}q_{13}v_{21} + q_{12}q_{23}v_{22} + q_{12}q_{33}v_{23} + q_{13}^2v_{31} + q_{13}q_{23}v_{32} + q_{13}q_{33}v_{33} - q_{13}}{q_{11} - q_{11}q_{33}}$$
$$v_{21} = \frac{q_{21}q_{11}v_{11} + q_{21}^2v_{12} + q_{21}q_{31}v_{13} + q_{22}q_{21}v_{22} + q_{22}q_{31}v_{23} + q_{23}q_{11}v_{31} + q_{23}q_{21}v_{32} + q_{23}q_{31}v_{33} - q_{21}}{q_{22}(1-q_{11})}$$
$$v_{22} = \frac{q_{21}q_{12}v_{11} + q_{21}q_{22}v_{12} + q_{21}q_{32}v_{13} + q_{12}q_{22}v_{21} + q_{22}q_{32}v_{23} + q_{23}q_{12}v_{31} + q_{23}q_{22}v_{32} + q_{23}q_{32}v_{33}}{q_{22} - q_{22}^2}$$
$$v_{23} = \frac{q_{21}q_{13}v_{11} + q_{21}q_{23}v_{12} + q_{21}q_{33}v_{13} + q_{22}q_{13}v_{21} + q_{22}q_{23}v_{22} + q_{23}q_{13}v_{31} + q_{23}^2v_{32} + q_{23}q_{33}v_{33} - q_{23}}{q_{22} - q_{22}q_{33}}$$
$$v_{31} = \frac{q_{31}q_{11}v_{11} + q_{31}q_{21}v_{12} + q_{31}^2v_{13} + q_{32}q_{11}v_{21} + q_{32}q_{21}v_{22} + q_{32}q_{31}v_{23} + q_{33}q_{21}v_{32} + q_{33}q_{31}v_{33} - q_{31}}{q_{33}(1-q_{11})}$$
$$v_{32} = \frac{q_{31}q_{12}v_{11} + q_{31}q_{22}v_{12} + q_{31}q_{32}v_{13} + q_{32}q_{12}v_{21} + q_{32}q_{22}v_{22} + q_{32}^2v_{23} + q_{33}q_{12}v_{31} + q_{33}q_{32}v_{33} - q_{32}}{q_{33}(1-q_{22})}$$
$$v_{33} = \frac{q_{31}q_{13}v_{11} + q_{31}q_{23}v_{12} + q_{31}q_{33}v_{13} + q_{32}q_{13}v_{21} + q_{32}q_{23}v_{22} + q_{32}q_{33}v_{23} + q_{33}q_{13}v_{31} + q_{33}q_{23}v_{32}}{q_{33}(1-q_{33})}$$

To show that the system (4) is J-semi-homogeneous, that is

$$F(VX) = J(Q)VF(X)$$

By putting the value of the matrix V in (2), we have

$$F\left(\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

Therefore,

$$q_{11}\frac{(q_{21}q_{11}v_{12} + q_{11}q_{31}v_{13} + q_{12}q_{11}v_{21} + q_{21}q_{12}v_{22} + q_{12}q_{31}v_{23} + q_{13}q_{11}v_{31} + q_{13}q_{21}v_{32} + q_{13}q_{31}v_{33})}{q_{11}(1-q_{11})}$$
$$= q_{11}^2\frac{(q_{21}q_{11}v_{12} + q_{11}q_{31}v_{13} + q_{12}q_{11}v_{21} + q_{21}q_{12}v_{22} + q_{12}q_{31}v_{23} + q_{13}q_{11}v_{31} + q_{13}q_{21}v_{32} + q_{13}q_{31}v_{33})}{q_{11}(1-q_{11})}$$
$$+ q_{21}q_{11}v_{12} + q_{11}q_{31}v_{13} + q_{12}q_{11}v_{21} + q_{21}q_{12}v_{22} + q_{12}q_{31}v_{23} + q_{13}q_{11}v_{31} + q_{13}q_{21}v_{32} + q_{13}q_{31}v_{33}$$
$$(q_{21}q_{11}v_{12} + q_{11}q_{31}v_{13} + q_{12}q_{11}v_{21} + q_{21}q_{12}v_{22} + q_{12}q_{31}v_{23} + q_{13}q_{11}v_{31} + q_{13}q_{21}v_{32} + q_{13}q_{31}v_{33})$$
$$= (q_{21}q_{11}v_{12} + q_{11}q_{31}v_{13} + q_{12}q_{11}v_{21} + q_{21}q_{12}v_{22} + q_{12}q_{31}v_{23} + q_{13}q_{11}v_{31} + q_{13}q_{21}v_{32} + q_{13}q_{31}v_{33})$$

and

$$q_{11}\frac{q_{11}q_{12}v_{11}+q_{11}q_{32}v_{13}+q_{12}^2v_{21}+q_{12}q_{22}v_{22}+q_{12}q_{32}v_{23}+q_{13}q_{12}v_{31}+q_{13}q_{22}v_{32}+q_{13}q_{32}v_{33}-q_{12}}{q_{11}-q_{11}q_{22}}+q_{12}$$
$$=q_{11}q_{12}v_{11}$$
$$+q_{11}q_{22}\frac{q_{11}q_{12}v_{11}+q_{11}q_{32}v_{13}+q_{12}^2v_{21}+q_{12}q_{22}v_{22}+q_{12}q_{32}v_{23}+q_{13}q_{12}v_{31}+q_{13}q_{22}v_{32}+q_{13}q_{32}v_{33}-q_{12}}{q_{11}\left(1-q_{22}\right)}$$
$$+q_{11}q_{32}v_{13}+q_{12}^2v_{21}+q_{12}q_{22}v_{22}+q_{12}q_{32}v_{23}+q_{13}q_{12}v_{31}+q_{13}q_{22}v_{32}+q_{13}q_{32}v_{33}$$
$$q_{11}q_{12}v_{11}+q_{11}q_{32}v_{13}+q_{12}^2v_{21}+q_{12}q_{22}v_{22}+q_{12}q_{32}v_{23}+q_{13}q_{12}v_{31}+q_{13}q_{22}v_{32}+q_{13}q_{32}v_{33}-q_{12}q_{22}$$
$$=q_{11}q_{12}v_{11}+q_{11}q_{32}v_{13}+q_{12}^2v_{21}+q_{12}q_{22}v_{22}+q_{12}q_{32}v_{23}+q_{13}q_{12}v_{31}+q_{13}q_{22}v_{32}+q_{13}q_{32}v_{33}-q_{12}q_{22}$$

and

$$q_{11}\frac{q_{11}q_{13}v_{11}+q_{11}q_{23}v_{12}+q_{12}q_{13}v_{21}+q_{12}q_{23}v_{22}+q_{12}q_{33}v_{23}+q_{13}^2v_{31}+q_{13}q_{23}v_{32}+q_{13}q_{33}v_{33}-q_{13}}{q_{11}-q_{11}q_{33}}+q_{13}$$
$$=q_{11}q_{13}v_{11}+q_{11}q_{23}v_{12}$$
$$+q_{11}q_{33}\frac{q_{11}q_{13}v_{11}+q_{11}q_{23}v_{12}+q_{12}q_{13}v_{21}+q_{12}q_{23}v_{22}+q_{12}q_{33}v_{23}+q_{13}^2v_{31}+q_{13}q_{23}v_{32}+q_{13}q_{33}v_{33}-q_{13}}{q_{11}-q_{11}q_{33}}$$
$$q_{11}q_{13}v_{11}+q_{11}q_{23}v_{12}+q_{12}q_{13}v_{21}+q_{12}q_{23}v_{22}+q_{12}q_{33}v_{23}+q_{13}^2v_{31}+q_{13}q_{23}v_{32}+q_{13}q_{33}v_{33}-q_{13}q_{33}$$
$$=q_{11}q_{13}v_{11}+q_{11}q_{23}v_{12}+q_{12}q_{13}v_{21}+q_{12}q_{23}v_{22}+q_{12}q_{33}v_{23}+q_{13}^2v_{31}+q_{13}q_{23}v_{32}+q_{13}q_{33}v_{33}-q_{13}q_{33}$$

And

$$q_{21}+q_{22}\frac{q_{21}q_{11}v_{11}+q_{21}^2v_{12}+q_{21}q_{31}v_{13}+q_{22}q_{21}v_{22}+q_{22}q_{31}v_{23}+q_{23}q_{11}v_{31}+q_{23}q_{21}v_{32}+q_{23}q_{31}v_{33}-q_{21}}{q_{22}\left(1-q_{11}\right)}$$
$$=q_{21}q_{11}v_{11}+q_{21}^2v_{12}+q_{21}q_{31}v_{13}$$
$$+q_{22}q_{11}\frac{q_{21}q_{11}v_{11}+q_{21}^2v_{12}+q_{21}q_{31}v_{13}+q_{22}q_{21}v_{22}+q_{22}q_{31}v_{23}+q_{23}q_{11}v_{31}+q_{23}q_{21}v_{32}+q_{23}q_{31}v_{33}-q_{21}}{q_{22}\left(1-q_{11}\right)}$$
$$+q_{22}q_{21}v_{22}+q_{22}q_{31}v_{23}+q_{23}q_{11}v_{31}+q_{23}q_{21}v_{32}+q_{23}q_{31}v_{33}$$
$$q_{21}q_{11}v_{11}+q_{21}^2v_{12}+q_{21}q_{31}v_{13}+q_{22}q_{21}v_{22}+q_{22}q_{31}v_{23}+q_{23}q_{11}v_{31}+q_{23}q_{21}v_{32}+q_{23}q_{31}v_{33}-q_{11}q_{21}$$
$$=q_{21}q_{11}v_{11}+q_{21}^2v_{12}+q_{21}q_{31}v_{13}+q_{22}q_{21}v_{22}+q_{22}q_{31}v_{23}+q_{23}q_{11}v_{31}+q_{23}q_{21}v_{32}+q_{23}q_{31}v_{33}-q_{11}q_{21}$$

and

$$q_{22}\frac{q_{21}q_{12}v_{11}+q_{21}q_{22}v_{12}+q_{21}q_{32}v_{13}+q_{12}q_{22}v_{21}+q_{22}q_{32}v_{23}+q_{23}q_{12}v_{31}+q_{23}q_{22}v_{32}+q_{23}q_{32}v_{33}}{q_{22}(1-q_{22})}=q_{21}q_{12}v_{11}$$
$$+q_{21}q_{22}v_{12}+q_{21}q_{32}v_{13}+q_{12}q_{22}v_{21}$$
$$+q_{22}^2\frac{q_{21}q_{12}v_{11}+q_{21}q_{22}v_{12}+q_{21}q_{32}v_{13}+q_{12}q_{22}v_{21}+q_{22}q_{32}v_{23}+q_{23}q_{12}v_{31}+q_{23}q_{22}v_{32}+q_{23}q_{32}v_{33}}{q_{22}(1-q_{22})}$$
$$+q_{22}q_{32}v_{23}+q_{23}q_{12}v_{31}+q_{23}q_{22}v_{32}+q_{23}q_{32}v_{33}$$
$$q_{21}q_{12}v_{11}+q_{21}q_{22}v_{12}+q_{21}q_{32}v_{13}+q_{12}q_{22}v_{21}+q_{22}q_{32}v_{23}+q_{23}q_{12}v_{31}+q_{23}q_{22}v_{32}+q_{23}q_{32}v_{33}$$
$$=q_{21}q_{12}v_{11}+q_{21}q_{22}v_{12}+q_{21}q_{32}v_{13}+q_{12}q_{22}v_{21}+q_{22}q_{32}v_{23}+q_{23}q_{12}v_{31}+q_{23}q_{22}v_{32}+q_{23}q_{32}v_{33}$$

and

$$q_{22}\frac{q_{21}q_{13}v_{11}+q_{21}q_{23}v_{12}+q_{21}q_{33}v_{13}+q_{22}q_{13}v_{21}+q_{22}q_{23}v_{22}+q_{23}q_{13}v_{31}+q_{23}^2v_{32}+q_{23}q_{33}v_{33}-q_{23}}{q_{22}-q_{22}q_{33}}$$
$$+q_{23}=q_{21}q_{13}v_{11}+q_{21}q_{23}v_{12}+q_{21}q_{33}v_{13}+q_{22}q_{13}v_{21}+q_{22}q_{23}v_{22}$$

$$+ q_{22}q_{33}\frac{q_{21}q_{13}v_{11} + q_{21}q_{23}v_{12} + q_{21}q_{33}v_{13} + q_{22}q_{13}v_{21} + q_{22}q_{23}v_{22} + q_{23}q_{13}v_{31} + q_{23}^2v_{32} + q_{23}q_{33}v_{33} - q_{23}}{q_{22} - q_{22}q_{33}}$$
$$+ q_{23}q_{13}v_{31} + q_{23}^2v_{32} + q_{23}q_{33}v_{33}$$
$$q_{21}q_{13}v_{11} + q_{21}q_{23}v_{12} + q_{21}q_{33}v_{13} + q_{22}q_{13}v_{21} + q_{22}q_{23}v_{22} + q_{23}q_{13}v_{31} + q_{23}^2v_{32} + q_{23}q_{33}v_{33} - q_{23}q_{33}$$
$$= q_{21}q_{13}v_{11} + q_{21}q_{23}v_{12} + q_{21}q_{33}v_{13} + q_{22}q_{13}v_{21} + q_{22}q_{23}v_{22} + q_{23}q_{13}v_{31} + q_{23}^2v_{32} + q_{23}q_{33}v_{33} - q_{23}q_{33}$$

and

$$q_{31} + q_{33}\frac{q_{31}q_{11}v_{11} + q_{31}q_{21}v_{12} + q_{31}^2v_{13} + q_{32}q_{11}v_{21} + q_{32}q_{21}v_{22} + q_{32}q_{31}v_{23} + q_{33}q_{21}v_{32} + q_{33}q_{31}v_{33} - q_{31}}{q_{33}\,(1 - q_{11})}$$
$$= q_{31}q_{11}v_{11} + q_{31}q_{21}v_{12} + q_{31}^2v_{13} + q_{32}q_{11}v_{21} + q_{32}q_{21}v_{22} + q_{32}q_{31}v_{23}$$
$$+ q_{33}q_{11}\frac{q_{31}q_{11}v_{11} + q_{31}q_{21}v_{12} + q_{31}^2v_{13} + q_{32}q_{11}v_{21} + q_{32}q_{21}v_{22} + q_{32}q_{31}v_{23} + q_{33}q_{21}v_{32} + q_{33}q_{31}v_{33} - q_{31}}{q_{33}\,(1 - q_{11})}$$
$$q_{31}q_{11}v_{11} + q_{31}q_{21}v_{12} + q_{31}^2v_{13} + q_{32}q_{11}v_{21} + q_{32}q_{21}v_{22} + q_{32}q_{31}v_{23} + q_{33}q_{21}v_{32} + q_{33}q_{31}v_{33} - q_{31}q_{11}$$
$$= q_{31}q_{11}v_{11} + q_{31}q_{21}v_{12} + q_{31}^2v_{13} + q_{32}q_{11}v_{21} + q_{32}q_{21}v_{22} + q_{32}q_{31}v_{23} + q_{33}q_{21}v_{32} + q_{33}q_{31}v_{33} - q_{31}q_{11}$$

and

$$q_{33}\frac{q_{31}q_{12}v_{11} + q_{31}q_{22}v_{12} + q_{31}q_{32}v_{13} + q_{32}q_{12}v_{21} + q_{32}q_{22}v_{22} + q_{32}^2v_{23} + q_{33}q_{12}v_{31} + q_{33}q_{32}v_{33} - q_{32}}{q_{33}\,(1 - q_{22})} + q_{32}$$
$$= q_{31}q_{12}v_{11} + q_{31}q_{22}v_{12} + q_{31}q_{32}v_{13} + q_{32}q_{12}v_{21} + q_{32}q_{22}v_{22} + q_{32}^2v_{23} + q_{33}q_{12}v_{31}$$
$$+ q_{33}q_{22}\frac{q_{31}q_{12}v_{11} + q_{31}q_{22}v_{12} + q_{31}q_{32}v_{13} + q_{32}q_{12}v_{21} + q_{32}q_{22}v_{22} + q_{32}^2v_{23} + q_{33}q_{12}v_{31} + q_{33}q_{32}v_{33} - q_{32}}{q_{33}\,(1 - q_{22})}$$
$$+ q_{33}q_{32}v_{33}$$
$$q_{31}q_{12}v_{11} + q_{31}q_{22}v_{12} + q_{31}q_{32}v_{13} + q_{32}q_{12}v_{21} + q_{32}q_{22}v_{22} + q_{32}^2v_{23} + q_{33}q_{12}v_{31} + q_{33}q_{32}v_{33} - q_{32}q_{22}$$
$$= q_{31}q_{12}v_{11} + q_{31}q_{22}v_{12} + q_{31}q_{32}v_{13} + q_{32}q_{12}v_{21} + q_{32}q_{22}v_{22} + q_{32}^2v_{23} + q_{33}q_{12}v_{31} + q_{33}q_{32}v_{33} - q_{32}q_{22}$$

and

$$q_{33}\frac{q_{31}q_{13}v_{11} + q_{31}q_{23}v_{12} + q_{31}q_{33}v_{13} + q_{32}q_{13}v_{21} + q_{32}q_{23}v_{22} + q_{32}q_{33}v_{23} + q_{33}q_{13}v_{31} + q_{33}q_{23}v_{32}}{q_{33}\,(1 - q_{33})}$$
$$= q_{31}q_{13}v_{11} + q_{31}q_{23}v_{12} + q_{31}q_{33}v_{13} + q_{32}q_{13}v_{21} + q_{32}q_{23}v_{22} + q_{32}q_{33}v_{23} + q_{33}q_{13}v_{31} + q_{33}q_{23}v_{32}$$
$$+ q_{33}^2\frac{q_{31}q_{13}v_{11} + q_{31}q_{23}v_{12} + q_{31}q_{33}v_{13} + q_{32}q_{13}v_{21} + q_{32}q_{23}v_{22} + q_{32}q_{33}v_{23} + q_{33}q_{13}v_{31} + q_{33}q_{23}v_{32}}{q_{33}\,(1 - q_{33})}$$
$$q_{31}q_{13}v_{11} + q_{31}q_{23}v_{12} + q_{31}q_{33}v_{13} + q_{32}q_{13}v_{21} + q_{32}q_{23}v_{22} + q_{32}q_{33}v_{23} + q_{33}q_{13}v_{31} + q_{33}q_{23}v_{32}$$
$$= q_{31}q_{13}v_{11} + q_{31}q_{23}v_{12} + q_{31}q_{33}v_{13} + q_{32}q_{13}v_{21} + q_{32}q_{23}v_{22} + q_{32}q_{33}v_{23} + q_{33}q_{13}v_{31} + q_{33}q_{23}v_{32}$$

That is, the left hand is equal to the right hand, and the system (4) is J-semi-homogenous.

Theorem 2: *The homogeneous system of differential equation is δ-semi homogeneous if and only if the matrix V equal to:*

$$\begin{bmatrix} \frac{q_{11}v_{12}+v_{12}-Jv_{12}q_{22}-Jv_{13}q_{32}}{Jq_{12}} & \frac{q_{11}v_{13}+q_{13}-Jv_{11}q_{13}-Jv_{13}q_{33}}{Jq_{23}} & \frac{q_{11}v_{11}-Jv_{11}q_{11}-Jv_{12}q_{21}}{Jq_{31}} \\ \frac{q_{22}v_{22}-Jv_{22}q_{22}-Jv_{22}q_{22}}{Jq_{12}} & \frac{q_{22}v_{23}+q_{23}-Jv_{21}q_{13}-Jv_{23}q_{33}}{Jq_{23}} & \frac{q_{21}+q_{22}v_{21}-Jv_{21}q_{11}-Jv_{22}q_{21}}{Jq_{31}} \\ \frac{q_{33}v_{33}-Jv_{32}q_{23}-Jv_{33}q_{33}}{Jq_{13}} & \frac{q_{32}+q_{33}v_{31}-Jv_{31}q_{11}-Jv_{33}q_{31}}{Jq_{21}} & \frac{q_{32}+q_{33}v_{32}-Jv_{31}q_{12}-Jv_{32}q_{22}}{Jq_{32}} \end{bmatrix}$$

Proof: Since F is δ-semi-homogeneous, there exists a non-zero matrix (for illustration, suppose V) that allows Eq. (3) to be held with $k = 1$.

$$F(VX) = det\ J(Q)VF(X)$$

$$F\left(\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = det\ J(Q)\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

$$\begin{pmatrix} f(v_{11}x + v_{12}y + v_{13}z) \\ g(v_{21}x + v_{22}y + v_{23}z) \\ h(v_{31}x + v_{32}y + v_{33}z) \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}\begin{pmatrix} v_{11}f(x) + v_{12}g(y) + v_{13}h(z) \\ v_{21}f(x) + v_{22}g(y) + v_{23}h(z) \\ v_{31}f(x) + v_{32}g(y) + v_{33}h(z) \end{pmatrix}$$

$$det\ J(Q) = q_{11}(q_{22}q_{33} - q_{23}q_{32}) - q_{12}(q_{21}q_{33} - q_{23}q_{31}) + q_{13}(q_{21}q_{32} - q_{22}q_{31})$$

The left hand is

$$\begin{pmatrix} q_{11}v_{11}x + (q_{11}v_{12} + q_{12})y + (q_{11}v_{13} + q_{13})z \\ (q_{22}v_{21} + q_{21})x + q_{22}v_{22}y + (q_{22}v_{23} + q_{23})z \\ (q_{31} + q_{33}v_{31})x + (q_{32} + q_{33}v_{32})y + q_{33}v_{33}z \end{pmatrix}$$

The right hand is

$$detJ(Q)\begin{pmatrix} (v_{11}q_{11} + v_{12}q_{21} + v_{13}q_{31})x + (v_{11}q_{12} + v_{12}q_{22} + v_{13}q_{32})y + (v_{11}q_{13} + v_{12}q_{23} + v_{13}q_{33})z \\ (v_{21}q_{11} + v_{22}q_{21} + v_{23}q_{31})x + (v_{21}q_{12} + v_{22}q_{22} + v_{23}q_{32})y + (v_{21}q_{13} + v_{22}q_{23} + v_{23}q_{33})z \\ (v_{31}q_{11} + v_{32}q_{21} + v_{33}q_{31})x + (v_{31}q_{12} + v_{32}q_{22} + v_{33}q_{32})y + (v_{31}q_{13} + v_{32}q_{23} + v_{33}q_{33})z \end{pmatrix}$$

$$q_{11}v_{11} = J\left[v_{11}q_{11} + v_{12}q_{21} + v_{13}q_{31}\right]$$
$$q_{11}v_{12} + q_{12} = J\left[v_{11}q_{12} + v_{12}q_{22} + v_{13}q_{32}\right]$$
$$q_{11}v_{13} + q_{13} = J\left[v_{11}q_{13} + v_{12}q_{23} + v_{13}q_{33}\right]$$
$$q_{22}v_{21} + q_{21} = J\left[v_{21}q_{11} + v_{22}q_{21} + v_{23}q_{31}\right]$$
$$q_{22}v_{22} = J\left[v_{21}q_{12} + v_{22}q_{22} + v_{23}q_{32}\right]$$
$$q_{22}v_{23} + q_{23} = J\left[v_{21}q_{13} + v_{22}q_{23} + v_{23}q_{33}\right]$$
$$q_{31} + q_{33}v_{31} = J\left[v_{31}q_{11} + v_{32}q_{21} + v_{33}q_{31}\right]$$
$$q_{32} + q_{33}v_{32} = J\left[v_{31}q_{12} + v_{32}q_{22} + v_{33}q_{32}\right]$$
$$q_{33}v_{33} = J\left[v_{31}q_{13} + v_{32}q_{23} + v_{33}q_{33}\right]$$

Matrix V is easily demonstrated to equal:

$$\begin{bmatrix} \frac{q_{11}v_{12}+v_{12}-Jv_{12}q_{22}-Jv_{13}q_{32}}{Jq_{12}} & \frac{q_{11}v_{13}+q_{13}-Jv_{11}q_{13}-Jv_{13}q_{33}}{Jq_{23}} & \frac{q_{11}v_{11}-Jv_{11}q_{11}-Jv_{12}q_{21}}{Jq_{31}} \\ \frac{q_{22}v_{22}-Jv_{22}q_{22}-Jv_{22}q_{22}}{Jq_{12}} & \frac{q_{22}v_{23}+q_{23}-Jv_{21}q_{13}-Jv_{23}q_{33}}{Jq_{23}} & \frac{q_{21}+q_{22}v_{21}-Jv_{21}q_{11}-Jv_{22}q_{21}}{Jq_{31}} \\ \frac{q_{33}v_{33}-Jv_{32}q_{23}-Jv_{33}q_{33}}{Jq_{13}} & \frac{q_{32}+q_{33}v_{31}-Jv_{31}q_{11}-Jv_{33}q_{31}}{Jq_{21}} & \frac{q_{32}+q_{33}v_{32}-Jv_{31}q_{12}-Jv_{32}q_{22}}{Jq_{32}} \end{bmatrix}$$

Conversely
Suppose that there is a matrix V equal to

$$\begin{bmatrix} \frac{q_{11}v_{12}+v_{12}-Jv_{12}q_{22}-Jv_{13}q_{32}}{Jq_{12}} & \frac{q_{11}v_{13}+q_{13}-Jv_{11}q_{13}-Jv_{13}q_{33}}{Jq_{23}} & \frac{q_{11}v_{11}-Jv_{11}q_{11}-Jv_{12}q_{21}}{Jq_{31}} \\ \frac{q_{22}v_{22}-Jv_{22}q_{22}-Jv_{22}q_{22}}{Jq_{12}} & \frac{q_{22}v_{23}+q_{23}-Jv_{21}q_{13}-Jv_{23}q_{33}}{Jq_{23}} & \frac{q_{21}+q_{22}v_{21}-Jv_{21}q_{11}-Jv_{22}q_{21}}{Jq_{31}} \\ \frac{q_{33}v_{33}-Jv_{32}q_{23}-Jv_{33}q_{33}}{Jq_{13}} & \frac{q_{32}+q_{33}v_{31}-Jv_{31}q_{11}-Jv_{33}q_{31}}{Jq_{21}} & \frac{q_{32}+q_{33}v_{32}-Jv_{31}q_{12}-Jv_{32}q_{22}}{Jq_{32}} \end{bmatrix}$$

To show that the system (4) is δ-semi-homogeneous, that is

$$F\left(V\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \det J(Q)\mathrm{V}\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

By putting the value of the matrix V in (3), we have

$$F\left(\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

Therefore,

$$q_{11}v_{11} = J\left[v_{11}q_{11} + v_{12}q_{21} + q_{31}\frac{q_{11}v_{11} - Jv_{11}q_{11} - Jv_{12}q_{21}}{Jq_{31}}\right]$$
$$v_{11}q_{11} = v_{11}q_{11}$$
$$q_{11}v_{12} + q_{12} = J\left[q_{12}\frac{q_{11}v_{12} + q_{12} - Jv_{12}q_{22} - Jv_{13}q_{32}}{Jq_{12}} + v_{12}q_{22} + v_{13}q_{32}\right]$$
$$q_{11}v_{12} + q_{12} = q_{11}v_{12} + q_{12}$$
$$q_{13} + q_{11}v_{13} = J\left[v_{11}q_{13} + q_{23}\frac{q_{11}v_{13} + q_{13} - Jv_{11}q_{13} - Jv_{13}q_{33}}{Jq_{23}} + v_{13}q_{33}\right]$$
$$q_{11}v_{13} + q_{13} = q_{11}v_{13} + q_{13}$$
$$q_{21} + q_{22}v_{21} = J\left[v_{21}q_{11} + v_{22}q_{21} + q_{31}\frac{q_{21} + q_{22}v_{21} - Jv_{21}q_{11} - Jv_{22}q_{21}}{Jq_{31}}\right]$$
$$q_{21} + q_{22}v_{21} = q_{21} + q_{22}v_{21}$$
$$q_{22}v_{22} = J\left[q_{12}\frac{q_{22}v_{22} - Jv_{22}q_{22} - Jv_{23}q_{32}}{Jq_{12}} + v_{22}q_{22} + v_{23}q_{33}\right]$$
$$q_{22}v_{22} = q_{22}v_{22}$$
$$q_{22}v_{23} + q_{23} = J\left[v_{21}q_{13} + q_{23}\frac{q_{22}v_{23} + q_{23} - Jv_{21}q_{13} - Jv_{23}q_{33}}{Jq_{23}} + v_{23}q_{33}\right]$$
$$q_{22}v_{23} + q_{23} = q_{22}v_{23} + q_{23}$$
$$q_{31} + q_{33}v_{31} = J\left[v_{31}q_{11} + v_{33}q_{31} + q_{21}\frac{q_{31} + q_{33}v_{31} - Jv_{31}q_{11} - Jv_{33}q_{31}}{Jq_{21}}\right]$$
$$q_{31} + q_{33}v_{31} = q_{31} + q_{33}v_{31}$$
$$q_{32} + q_{33}v_{32} = J\left[v_{31}q_{12} + v_{32}q_{22} + q_{32}\frac{q_{32} + q_{33}v_{32} - Jv_{31}q_{12} - Jv_{32}q_{22}}{Jq_{32}}\right]$$
$$q_{32} + q_{33}v_{32} = q_{32} + q_{33}v_{32}$$
$$q_{33}v_{33} = J\left[q_{13}\frac{q_{33}v_{33} - Jv_{32}q_{23} - Jv_{33}q_{33}}{Jq_{13}} + v_{32}q_{23} + v_{33}q_{33}\right]$$
$$q_{33}v_{33} = q_{33}v_{33}$$

That is, the left hand is equal to the right hand, and the system (1) is δ-semi-homogenous.

Corollary 1:

1- A homogeneous system of a differential equations is J-semi-homogenous, if the following is held:

$$q_{11}v_{11} = q_{11}^2v_{11} + q_{21}q_{11}v_{12} + q_{11}q_{31}v_{13} + q_{12}q_{11}v_{21} + q_{21}q_{12}v_{22} + q_{12}q_{31}v_{23} + q_{13}q_{11}v_{31} + q_{13}q_{21}v_{32} + q_{13}q_{31}v_{33}$$
$$q_{11}v_{12} + q_{12} = q_{11}q_{12}v_{11} + q_{11}q_{22}v_{12} + q_{11}q_{32}v_{13} + q_{12}^2v_{21} + q_{12}q_{22}v_{22} + q_{12}q_{32}v_{23} + q_{13}q_{12}v_{31} + q_{13}q_{22}v_{32} + q_{13}q_{32}v_{33}$$
$$q_{13} + q_{11}v_{13} = q_{11}q_{13}v_{11} + q_{11}q_{23}v_{12} + q_{11}q_{33}v_{13} + q_{12}q_{13}v_{21} + q_{12}q_{23}v_{22} + q_{12}q_{33}v_{23} + q_{13}^2v_{31} + q_{13}q_{23}v_{32} + q_{13}q_{33}v_{33}$$
$$q_{21} + q_{22}v_{21} = q_{21}q_{11}v_{11} + q_{21}^2v_{12} + q_{21}q_{31}v_{13} + q_{22}q_{11}v_{21} + q_{22}q_{21}v_{22} + q_{22}q_{31}v_{23} + q_{23}q_{11}v_{31} + q_{23}q_{21}v_{32} + q_{23}q_{31}v_{33}$$
$$q_{22}v_{22} = q_{21}q_{12}v_{11} + q_{21}q_{22}v_{12} + q_{21}q_{32}v_{13} + q_{12}q_{22}v_{21} + q_{22}^2v_{22} + q_{22}q_{32}v_{23} + q_{23}q_{12}v_{31} + q_{23}q_{22}v_{32} + q_{23}q_{32}v_{33}$$

$$q_{22}v_{23} + q_{23} = q_{21}q_{13}v_{11} + q_{21}q_{23}v_{12} + q_{21}q_{33}v_{13} + q_{22}q_{13}v_{21} + q_{22}q_{23}v_{22} + q_{22}q_{33}v_{23} + q_{23}q_{13}v_{31} + q_{23}^2v_{32} + q_{23}q_{33}v_{33}$$
$$q_{31} + q_{33}v_{31} = q_{31}q_{11}v_{11} + q_{31}q_{21}v_{12} + q_{31}^2v_{13} + q_{32}q_{11}v_{21} + q_{32}q_{21}v_{22} + q_{32}q_{31}v_{23} + q_{33}q_{11}v_{31} + q_{33}q_{21}v_{32} + q_{33}q_{31}v_{33}$$
$$q_{33}v_{32} + q_{32} = q_{31}q_{12}v_{11} + q_{31}q_{22}v_{12} + q_{31}q_{32}v_{13} + q_{32}q_{12}v_{21} + q_{32}q_{22}v_{22} + q_{32}^2v_{23} + q_{33}q_{12}v_{31} + q_{33}q_{22}v_{32} + q_{33}q_{32}v_{33}$$
$$q_{33}v_{33} = q_{31}q_{13}v_{11} + q_{31}q_{23}v_{12} + q_{31}q_{33}v_{13} + q_{32}q_{13}v_{21} + q_{32}q_{23}v_{22} + q_{32}q_{33}v_{23} + q_{33}q_{13}v_{31} + q_{33}q_{23}v_{32} + q_{33}^2v_{33}$$

2- A homogeneous system of differential equations is δ-semi-homogenous if the following is held:

$$q_{11}v_{11} = J\left[v_{11}q_{11} + v_{12}q_{21} + v_{13}q_{31}\right]$$
$$q_{11}v_{12} + q_{12} = J\left[v_{11}q_{12} + v_{12}q_{22} + v_{13}q_{32}\right]$$
$$q_{11}v_{13} + q_{13} = J\left[v_{11}q_{13} + v_{12}q_{23} + v_{13}q_{33}\right]$$
$$q_{22}v_{21} + q_{21} = J\left[v_{21}q_{11} + v_{22}q_{21} + v_{23}q_{31}\right]$$
$$q_{22}v_{22} = J\left[v_{21}q_{12} + v_{22}q_{22} + v_{23}q_{32}\right]$$
$$q_{22}v_{23} + q_{23} = J\left[v_{21}q_{13} + v_{22}q_{23} + v_{23}q_{33}\right]$$
$$q_{31} + q_{33}v_{31} = J\left[v_{31}q_{11} + v_{32}q_{21} + v_{33}q_{31}\right]$$
$$q_{32} + q_{33}v_{32} = J\left[v_{31}q_{12} + v_{32}q_{22} + v_{33}q_{32}\right]$$
$$q_{33}v_{33} = J\left[v_{31}q_{13} + v_{32}q_{23} + v_{33}q_{33}\right]$$

Proof: Direct from Theorems 1 and 2.

Example 2:

1- Consider the system $F(X) = QX$ where $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$, there exists a matrix V equal to $\begin{pmatrix} \frac{60}{82} & \frac{-26}{82} & \frac{10}{82} \\ \frac{-34}{82} & \frac{12}{82} & \frac{8}{82} \\ \frac{68}{82} & \frac{-24}{82} & \frac{-16}{82} \end{pmatrix}$, then this system is J-semi-homogenous.

2- Consider the system $F(X) = QX$ where $Q = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ then there is a matrix V equal to $\begin{pmatrix} 0 & \frac{-1}{4} & \frac{-1}{8} \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, then this system is δ-semi-homogenous. To Justify that by using Corollary (2.5),

Solution:

1-

$$q_{11}v_{11} = 1\left(\frac{60}{82}\right) = \frac{30}{41}$$
$$q_{11}^2v_{11} + q_{21}q_{11}v_{12} + q_{11}q_{31}v_{13} + q_{12}q_{11}v_{21} + q_{21}q_{12}v_{22} + q_{12}q_{31}v_{23} + q_{13}q_{11}v_{31} + q_{13}q_{21}v_{32} + q_{13}q_{31}v_{33}$$
$$= \frac{60}{82} - \frac{26}{82} + 2\left(\frac{10}{82}\right) - \frac{34}{82} + \frac{12}{82} + 2\left(\frac{8}{82}\right) + \frac{68}{82} - \frac{24}{82} - 2\left(\frac{16}{82}\right) = \frac{30}{41}$$
$$q_{11}v_{12} + q_{12} = -\frac{26}{82} + 1 = \frac{28}{41}$$
$$q_{11}q_{12}v_{11} + q_{11}q_{22}v_{12} + q_{11}q_{32}v_{13} + q_{12}^2v_{21} + q_{12}q_{22}v_{22} + q_{12}q_{32}v_{23} + q_{13}q_{12}v_{31} + q_{13}q_{22}v_{32} + q_{13}q_{32}v_{33}$$
$$\frac{60}{82} - \frac{26}{82} + 0 - \frac{34}{82} + \frac{12}{82} + 0 + \frac{68}{82} - \frac{24}{82} + 0 = \frac{28}{41}$$
$$q_{13} + q_{11}v_{13} = 1 + \frac{10}{82} = \frac{46}{41}$$
$$q_{11}q_{13}v_{11} + q_{11}q_{23}v_{12} + q_{11}q_{33}v_{13} + q_{12}q_{13}v_{21} + q_{12}q_{23}v_{22} + q_{12}q_{33}v_{23} + q_{13}^2v_{31} + q_{13}q_{23}v_{32} + q_{13}q_{33}v_{33}$$
$$\frac{60}{82} + 0 - \frac{10}{82} - \frac{34}{82} + 0 - \frac{8}{82} + \frac{68}{82} + 0 + \frac{16}{82} = \frac{46}{41}$$
$$q_{21} + q_{22}v_{21} = 1 - \frac{34}{82} = \frac{24}{41}$$

$$q_{21}q_{11}v_{11} + q_{21}^2v_{12} + q_{21}q_{31}v_{13} + q_{22}q_{11}v_{21} + q_{22}q_{21}v_{22} + q_{22}q_{31}v_{23} + q_{23}q_{11}v_{31} + q_{23}q_{21}v_{32} + q_{23}q_{31}v_{33}$$
$$\frac{60}{82} - \frac{26}{82} + 2\left(\frac{10}{82}\right) - \frac{34}{82} + \frac{12}{82} + 2\left(\frac{8}{82}\right) + 0 + 0 + 0 = \frac{24}{41}$$
$$q_{22}v_{22} = \frac{12}{82} = \frac{6}{41}$$
$$q_{21}q_{12}v_{11} + q_{21}q_{22}v_{12} + q_{21}q_{32}v_{13} + q_{12}q_{22}v_{21} + q_{22}^2v_{22} + q_{22}q_{32}v_{23} + q_{23}q_{12}v_{31} + q_{23}q_{22}v_{32} + q_{23}q_{32}v_{33}$$
$$\frac{60}{82} - \frac{26}{82} + 0 - \frac{34}{82} + \frac{12}{82} + 0 + 0 + 0 + 0 = \frac{6}{41}$$
$$q_{22}v_{23} + q_{23} = \frac{8}{82} = \frac{4}{41}$$
$$q_{21}q_{13}v_{11} + q_{21}q_{23}v_{12} + q_{21}q_{33}v_{13} + q_{22}q_{13}v_{21} + q_{22}q_{23}v_{22} + q_{22}q_{33}v_{23} + q_{23}q_{13}v_{31} + q_{23}^2v_{32} + q_{23}q_{33}v_{33}$$
$$\frac{60}{82} + 0 - \frac{10}{82} - \frac{34}{82} + 0 - \frac{8}{82} + 0 + 0 + 0 = \frac{4}{41}$$
$$q_{31} + q_{33}v_{31} = 2 - \frac{68}{82} = \frac{48}{41}$$
$$q_{31}q_{11}v_{11} + q_{31}q_{21}v_{12} + q_{31}^2v_{13} + q_{32}q_{11}v_{21} + q_{32}q_{21}v_{22} + q_{32}q_{31}v_{23} + q_{33}q_{11}v_{31} + q_{33}q_{21}v_{32} + q_{33}q_{31}v_{33}$$
$$= 2\left(\frac{60}{82}\right) - 2\left(\frac{26}{82}\right) + 4\left(\frac{10}{82}\right) + 0 + 0 + 0 - \frac{68}{82} + \frac{24}{82} + 2\left(\frac{16}{82}\right) = \frac{48}{41}$$
$$q_{33}v_{32} + q_{32} = \frac{24}{82} = \frac{12}{41}$$
$$q_{31}q_{12}v_{11} + q_{31}q_{22}v_{12} + q_{31}q_{32}v_{13} + q_{32}q_{12}v_{21} + q_{32}q_{22}v_{22} + q_{32}^2v_{23} + q_{33}q_{12}v_{31} + q_{33}q_{22}v_{32} + q_{33}q_{32}v_{33}$$
$$= 2\left(\frac{60}{82}\right) - 2\left(\frac{26}{82}\right) + 0 + 0 + 0 + 0 - \frac{68}{82} + \frac{24}{82} + 0 = \frac{12}{41}$$
$$q_{33}v_{33} = \frac{16}{82} = \frac{8}{41}$$
$$q_{31}q_{13}v_{11} + q_{31}q_{23}v_{12} + q_{31}q_{33}v_{13} + q_{32}q_{13}v_{21} + q_{32}q_{23}v_{22} + q_{32}q_{33}v_{23} + q_{33}q_{13}v_{31} + q_{33}q_{23}v_{32} + q_{33}^2v_{33}$$
$$= 2\left(\frac{60}{82}\right) + 0 - 2\left(\frac{10}{82}\right) + 0 + 0 + 0 - \frac{68}{82} + 0 - \frac{16}{82} = \frac{8}{41}$$

In the same way, this can be proved in example two.

Definition 2: System (1) is called J-adjoint (δ-adjoint)–semi–homogeneous if there are two non-zero matrices V and C such that the following equation holds:

$$F(VX) = J(Q)CF(X)\left(F(VX) = det\ J(Q)CF(X)\right),$$

respectively

Theorem 3: *A homogeneous system of differential equations is J-adjoint (δ-adjoint)-semi-homogeneous if and only if the matrix V equal to* $\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}$

$$v_{11} = \frac{q_{11}^2c_{11} + q_{21}q_{11}c_{12} + q_{11}q_{31}c_{13} + q_{12}q_{11}c_{21} + q_{21}q_{12}c_{22} + q_{12}q_{31}c_{23} + q_{13}q_{11}c_{31} + q_{13}q_{21}c_{32} + q_{13}q_{31}c_{33}}{q_{11}}$$
$$v_{12} = \frac{q_{11}q_{12}c_{11} + q_{11}q_{22}c_{12} + q_{11}q_{32}c_{13} + q_{12}^2c_{21} + q_{12}q_{22}c_{22} + q_{12}q_{32}c_{23} + q_{13}q_{12}c_{31} + q_{13}q_{22}c_{32} + q_{13}q_{32}c_{33} - q_{12}}{q_{11}}$$
$$v_{13} = \frac{q_{11}q_{13}c_{11} + q_{11}q_{23}c_{12} + q_{11}q_{33}c_{13} + q_{12}q_{13}c_{21} + q_{12}q_{23}c_{22} + q_{12}q_{33}c_{23} + q_{13}^2c_{31} + q_{13}q_{23}c_{32} + q_{13}q_{33}c_{33} - q_{13}}{q_{11}}$$

$$v_{21} = \frac{q_{21}q_{11}c_{11}+q_{21}^2c_{12}+q_{21}q_{31}c_{13}+q_{22}q_{11}c_{21}c_{21}+q_{22}q_{21}c_{22}+q_{22}q_{31}c_{23}+q_{23}q_{11}c_{31}+q_{23}q_{21}c_{32}+q_{23}q_{31}c_{33}-q_{21}}{q_{22}}$$

$$v_{22} = \frac{q_{21}q_{12}c_{11}+q_{21}q_{22}c_{12}+q_{21}q_{32}c_{13}+q_{12}q_{22}c_{21}+q_{22}^2c_{22}+q_{22}q_{32}c_{23}+q_{23}q_{12}c_{31}+q_{23}q_{22}c_{32}+q_{23}q_{32}c_{33}}{q_{22}}$$

$$v_{23} = \frac{q_{21}q_{13}c_{11}+q_{21}q_{23}c_{12}+q_{21}q_{33}c_{13}+q_{22}q_{13}c_{21}+q_{22}q_{23}c_{22}+q_{22}q_{33}c_{23}+q_{23}q_{13}c_{31}+q_{23}^2c_{32}+q_{23}q_{33}c_{33}-q_{23}}{q_{22}}$$

$$v_{31} = \frac{q_{31}q_{11}c_{11}+q_{31}q_{21}c_{12}+q_{31}^2c_{13}+q_{32}q_{11}c_{21}+q_{32}q_{21}c_{22}+q_{32}q_{31}c_{23}+q_{33}q_{11}c_{31}+q_{33}q_{21}c_{32}+q_{33}q_{31}c_{33}-q_{31}}{q_{33}}$$

$$v_{32} = \frac{q_{31}q_{12}c_{11}+q_{31}q_{22}c_{12}+q_{31}q_{32}c_{13}+q_{32}q_{12}c_{21}+q_{32}q_{22}c_{22}+q_{32}^2c_{23}+q_{33}q_{12}c_{31}+q_{33}q_{22}c_{32}+q_{33}q_{32}c_{33}-q_{32}}{q_{33}}$$

$$v_{33} = \frac{q_{31}q_{13}c_{11}+q_{31}q_{23}c_{12}+q_{31}q_{33}c_{13}+q_{32}q_{13}c_{21}+q_{32}q_{23}c_{22}+q_{32}q_{33}c_{23}+q_{33}q_{13}c_{31}+q_{33}q_{23}c_{32}+q_{33}^2c_{33}}{q_{33}}$$

$$\left[\begin{pmatrix} \frac{J(c_{11}q_{11}+c_{12}q_{21}+c_{13}q_{31})}{q_{11}} & \frac{J(c_{11}q_{12}+c_{12}q_{22}+c_{13}q_{32})-q_{12}}{q_{11}} & \frac{J(c_{11}q_{13}+c_{12}q_{23}+c_{13}q_{33})-q_{13}}{q_{11}} \\ \frac{J(c_{21}q_{11}+c_{22}q_{21}+c_{23}q_{31})-q_{21}}{q_{22}} & \frac{J(c_{21}q_{12}+c_{22}q_{22}+c_{23}q_{32})}{q_{22}} & \frac{J(c_{21}q_{13}+c_{22}q_{23}+c_{23}q_{33})-q_{23}}{q_{22}} \\ \frac{J(c_{31}q_{11}+c_{32}q_{21}+c_{33}q_{31})-q_{31}}{q_{33}} & \frac{J(c_{31}q_{12}+c_{32}q_{22}+c_{33}q_{32})-q_{32}}{q_{33}} & \frac{J(c_{31}q_{13}+c_{32}q_{23}+c_{33}q_{33})}{q_{33}} \end{pmatrix}\right],$$

respectively.

Proof: The proof for J-adjoint-semi homogeneous as the following:

Since F is a homogeneous system of degree 1, there exists a non-zero matrix (for example, let's say V) that allows the Eq. (2) to be held with $k = 1$.

$$F(VX) = J(Q)VF(X)$$

$$F\left(\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = J(Q)\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

$$\begin{pmatrix} f(v_{11}x+v_{12}y+v_{13}z) \\ g\left(v_{21}x+v_{22}y+v_{23}z\right) \\ h(v_{31}x+v_{32}y+v_{33}z) \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}\begin{pmatrix} c_{11}f(x)+c_{12}g(y)+c_{13}h(z) \\ c_{21}f(x)+c_{22}g(y)+c_{23}h(z) \\ c_{31}f(x)+c_{32}g(y)+c_{33}h(z) \end{pmatrix}$$

The left hand is

$$\begin{pmatrix} q_{11}v_{11}x+(q_{11}v_{12}+q_{12})y+(q_{11}v_{13}+q_{13})z \\ (q_{22}v_{21}+q_{21})x+q_{22}v_{22}y+(q_{22}v_{23}+q_{23})z \\ (q_{31}+q_{33}v_{31})x+(q_{32}+q_{33}v_{32})y+q_{33}v_{33}z \end{pmatrix}$$

The right hand is $\begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}$ where

$$\begin{aligned} G_1 = {} & q_{11}^2c_{11}x+q_{11}c_{11}q_{12}y+q_{11}c_{11}q_{13}z+q_{11}c_{12}q_{21}x+q_{11}c_{12}q_{22}y+q_{11}q_{23}c_{12}z+q_{11}c_{13}q_{31}x+q_{11}c_{13}q_{32}y \\ & +q_{11}c_{13}q_{33}z+q_{12}c_{21}q_{11}x+q_{12}^2c_{21}y+q_{12}c_{21}q_{13}z+q_{12}c_{22}q_{21}x+q_{12}c_{22}q_{22}\,y+q_{12}c_{22}q_{23}z+q_{12}c_{23}q_{31}x \\ & +q_{12}c_{23}q_{32}y+q_{12}c_{23}q_{33}z+q_{13}c_{31}q_{11}x+q_{13}c_{31}q_{12}y+q_{13}^2c_{31}z+q_{13}c_{32}q_{21}x+q_{13}c_{32}q_{22}y+q_{13}c_{32}q_{23}z \\ & +q_{13}c_{33}q_{31}x+q_{13}c_{33}q_{22}y+q_{13}c_{33}q_{33}z \end{aligned}$$

$$G_2 = q_{21}c_{11}q_{11}x + q_{12}c_{11}q_{21}y + q_{21}c_{11}q_{13}z + q_{21}^2c_{12}x + q_{21}c_{12}q_{22}y + q_{21}q_{23}c_{12}z + q_{21}c_{13}q_{31}x + q_{21}c_{13}q_{32}y + q_{21}c_{13}q_{33}z + q_{22}c_{21}q_{11}x + q_{22}c_{21}q_{12}y + q_{22}c_{21}q_{13}z + q_{22}c_{22}q_{21}x + q_{22}^2c_{22}y + q_{22}c_{22}q_{23}z + q_{22}c_{23}q_{31}x + q_{22}c_{23}q_{32}y + q_{22}c_{23}q_{33}z + q_{23}c_{31}q_{11}x + q_{23}c_{31}q_{12}y + q_{23}c_{31}q_{13}z + q_{23}c_{32}q_{21}x + q_{23}c_{32}q_{22}y + q_{23}^2c_{32}z + q_{23}c_{33}q_{31}x + q_{23}c_{33}q_{22}y + q_{23}c_{33}q_{33}z$$

$$G_3 = q_{31}c_{11}q_{11}x + q_{12}c_{11}q_{31}y + q_{31}c_{11}q_{13}z + q_{21}c_{12}q_{21}x + q_{31}c_{12}q_{22}y + q_{31}q_{23}c_{12}z + q_{31}^2c_{13}x + q_{31}c_{13}q_{32}y + q_{31}c_{13}q_{33}z + q_{32}c_{21}q_{11}x + q_{32}c_{21}q_{12}y + q_{32}c_{21}q_{13}z + q_{32}c_{22}q_{21}x + q_{32}c_{22}q_{22}y + q_{32}c_{22}q_{23}z + q_{32}c_{23}q_{31}x + q_{32}^2c_{23}y + q_{32}c_{23}q_{33}z + q_{33}c_{31}q_{11}x + q_{33}c_{31}q_{12}y + q_{33}c_{31}q_{13}z + q_{33}c_{32}q_{21}x + q_{33}c_{32}q_{22}y + q_{33}c_{32}q_{23}z + q_{33}c_{33}q_{31}x + q_{33}c_{33}q_{32}y + q_{33}^2c_{33}z$$

$$q_{11}v_{11} = q_{11}^2c_{11} + q_{21}q_{11}c_{12} + q_{11}q_{31}c_{13} + q_{12}q_{11}c_{21} + q_{21}q_{12}c_{22} + q_{12}q_{31}c_{23} + q_{13}q_{11}c_{31} + q_{13}q_{21}c_{32} + q_{13}q_{31}c_{33}$$

$$q_{11}v_{12} + q_{12} = q_{11}q_{12}c_{11} + q_{11}q_{22}c_{12} + q_{11}q_{32}c_{13} + q_{12}^2c_{21} + q_{12}q_{22}c_{22} + q_{12}q_{32}c_{23} + q_{13}q_{12}c_{31} + q_{13}q_{22}c_{32} + q_{13}q_{32}c_{33}$$

$$q_{13} + q_{11}v_{13} = q_{11}q_{13}c_{11} + q_{11}q_{23}c_{12} + q_{11}q_{33}c_{13} + q_{12}q_{13}c_{21} + q_{12}q_{23}c_{22} + q_{12}q_{33}c_{23} + q_{13}^2c_{31} + q_{13}q_{23}c_{32} + q_{13}q_{33}c_{33}$$

$$q_{21} + q_{22}v_{21} = q_{21}q_{11}c_{11} + q_{21}^2c_{12} + q_{21}q_{31}c_{13} + q_{22}q_{11}c_{21} + q_{22}q_{21}c_{22} + q_{22}q_{31}c_{23} + q_{23}q_{11}c_{31} + q_{23}q_{21}c_{32} + q_{23}q_{31}c_{33}$$

$$q_{22}v_{22} = q_{21}q_{12}c_{11} + q_{21}q_{22}c_{12} + q_{21}q_{32}c_{13} + q_{12}q_{22}c_{21} + q_{22}^2c_{22} + q_{22}q_{32}c_{23} + q_{23}q_{12}c_{31} + q_{23}q_{22}c_{32} + q_{23}q_{22}c_{33}$$

$$q_{22}v_{23} + q_{23} = q_{21}q_{13}c_{11} + q_{21}q_{23}c_{12} + q_{21}q_{33}c_{13} + q_{22}q_{13}c_{21} + q_{22}q_{23}c_{22} + q_{22}q_{33}c_{23} + q_{23}q_{13}c_{31} + q_{23}^2c_{32} + q_{23}q_{33}c_{33}$$

$$q_{31} + q_{33}v_{31} = q_{31}q_{11}c_{11} + q_{31}q_{21}c_{12} + q_{31}^2c_{13} + q_{32}q_{11}c_{21} + q_{32}q_{21}c_{22} + q_{32}q_{31}c_{23} + q_{33}q_{11}c_{31} + q_{33}q_{21}c_{32} + q_{33}q_{31}c_{33}$$

$$q_{33}v_{32} + q_{32} = q_{31}q_{12}c_{11} + q_{31}q_{22}c_{12} + q_{31}q_{32}c_{13} + q_{32}q_{12}c_{21} + q_{32}q_{22}c_{22} + q_{32}^2c_{23} + q_{33}q_{12}c_{31} + q_{33}q_{22}c_{32} + q_{33}q_{32}c_{33}$$

$$q_{33}v_{33} = q_{31}q_{13}c_{11} + q_{31}q_{23}c_{12} + q_{31}q_{33}c_{13} + q_{32}q_{13}c_{21} + q_{32}q_{23}c_{22} + q_{32}q_{33}c_{23} + q_{33}q_{13}c_{31} + q_{33}q_{23}c_{32} + q_{33}^2c_{33}$$

It is simple to show that matrix V equals:

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}$$

Conversely

Suppose that there is a matrix V equal to

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}$$

To show that system (3) is J-adjoint-semi-homogeneous, that is

$$F(VX) = J(Q)CF(X)$$

By putting the value of the matrix V in 3, we have

$$F\left(\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = J(Q)\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

Therefore,

$$q_{11}\frac{q_{11}^2c_{11}+q_{21}q_{11}c_{12}+q_{11}q_{31}c_{13}+q_{12}q_{11}c_{21}+q_{21}q_{12}c_{22}+q_{12}q_{31}c_{23}+q_{13}q_{11}c_{31}+q_{13}q_{21}c_{32}+q_{13}q_{31}c_{33}}{q_{11}}$$
$$= q_{11}^2c_{11}+q_{21}q_{11}c_{12}+q_{11}q_{31}c_{13}+q_{12}q_{11}c_{21}+q_{21}q_{12}c_{22}+q_{12}q_{31}c_{23}+q_{13}q_{11}c_{31}+q_{13}q_{21}c_{32}+q_{13}q_{31}c_{33}$$
$$q_{11}^2c_{11}+q_{21}q_{11}c_{12}+q_{11}q_{31}c_{13}+q_{12}q_{11}c_{21}+q_{21}q_{12}c_{22}+q_{12}q_{31}c_{23}+q_{13}q_{11}c_{31}+q_{13}q_{21}c_{32}+q_{13}q_{31}c_{33}$$
$$= q_{11}^2c_{11}+q_{21}q_{11}c_{12}+q_{11}q_{31}c_{13}+q_{12}q_{11}c_{21}+q_{21}q_{12}c_{22}+q_{12}q_{31}c_{23}+q_{13}q_{11}c_{31}+q_{13}q_{21}c_{32}+q_{13}q_{31}c_{33}$$

and

$$q_{11}\frac{q_{11}q_{12}c_{11}+q_{11}q_{22}c_{12}+q_{11}q_{32}c_{13}+q_{12}^2c_{21}+q_{12}q_{22}c_{22}+q_{12}q_{32}c_{23}+q_{13}q_{12}c_{31}+q_{13}q_{22}c_{32}+q_{13}q_{32}c_{33}-q_{12}}{q_{11}}$$
$$+q_{12} = q_{11}q_{12}c_{11}+q_{11}q_{22}c_{12}+q_{11}q_{32}c_{13}+q_{12}^2c_{21}+q_{12}q_{22}c_{22}+q_{12}q_{32}c_{23}+q_{13}q_{12}c_{31}+q_{13}q_{22}c_{32}+q_{13}q_{32}c_{33}$$
$$q_{11}q_{12}c_{11}+q_{11}q_{22}c_{12}+q_{11}q_{32}c_{13}+q_{12}^2c_{21}+q_{12}q_{22}c_{22}+q_{12}q_{32}c_{23}+q_{13}q_{12}c_{31}+q_{13}q_{22}c_{32}+q_{13}q_{32}c_{33}$$
$$= q_{11}q_{12}c_{11}+q_{11}q_{22}c_{12}+q_{11}q_{32}c_{13}+q_{12}^2c_{21}+q_{12}q_{22}c_{22}+q_{12}q_{32}c_{23}+q_{13}q_{12}c_{31}+q_{13}q_{22}c_{32}+q_{13}q_{32}c_{33}$$

and

$$q_{13}+q_{11}$$
$$\times\frac{q_{11}q_{13}c_{11}+q_{11}q_{23}c_{12}+q_{11}q_{33}c_{13}+q_{12}q_{13}c_{21}+q_{12}q_{23}c_{22}+q_{12}q_{33}c_{23}+q_{13}^2c_{31}+q_{13}q_{23}c_{32}+q_{13}q_{33}c_{33}-q_{13}}{q_{11}}$$
$$= q_{11}q_{13}c_{11}+q_{11}q_{23}c_{12}+q_{11}q_{33}c_{13}+q_{12}q_{13}c_{21}+q_{12}q_{23}c_{22}+q_{12}q_{33}c_{23}+q_{13}^2c_{31}+q_{13}q_{23}c_{32}+q_{13}q_{33}c_{33}$$
$$q_{11}q_{13}c_{11}+q_{11}q_{23}c_{12}+q_{11}q_{33}c_{13}+q_{12}q_{13}c_{21}+q_{12}q_{23}c_{22}+q_{12}q_{33}c_{23}+q_{13}^2c_{31}+q_{13}q_{23}c_{32}+q_{13}q_{33}c_{33}$$
$$= q_{11}q_{13}c_{11}+q_{11}q_{23}c_{12}+q_{11}q_{33}c_{13}+q_{12}q_{13}c_{21}+q_{12}q_{23}c_{22}+q_{12}q_{33}c_{23}+q_{13}^2c_{31}+q_{13}q_{23}c_{32}+q_{13}q_{33}c_{33}$$

and

$$q_{21}+q_{22}$$
$$\times\frac{q_{21}q_{11}c_{11}+q_{21}^2c_{12}+q_{21}q_{31}c_{13}+q_{22}q_{11}c_{21}c_{21}+q_{22}q_{21}c_{22}+q_{22}q_{31}c_{23}+q_{23}q_{11}c_{31}+q_{23}q_{21}c_{32}+q_{23}q_{31}c_{33}-q_{21}}{q_{22}}$$
$$= q_{21}q_{11}c_{11}+q_{21}^2c_{12}+q_{21}q_{31}c_{13}+q_{22}q_{11}c_{21}c_{21}+q_{22}q_{21}c_{22}+q_{22}q_{31}c_{23}+q_{23}q_{11}c_{31}+q_{23}q_{21}c_{32}+q_{23}q_{31}c_{33}$$
$$q_{21}q_{11}c_{11}+q_{21}^2c_{12}+q_{21}q_{31}c_{13}+q_{22}q_{11}c_{21}c_{21}+q_{22}q_{21}c_{22}+q_{22}q_{31}c_{23}+q_{23}q_{11}c_{31}+q_{23}q_{21}c_{32}+q_{23}q_{31}c_{33}$$
$$= q_{21}q_{11}c_{11}+q_{21}^2c_{12}+q_{21}q_{31}c_{13}+q_{22}q_{11}c_{21}c_{21}+q_{22}q_{21}c_{22}+q_{22}q_{31}c_{23}+q_{23}q_{11}c_{31}+q_{23}q_{21}c_{32}+q_{23}q_{31}c_{33}$$

and

$$q_{22}\frac{q_{21}q_{12}c_{11}+q_{21}q_{22}c_{12}+q_{21}q_{32}c_{13}+q_{12}q_{22}c_{21}+q_{22}^2c_{22}+q_{22}q_{32}c_{23}+q_{23}q_{12}c_{31}+q_{23}q_{22}c_{32}+q_{23}q_{22}c_{33}}{q_{22}}$$
$$= q_{21}q_{12}c_{11}+q_{21}q_{22}c_{12}+q_{21}q_{32}c_{13}+q_{12}q_{22}c_{21}+q_{22}^2c_{22}+q_{22}q_{32}c_{23}+q_{23}q_{12}c_{31}+q_{23}q_{22}c_{32}+q_{23}q_{32}c_{33}$$
$$q_{21}q_{12}c_{11}+q_{21}q_{22}c_{12}+q_{21}q_{32}c_{13}+q_{12}q_{22}c_{21}+q_{22}^2c_{22}+q_{22}q_{32}c_{23}+q_{23}q_{12}c_{31}+q_{23}q_{22}c_{32}+q_{23}q_{32}c_{33}$$
$$= q_{21}q_{12}c_{11}+q_{21}q_{22}c_{12}+q_{21}q_{32}c_{13}+q_{12}q_{22}c_{21}+q_{22}^2c_{22}+q_{22}q_{32}c_{23}+q_{23}q_{12}c_{31}+q_{23}q_{22}c_{32}+q_{23}q_{32}c_{33}$$

and

$$q_{22}\frac{q_{21}q_{13}c_{11}+q_{21}q_{23}c_{12}+q_{21}q_{33}c_{13}+q_{22}q_{13}c_{21}+q_{22}q_{23}c_{22}+q_{22}q_{33}c_{23}+q_{23}q_{13}c_{31}+q_{23}^2c_{32}+q_{23}q_{33}c_{33}-q_{23}}{q_{22}}$$
$$+q_{23} = q_{21}q_{13}c_{11}+q_{21}q_{23}c_{12}+q_{21}q_{33}c_{13}+q_{22}q_{13}c_{21}+q_{22}q_{23}c_{22}+q_{22}q_{33}c_{23}+q_{23}q_{13}c_{31}+q_{23}^2c_{32}+q_{23}q_{33}c_{33}$$
$$q_{21}q_{13}c_{11}+q_{21}q_{23}c_{12}+q_{21}q_{33}c_{13}+q_{22}q_{13}c_{21}+q_{22}q_{23}c_{22}+q_{22}q_{33}c_{23}+q_{23}q_{13}c_{31}+q_{23}^2c_{32}+q_{23}q_{33}c_{33}$$
$$= q_{21}q_{13}c_{11}+q_{21}q_{23}c_{12}+q_{21}q_{33}c_{13}+q_{22}q_{13}c_{21}+q_{22}q_{23}c_{22}+q_{22}q_{33}c_{23}+q_{23}q_{13}c_{31}+q_{23}^2c_{32}+q_{23}q_{33}c_{33}$$

and

$$q_{31}+q_{33}\times\frac{q_{31}q_{11}c_{11}+q_{31}q_{21}c_{12}+q_{31}^{2}c_{13}+q_{32}q_{11}c_{21}+q_{32}q_{21}c_{22}+q_{32}q_{31}c_{23}+q_{33}q_{11}c_{31}+q_{33}q_{21}c_{32}+q_{33}q_{31}c_{33}-q_{31}}{q_{33}}$$
$$=q_{31}q_{11}c_{11}+q_{31}q_{21}c_{12}+q_{31}^{2}c_{13}+q_{32}q_{11}c_{21}+q_{32}q_{21}c_{22}+q_{32}q_{31}c_{23}+q_{33}q_{11}c_{31}+q_{33}q_{21}c_{32}+q_{33}q_{31}c_{33}$$
$$q_{31}q_{11}c_{11}+q_{31}q_{21}c_{12}+q_{31}^{2}c_{13}+q_{32}q_{11}c_{21}+q_{32}q_{21}c_{22}+q_{32}q_{31}c_{23}+q_{33}q_{11}c_{31}+q_{33}q_{21}c_{32}+q_{33}q_{31}c_{33}$$
$$=q_{31}q_{11}c_{11}+q_{31}q_{21}c_{12}+q_{31}^{2}c_{13}+q_{32}q_{11}c_{21}+q_{32}q_{21}c_{22}+q_{32}q_{31}c_{23}+q_{33}q_{11}c_{31}+q_{33}q_{21}c_{32}+q_{33}q_{31}c_{33}$$

and

$$q_{33}\frac{q_{31}q_{12}c_{11}+q_{31}q_{22}c_{12}+q_{31}q_{32}c_{13}+q_{32}q_{12}c_{21}+q_{32}q_{22}c_{22}+q_{32}^{2}c_{23}+q_{33}q_{12}c_{31}+q_{33}q_{22}c_{32}+q_{33}q_{32}c_{33}-q_{32}}{q_{33}}$$
$$+q_{32}=q_{31}q_{12}c_{11}+q_{31}q_{22}c_{12}+q_{31}q_{32}c_{13}+q_{32}q_{12}c_{21}+q_{32}q_{22}c_{22}+q_{32}^{2}c_{23}+q_{33}q_{12}c_{31}+q_{33}q_{22}c_{32}+q_{33}q_{32}c_{33}$$
$$q_{31}q_{12}c_{11}+q_{31}q_{22}c_{12}+q_{31}q_{32}c_{13}+q_{32}q_{12}c_{21}+q_{32}q_{22}c_{22}+q_{32}^{2}c_{23}+q_{33}q_{12}c_{31}+q_{33}q_{22}c_{32}+q_{33}q_{32}c_{33}$$
$$=q_{31}q_{12}c_{11}+q_{31}q_{22}c_{12}+q_{31}q_{32}c_{13}+q_{32}q_{12}c_{21}+q_{32}q_{22}c_{22}+q_{32}^{2}c_{23}+q_{33}q_{12}c_{31}+q_{33}q_{22}c_{32}+q_{33}q_{32}c_{33}$$

and

$$q_{33}\frac{q_{31}q_{13}c_{11}+q_{31}q_{23}c_{12}+q_{31}q_{33}c_{13}+q_{32}q_{13}c_{21}+q_{32}q_{23}c_{22}+q_{32}q_{33}c_{23}+q_{33}q_{13}c_{31}+q_{33}q_{23}c_{32}+q_{33}^{2}c_{33}}{q_{33}}$$
$$=q_{31}q_{13}c_{11}+q_{31}q_{23}c_{12}+q_{31}q_{33}c_{13}+q_{32}q_{13}c_{21}+q_{32}q_{23}c_{22}+q_{32}q_{33}c_{23}+q_{33}q_{13}c_{31}+q_{33}q_{23}c_{32}+q_{33}^{2}c_{33}$$
$$q_{31}q_{13}c_{11}+q_{31}q_{23}c_{12}+q_{31}q_{33}c_{13}+q_{32}q_{13}c_{21}+q_{32}q_{23}c_{22}+q_{32}q_{33}c_{23}+q_{33}q_{13}c_{31}+q_{33}q_{23}c_{32}+q_{33}^{2}c_{33}$$
$$=q_{31}q_{13}c_{11}+q_{31}q_{23}c_{12}+q_{31}q_{33}c_{13}+q_{32}q_{13}c_{21}+q_{32}q_{23}c_{22}+q_{32}q_{33}c_{23}+q_{33}q_{13}c_{31}+q_{33}q_{23}c_{32}+q_{33}^{2}c_{33}$$

That, the left hand is equal to the right hand, and the system (4) is J-adjoint-semi-homogenous.

Corollary 2: *A homogeneous system of a differential Eq. (4) is J-adjoint-homogenous if the following is held:*

$$q_{11}v_{11}=q_{11}^{2}c_{11}+q_{21}q_{11}c_{12}+q_{11}q_{31}c_{13}+q_{12}q_{11}c_{21}+q_{21}q_{12}c_{22}+q_{12}q_{31}c_{23}+q_{13}q_{11}c_{31}+q_{13}q_{21}c_{32}+q_{13}q_{31}c_{33}$$
$$q_{11}v_{12}+q_{12}=q_{11}q_{12}c_{11}+q_{11}q_{22}c_{12}+q_{11}q_{32}c_{13}+q_{12}^{2}c_{21}+q_{12}q_{22}c_{22}+q_{12}q_{32}c_{23}+q_{13}q_{12}c_{31}+q_{13}q_{22}c_{32}+q_{13}q_{32}c_{33}$$
$$q_{13}+q_{11}v_{13}=q_{11}q_{13}c_{11}+q_{11}q_{23}c_{12}+q_{11}q_{33}c_{13}+q_{12}q_{13}c_{21}+q_{12}q_{23}c_{22}+q_{12}q_{33}c_{23}+q_{13}^{2}c_{31}+q_{13}q_{23}c_{32}+q_{13}q_{33}c_{33}$$
$$q_{21}+q_{22}v_{21}=q_{21}q_{11}c_{11}+q_{21}^{2}c_{12}+q_{21}q_{31}c_{13}+q_{22}q_{11}c_{21}+q_{22}q_{21}c_{22}+q_{22}q_{31}c_{23}+q_{23}q_{11}c_{31}+q_{23}q_{21}c_{32}+q_{23}q_{31}c_{33}$$
$$q_{22}v_{22}=q_{21}q_{12}c_{11}+q_{21}q_{22}c_{12}+q_{21}q_{32}c_{13}+q_{12}q_{22}c_{21}+q_{22}^{2}c_{22}+q_{22}q_{32}c_{23}+q_{23}q_{12}c_{31}+q_{23}q_{22}c_{32}+q_{23}q_{22}c_{33}$$
$$q_{22}v_{23}+q_{23}=q_{21}q_{13}c_{11}+q_{21}q_{23}c_{12}+q_{21}q_{33}c_{13}+q_{22}q_{13}c_{21}+q_{22}q_{23}c_{22}+q_{22}q_{33}c_{23}+q_{23}q_{13}c_{31}+q_{23}^{2}c_{32}+q_{23}q_{33}c_{33}$$
$$q_{31}+q_{33}v_{31}=q_{31}q_{11}c_{11}+q_{31}q_{21}c_{12}+q_{31}^{2}c_{13}+q_{32}q_{11}c_{21}+q_{32}q_{21}c_{22}+q_{32}q_{31}c_{23}+q_{33}q_{11}c_{31}+q_{33}q_{21}c_{32}+q_{33}q_{31}c_{33}$$

$$q_{33}v_{32}+q_{32}=q_{31}q_{12}c_{11}+q_{31}q_{22}c_{12}+q_{31}q_{32}c_{13}+q_{32}q_{12}c_{21}+q_{32}q_{22}c_{22}+q_{32}^{2}c_{23}+q_{33}q_{12}c_{31}+q_{33}q_{22}c_{32}+q_{33}q_{32}c_{33}$$
$$q_{33}v_{33}=q_{31}q_{13}c_{11}+q_{31}q_{23}c_{12}+q_{31}q_{33}c_{13}+q_{32}q_{13}c_{21}+q_{32}q_{23}c_{22}+q_{32}q_{33}c_{23}+q_{33}q_{13}c_{31}+q_{33}q_{23}c_{32}+q_{33}^{2}c_{33}$$

Proof: Direct from Theorem 3

Example 3:

1- Consider the system $F(X) = QX$ *where* $Q = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & -1 \\ 3 & 0 & 2 \end{bmatrix}$, *then there are two matrices* $V = \begin{bmatrix} \frac{19}{2} & 2 & \frac{9}{2} \\ 1 & 0 & -4 \\ 11 & 4 & \frac{21}{2} \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ *that satisfy the equation:*

$$F(VX) = J(Q)CF(X)$$

2- Consider the system $F(X) = QX$ *where* $Q = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ *Then there are two matrices,* $V = \begin{bmatrix} 2 & -8 & -3 \\ \frac{3}{2} & 0 & 0 \\ 2 & 4 & 2 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}$ *that satisfy the equation:*

$$F(VX) = \det J(Q)CF(X),$$

then the systems in 1 and 2 are J-adjoint (δ-adjoint)–semi–homogeneous, respectively.

Results and discussion

In this study, we obtained a definition equivalent to the definition of a semi-homogeneous system of type J, as well as a definition equivalent to that of type δ, as mentioned in Theorems 1 and 2. Examples of these equivalents were given and the relationship between a semi-homogeneous system of type J and a semi-homogeneous system of type δ was studied. Necessary and sufficient conditions were found for the system of differential equations to be a semi-homogeneous system of type J and δ.

Conclusion

In this work, the classification of three-dimensional differential equation systems was presented based on the definition of the Jacobean matrix and its determinant, the two homogeneous systems were defined, called the J-semi-homogeneous system and the other δ-semi-homogeneous system, where the first definition was based on the Jacobian matrix, demonstrating the generalizability of the concept of homogeneity, on both definitions, examples were provided, and their interrelationship was examined. in addition to discovering an equivalent for these two definitions.

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Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Mustansiriyah University.

Authors' contribution statement

B. J. Al. and A. H. H. contributed to the design and implementation of the research, the proof of the theorems, and the writing of the manuscript.

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حول الأنظمة $(\delta - J)$ شبه المتجانسة للمعادلات التفاضلية

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الخلاصة

مفهوم التجانس في المعادلات التفاضلية يمكن ان يعمم على أنظمة المعادلات التفاضلية، حيث تم في هذا العمل عرض تصنيف لأنظمة المعادلات التفاضلية ثلاثية الأبعاد بناءً على تعريف مصفوفة الجاكوبيان ومحدداتها، حيث تم تعريف نظامين من النظام المتجانس، ويسمى النظام شبه المتجانس من النمط δ, والنظام الاخر شبه المتجانس من النمط J, حيث اعتمد التعريف الأول على مصفوفة الجاكوبيان، بينما اعتمد الثاني على محدد المصفوفة الجاكوبية. وتم تقديم أمثلة على كلا التعريفين، ودراسة العلاقة بينهما. بالإضافة إلى إيجاد معادلات لهذين التعريفين، وقد تم أيضًا إثبات بعض النتائج لهذين التعريفين.

الكلمات المفتاحية: النظام شبه المتجانس من النمط δ، مصفوفة الجاكوبيان، النظام شبه المتجانس من النمط J، شبه متجانس، نظام المعادلات التفاضلية.