Published Online First: March, 2024
https://doi.org/10.21123/bsj.2024.9619
P-ISSN: 2078-8665 - E-ISSN: 2411-7986

Baghdad Science Journal

Third-Order Differential Subordination for Generalized Struve Function Associated with Meromorphic Functions

Suha J. Hammad *[1], ***Abdul Rahman S. Juma*** [2], ***Hassan H. Ebrahim*** [3]

[1]Department of Mathematics, College of Education for Pure Sciences, University of Tikrit, Tikrit, Iraq.
[2]Department of Mathematics, College of Education for Pure Sciences, University of Anbar, Anbar, Iraq.
[3]Department of Mathematics, College of Computer Science and Mathematics, University of Tikrit, Tikrit, Iraq.
*Corresponding Author.

Received 23/09/2023, Revised 22/12/2023, Accepted 24/12/2023, Published Online First 20/03/2024

Abstract

Previously, many works dealt with the study of the $1^{st}-$ order differential subordination and shortly after that other studies dealt with the $2^{nd}-$order differential subordination in the unit disc. Recently the $3^{rd}-$ order differential subordination was presented by Antonino and Miller (2011). This paper looks at a considerably broader class of $3^{rd}-$order differential inequalities and subordination. The authors define the criteria on an admissible class of operators, implying that $3^{rd}-$order differentiale subordination exists. Meromorphic in D is a function that is holomorphic in domain D except for poles. If $D=\mathbb{C}$, it simply states the function is meromorphic. Meromorphic functions in $\mathbb{C}$ are those that may be represented as a quotient of two entire functions. Struve functions have applications in surface $-$wave and water $-$wave issues, unstable aerodynamics, optical direction and resistive MHD instability theory. Struve functions have lately appeared in a number of particle systems. The idea of differential subordination in $\mathbb{C}$ is a generalization of differential inequality in $\mathbb{R}$, and it was initiated in 1981 by the works of Miller, Mocanu and Reade. In this artical, appropriate classes of admissible functions are examined and the properties of $3^{rd}-$order differential subordination are established by using the operator $\check{S}_{c,p}^{\zeta,a}$ of meromorphic multivalent functions connected with generalized Struve function. In this study, there is a need to present many concepts including subordination, superordination, the dominant, the best dominant, convolution (or Hadamard product), meromorphic multivalent function, the Struve function addition to the concept of shifted factorial (or Pochhammer symbol) and admissible functions.

Keywords: Admissible Functions, Analytic Function, Convolution (or Hadamard product), Meromorphic Functions, Struve Function, $3^{rd}-$ Order Differential Subordination.

Introduction

Let $\underset{\sim}{\mathrm{H}}(UD)$ represent the class of analytic functions in $UD=\{z: z\in\mathbb{C}\ and\ |z|<1\}$ and let $\underset{\sim}{\mathrm{H}}[a,\imath]$ be the subclass of analytic functions defined by:

$$\underset{\sim}{\mathrm{H}}[a,\imath]=\left\{f\in\underset{\sim}{\mathrm{H}}(UD): f(z)=a+\sum_{\zeta=\imath}^{\infty}a_\zeta z^\zeta+\cdots\right\}$$

Assume that $\underset{\sim}{\mathrm{H}}[1,\imath]=\underset{\sim}{\mathrm{H}}_\imath$.

For two analytic function $\check{f}$ and $\breve{g}$ in UD, the function $\check{f}$ is said to be subordinate to the function $\breve{g}$ in UD, or the function $\breve{g}$ is said to be superordinate to the function $\check{f}$ in UD, and write

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$$\check{f}(z) \prec \breve{g}(z) \qquad (z \in UD),$$

if there exists a Schwarz function $\varpi(z)$ analytic in UD with $|\varpi(z)| < 1$ and $\varpi(0) = 0$ $(z \in UD)$ such that $\check{f}(z) = \breve{g}(\mu(z))$ $(z \in UD)$

It is well known that

$\check{f}(z) \prec \breve{g}(z)$ $(z \in UD) \Rightarrow \check{f}(0) = \breve{g}(0)$ and $\check{f}(UD) \subseteq \breve{g}(UD)$

Furthermore, if the function $\breve{g}$ is univalent in UD, the following equivalence hold.

$\check{f}(z) \prec \breve{g}(z)$ $(z \in UD) \Leftrightarrow \check{f}(0) = \breve{g}(0)$ and $\check{f}(UD) \subseteq \breve{g}(UD)$. [1-3]

Let $\Upsilon(t, u, v, w; z): \mathbb{C}^4 \text{x}\, UD \to \mathbb{C}$ and $\hbar(z)$ be univalent in unit disk UD. If $\mathcal{M}(z)$ is analytic in UD satisfies:

$$\Upsilon(\mathcal{M}(z), z\mathcal{M}'(z), z^2\mathcal{M}''(z), z^3\mathcal{M}'''(z); z) \prec \hbar(z). \qquad 1$$

Then, $\mathcal{M}(z)$ is the solution to the differential subordination Eq 1 described above. If $\mathcal{M}(z)$ is subordinate to $\mathbb{Q}(z)$ for all $\mathcal{M}(z)$ satisfying Eq 1, the univalent function $\mathbb{Q}(z)$ is said to be a dominant solution of Eq.1. A univalent dominant q satisfying $q \prec \mathbb{Q}$ for all dominants of Eq 1 is referred to as the best dominant .[4,5]

Let $\sum(p, i)$ be the class of all meromorphic functions of the form:

$$f(z) = z^{-p} + \sum_{\zeta=i}^{\infty} a_\zeta z^{\zeta-p} \quad (p, i \in N = \{1,2,3,\dots\}), \qquad 2$$

which are analytic and p −valent in the punctured disk $(UD)^* = UD\backslash\{0\} = \{z: z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$.

Let $f, l \in \sum(p, i)$ where f is given by Eq 2 and l is defined by

$$l(z) = z^{-p} + \sum_{\zeta=i}^{\infty} b_\zeta z^{\zeta-p} \quad (z \in (UD)^*), \qquad 3$$

the convolution (or Hadamard product) of two functions f and l is defined by

$$(f * l)(z) = z^{-p} + \sum_{\zeta=i}^{\infty} a_\zeta b_\zeta z^{\zeta-p} = (l * f)(z)$$

In this study, it was considered one of these functions, the series solution of an inhomogeneous second-order Bessel differential equation, which was presented and explored by Struve.[4] Struve functions and their generalization are used in a variety of areas of applied mathematics and physics.

After a series of mathematical operations, the Struve function of order has the following form [6]:

$$\acute{\check{U}}_{r,b,c}(z) = 2^r\sqrt{\pi}\,\Gamma\left(r + \frac{b+2}{2}\right) z^{\frac{-r-1}{2}} \sum_{\zeta=0}^{\infty} \frac{(-c)^\zeta (\sqrt{z}/2)^{2\zeta+r+1}}{\Gamma(\zeta+3/2)\Gamma(r+\zeta+(b+2)/2)} = \sum_{\zeta=0}^{\infty} \frac{(-c/4)^\zeta}{(3/2)_\zeta (a)_\zeta} z^\zeta, \qquad (z \in UD)$$

and

$$\acute{\check{U}}_{a,c}(z) = z\acute{\check{U}}_{r,b,c}(z) = z + \sum_{\zeta=1}^{\infty} \frac{(-c/4)^\zeta}{(3/2)_\zeta (a)_\zeta} z^{\zeta+1},$$

where $r, b, c \in \mathbb{C}, a = r + (b+2)/2 \neq 0, -1, -2, \dots$ and $(a)_\zeta$ is the shifted factorial (or Pochhammer symbol) [6] expressed terms of the gamma function, by

$$(a)_\zeta = \frac{\Gamma(a+\zeta)}{\Gamma(a)} = \begin{cases} 1 & ; (\zeta = 0), \\ a(a+1)(a+2)\dots(a+\zeta-1) & ; (\zeta \in N = \{1,2,3,\dots\}) \end{cases}$$

Numerous authors have used the theory of 1^{st} and 2^{nd} order differential subordination and superorination to address a variety of issues in geometric function theory. It is difficult to take the dual issues for higher-order instances into consideration. The idea of 3^{rd} − order differential subordination was first introduced in the works of Ponnusamy and Juneje [7] , but more recently, work by Antonino and Miller [7] has wekindled interest in this field among researchers. Tang et al [8] .presented the concept of third-order differential subordination in 2014 as a generalization of the second-order case.

Many studies on the results of the 3^{rd} − order differential subordination and superordination in various contexts have been conducted in recent years.

By setting

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P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

$$\delta_{a,c,p}(z) = z^{-p} + \sum_{\zeta=i}^{\infty} \frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}\, z^{\zeta-p}$$

By using the convolution (or Hadamard product), in this work, it is defined a new operator

$$Š_{c,p}^{\zeta,a}: \sum(p,i) \to \sum(p,i)$$

Which is defined as follows

$$Š_{c,p}^{\zeta,a} f(z) = \delta_{a,c,p}(z) * f(z) = z^{-p} + \sum_{\zeta=i}^{\infty} \frac{(-c/4)^{\zeta}}{(3/2)_{\zeta}(a)_{\zeta}}\, a_{\zeta} z^{\zeta-p} \quad (z \in UD), \qquad 4$$

From Eq 4 it can be easily concluded that

$$z\left(Š_{c,p}^{\zeta,a} f(z)\right)' = (a+\zeta-p) Š_{c,p}^{\zeta,a} f(z) - (a+\zeta) Š_{c,p}^{\zeta,a+1} f(z) \qquad 5$$

To derive the results of this paper the following lemmas and definitions are needed.

Definition 1:[5,9] Let Ꝗ be the collection of analytic and univalent functions $ꝏ$ on $\overline{UD}\backslash E(ꝏ)$, where $E(ꝏ) = \left\{ ɼ \in \partial UD: \lim_{z\to ɼ} ꝏ(z) = \infty \right\}$ and $min|ꝏ'(ɼ)| = \partial > 0$ for $ɼ \in \partial UD\backslash E(ꝏ)$. Further, let $Ꝗ(a)$ denote the subclass of Ꝗ consisting of functions $ꝏ$ for which $ꝏ(0) = a$ and $Ꝗ(1) \equiv Ꝗ_1$.

Definition 2:[5] If $ʽΩ \subseteq \mathbb{C}, ꝏ \in Ꝗ$ and $i \geq 2$. Let $\Psi_i[ʽΩ, ꝏ]$ be the family of admissible functions that include functions $ʹϒ: \mathbb{C}^4 \text{x } UD \to \mathbb{C}$ that satisfy the requirement of acceptability as:

$$ʹϒ(t, u, v, w, ; z) \notin ʽΩ,$$

when

$$t = ꝏ(ɼ), u = 𝓃ɼ\, ꝏ'(ɼ),$$

$$Re\left\{\frac{v}{u} + 1\right\} > 𝓃 Re\left\{1 + \frac{ɼ\, ꝏ''(ɼ)}{ꝏ'(ɼ)}\right\}$$

and

$$Re\left\{\frac{w}{u}\right\} \geq 𝓃^2\, Re\left\{\frac{ɼ^2 ꝏ'''(ɼ)}{ꝏ'(ɼ)}\right\},$$

where $z \in UD, ɼ \in \partial UD\backslash E(ꝏ)$ and $𝓃 \geq i$.

Lemma1: [5] Let $\mathcal{M} \in Ḫ[a, i]$ with $i \geq 2$. Furthermore, let $ꝏ \in Ꝗ(a)$ and achieve the following requirements:

$$Re\left\{\frac{ɼ\, ꝏ''(ɼ)}{ꝏ'(ɼ)}\right\} \geq 0 \text{ and } \left|\frac{z\mathcal{M}'(z)}{ꝏ'(ɼ)}\right| \leq 𝓃,$$

where $z \in UD, ɼ \in \partial UD\backslash E(ꝏ)$ and $𝓃 \geq i$. If $ʽΩ$ is a set in $\mathbb{C}$, $ʹϒ \in \Psi_i[ʽΩ, ꝏ]$ and

$$ʹϒ\left(\mathcal{M}(z), z\mathcal{M}'(z), z^2\mathcal{M}''(z), z^3\mathcal{M}'''(z); z\right) \in ʽΩ,$$

then $\mathcal{M}(z) \prec ꝏ(z)$.

Several writers have derived many important conclusions involving numerous operators connected by differential superordination and differential subordination for example [2,3,10] .

Definition 3: If $ʽΩ \subseteq \mathbb{C}$ and $ꝏ \in Ꝗ_1 \cap Ḫ_i$. Let $ð_1[ʽΩ, ꝏ]$ be the family of admissible functions, which comprises the functions $\Psi: \mathbb{C}^4 \text{x } UD \to \mathbb{C}$ that meet the admissibility requirement:

$$\Psi(a', b', c', d'; z) \notin ʽΩ,$$

when

$$a' = ꝏ(ɼ),\ b' = ꝏ(ɼ) - \frac{1}{a+\zeta} 𝓃ɼ\, ꝏ'(ɼ),$$

$$Re\left\{\frac{c'(a+\zeta+1) + a'(a+\zeta) - b'(2a+2\zeta+1)}{a'-b'}\right\} > 𝓃 Re\left\{1 + \frac{ɼ\, ꝏ''(ɼ)}{ꝏ'(ɼ)}\right\}$$

and

$$Re\left\{\frac{3c'(a+\zeta+1)(a+\zeta) - a'(5a^2+5\zeta^2+10a\zeta-3a-3\zeta) - d'(a+\zeta+2)(a+\zeta+1) - 3b'(a+\zeta)(a+\zeta-2)}{a'-b'}\right\} \geq 𝓃^2\, Re\left\{1 + \frac{ɼ^2 ꝏ'''(ɼ)}{ꝏ'(ɼ)}\right\},$$

where $z \in UD, ɼ \in \partial UD\backslash E(ꝏ)$ and $𝓃 \geq i \geq 2$.

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https://doi.org/10.21123/bsj.2024.9619
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Results and Discussion

Theorem 1: Let $\grave{\Omega} \subseteq \mathbb{C}$ and $\Psi \in ð_1[\grave{\Omega}, \text{Ꞷ}]$. If $f \in \sum(p, i)$ and $\text{Ꞷ} \in Q_1$ achieve the following requirements:

$$Re\left\{\frac{\text{ŗ}\,\text{Ꞷ}''(\text{ŗ})}{\text{Ꞷ}'(\text{ŗ})}\right\} \geq 0 \text{ and } \left|z\left(z^p \check{S}_{c,p}^{\zeta,a} f(z)\right)'\right| \leq n|\text{Ꞷ}'(\text{ŗ})|, \qquad 6$$

then

$$\left\{\Psi\left(z^p \check{S}_{c,p}^{\zeta,a} f(z), z^p \check{S}_{c,p}^{\zeta,a+1} f(z), z^p \check{S}_{c,p}^{\zeta,a+2} f(z), z^p \check{S}_{c,p}^{\zeta,a+3} f(z); z\right) : z \in UD\right\} \subset \grave{\Omega} \qquad 7$$

which leads to

$$z^p \check{S}_{c,p}^{\zeta,a} f(z) \prec \text{Ꞷ}(\text{ŗ}).$$

Proof: In unit disk UD, define

$$z^p \check{S}_{c,p}^{\zeta,a} f(z) = \mathcal{M}(z) \quad (z \in UD). \qquad 8$$

To differentiate equation Eq 8 with respect to z and using Eq 5 and its recurrence,

$$z^p \check{S}_{c,p}^{\zeta,a+1} f(z) = \mathcal{M}(z) - \frac{1}{a+\zeta} z\mathcal{M}'(z) \qquad 9$$

Once again to differentiate equation Eq 9 through applying the recurrence relation Eq 5 and with regard to z,

$$z^p \check{S}_{c,p}^{\zeta,a+2} f(z) = \mathcal{M}(z) - \frac{2}{a+\zeta+1} z\mathcal{M}'(z) + \frac{1}{(a+\zeta+1)(a+\zeta)} z^2 \mathcal{M}''(z) \qquad 10$$

Additional calculations show that

$$z^p \check{S}_{c,p}^{\zeta,a+3} f(z) = \mathcal{M}(z) - \frac{3}{(a+\zeta+2)} z\mathcal{M}'(z) + \frac{3}{(a+\zeta+2)(a+\zeta+1)} z^2 \mathcal{M}''(z) - \frac{1}{(a+\zeta+2)(a+\zeta+1)(a+\zeta)} z^3 \mathcal{M}'''(z) \qquad 11$$

Let

$$\left.\begin{array}{l} a' = t, \quad b' = t - \frac{1}{a+\zeta} u, \quad c' = t - \frac{2}{a+\zeta+1} u + \frac{1}{(a+\zeta+1)(a+\zeta)} v, \\ d' = t - \frac{3}{(a+\zeta+2)} u + \frac{3}{(a+\zeta+2)(a+\zeta+1)} v - \frac{1}{(a+\zeta+2)(a+\zeta+1)(a+\zeta)} w \end{array}\right\} \qquad 12$$

Now, the transformation is defined $\Upsilon(t, u, v, w; z): \mathbb{C}^4 \text{x } UD \to \mathbb{C}$ by

$$\Upsilon(t, u, v, w; z) = \Psi(a', b', c', d'; z) = \Psi\left(t, t - \frac{1}{a+\zeta} u, t - \frac{2}{a+\zeta+1} u + \frac{1}{(a+\zeta+1)(a+\zeta)} v, t - \frac{3}{(a+\zeta+2)} u + \frac{3}{(a+\zeta+2)(a+\zeta+1)} v - \frac{1}{(a+\zeta+2)(a+\zeta+1)(a+\zeta)} w\right) \qquad 13$$

And by making use of the equations Eq 8 – Eq 11

$$\Upsilon(\mathcal{M}(z), z\mathcal{M}'(z), z^2\mathcal{M}''(z), z^3\mathcal{M}'''(z); z)$$

$$= \Psi\begin{pmatrix} z^p \check{S}_{c,p}^{\zeta,a} f(z), z^p \check{S}_{c,p}^{\zeta,a+1} f(z), \\ z^p \check{S}_{c,p}^{\zeta,a+2} f(z), z^p \check{S}_{c,p}^{\zeta,a+3} f(z); z \end{pmatrix} \qquad 14$$

Also, note that

$$\frac{v}{u} + 1 = \frac{c'(a+\zeta+1) + a'(a+\zeta) - b'(2a+2\zeta+1)}{a' - b'}$$

and

$$\frac{w}{u} = \frac{3c'(a+\zeta+1)(a+\zeta) - a'(5a^2 + 5\zeta^2 + 10a\zeta - 3a - 3\zeta) - d'(a+\zeta+2)(a+\zeta+1) - 3b'(a+\zeta)(a+\zeta-2)}{a' - b'}$$

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Theorem 1 is proved by relying on Lemma 1 because the requirements of admissibility for function $\Psi \in \eth_1[\Omega, \mathbb{Q}]$ of Definition 3 is equivalent to the requirements of admissibility for function $\Upsilon \in \Psi_\zeta[\Omega, \mathbb{Q}]$ in Definition 2.

When the function $\mathbb{Q}(z)$ has an unknown behavior on ∂UD, the following results will be an extension of Theorem 1.

Corollary 1: If $\Omega \subseteq \mathbb{C}$ and $\mathbb{Q}$ is univalent in UD with $\mathbb{Q} \in Q_1$. Let $\Psi \in \eth_1[\Omega, \mathbb{Q}_\partial]$ for some $\partial \in (0,1)$, where $\mathbb{Q}_\partial(z) = \mathbb{Q}(\partial z)$. If $f \in \sum(p, \imath)$ and $\mathbb{Q}_\partial$, satisfies the following condition:

$$Re\left\{\frac{\mathfrak{r}\,\mathbb{Q}_\partial''(\mathfrak{r})}{\mathbb{Q}_\partial'(\mathfrak{r})}\right\} \geq 0 \text{ and } \left|z\left(z^p \check{S}_{c,p}^{\zeta,a} f(z)\right)'\right| \leq n|\mathbb{Q}_\partial'(\mathfrak{r})|, \quad 15$$

then

$$\left\{\Psi\begin{pmatrix} z^p\check{S}_{c,p}^{\zeta,a}f(z), z^p\check{S}_{c,p}^{\zeta,a+1}f(z), \\ z^p\check{S}_{c,p}^{\zeta,a+2}f(z), z^p\check{S}_{c,p}^{\zeta,a+3}f(z); z\end{pmatrix} : z \in UD\right\} \subset \Omega,$$

which implies

$$z^p\check{S}_{c,p}^{\zeta,a}f(z) \prec \mathbb{Q}(z).$$

Proof: According to Theorem 1

$$z^p\check{S}_{c,p}^{\zeta,a}f(z) \prec \mathbb{Q}_\partial(z),$$

and since $\mathbb{Q}_\partial(z) \prec \mathbb{Q}(z)$, then $z^p\check{S}_{c,p}^{\zeta,a}f(z) \prec \mathbb{Q}(z)$.

If $\Omega \neq \mathbb{C}$ is a family-connected domain, there is a conformal mapping $\hbar: UD \to \mathbb{C}$ such that $\hbar(UD) = \mathbb{C}$. The class $\eth_1[\hbar(UD), \mathbb{Q}]$ is then denoted as $\eth_1[\hbar, \mathbb{Q}]$. The following two corollaries are direct implications of Theorem 1 as well as Corollary 1.

Corollary 2: If $\hbar$ is a univalent function in UD and let $\Psi \in \eth_1[\hbar, \mathbb{Q}]$, assume that $\mathbb{Q} \in Q_1$ satisfies Eq 6. Then,

$$\left\{\Psi\begin{pmatrix} z^p\check{S}_{c,p}^{\zeta,a}f(z), z^p\check{S}_{c,p}^{\zeta,a+1}f(z), \\ z^p\check{S}_{c,p}^{\zeta,a+2}f(z), z^p\check{S}_{c,p}^{\zeta,a+3}f(z); z\end{pmatrix} : z \in UD\right\} \prec \hbar(z), \quad 16$$

which implies

$$z^p\check{S}_{c,p}^{\zeta,a}f(z) \prec \mathbb{Q}(z).$$

Corollary 3: If $\mathbb{Q}$ is a univalent function in UD with $\mathbb{Q} \in Q_1$ and $\Psi \in \eth_1[\hbar, \mathbb{Q}_\partial]$ for some $\partial \in (0,1)$, where $\mathbb{Q}_\partial(z) = \mathbb{Q}(\partial z)$. If $\mathbb{Q}_\partial$ meets the requirements of Eq 15, then the subordination relation Eq 16 implies that

$$z^p\check{S}_{c,p}^{\zeta,a}f(z) \prec \mathbb{Q}(z).$$

The next corollary explains the connection between the solution of the related 3^{rd} – order differential equation and the best dominant of a 3^{rd} –order differential subordination.

Corollary 4: If $\hbar$ is a univalent function in UD and Υ is given by Eq 14 where $\Psi \in \eth_1[\hbar, \mathbb{Q}]$. If the differential equation

$$\Upsilon\left(\mathbb{Q}(z), z\mathbb{Q}'(z), z^2\mathbb{Q}''(z), z^3\mathbb{Q}'''(z); z\right) = \hbar(z)$$

has a solution $\mathbb{Q}$ with $\mathbb{Q} \in Q_1$ that meets the requirements Eq 6, then subordination Eq 16 implies that

$$z^p\check{S}_{c,p}^{\zeta,a}f(z) \prec \mathbb{Q}(z),$$

and $\mathbb{Q}$ is the best dominant of Eq 16.

Proof: Since

$$\Psi\begin{pmatrix} z^p\check{S}_{c,p}^{\zeta,a}f(z), z^p\check{S}_{c,p}^{\zeta,a+1}f(z), \\ z^p\check{S}_{c,p}^{\zeta,a+2}f(z), z^p\check{S}_{c,p}^{\zeta,a+3}f(z); z\end{pmatrix}$$

$$= \Upsilon\left(\mathcal{M}(z), z\mathcal{M}'(z), z^2\mathcal{M}''(z), z^3\mathcal{M}'''(z); z\right) \prec \hbar(z), \quad 17$$

then $\mathcal{M}(z)$ is a solution of Eq 17, and Corollary 2

$$\mathcal{M}(z) \prec \mathbb{Q}(z),$$

that is $\mathbb{Q}$ is a dominant of Eq 17. So is,

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https://doi.org/10.21123/bsj.2024.9619
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$$\Psi\begin{pmatrix} z^p \check{S}_{c,p}^{\zeta,a} f(z), z^p \check{S}_{c,p}^{\zeta,a+1} f(z), \\ z^p \check{S}_{c,p}^{\zeta,a+2} f(z), z^p \check{S}_{c,p}^{\zeta,a+3} f(z); z \end{pmatrix}$$

$$= \Upsilon\left(\mathcal{M}(z), z\mathcal{M}'(z), z^2\mathcal{M}''(z), z^3\mathcal{M}'''(z); z\right) \prec \hbar(z)$$

$$= \Upsilon\left(\mathbb{Q}(z), z\mathbb{Q}'(z), z^2\mathbb{Q}''(z), z^3\mathbb{Q}'''(z); z\right),$$

that is $\mathbb{Q}$ is the best dominant of Eq 17.

When $\mathbb{Q}(z) = 1 + \mathcal{B}e^{i\theta}, \mathcal{B} > 0$, by applying Theorem 1. According to Definition 3, the family of admissible functions $\eth_1[\Omega, \mathbb{Q}]$ is now denoted by $\eth_1[\Omega, \mathcal{B}]$ as follows:

Definition 4: If $\Omega \subseteq \mathbb{C}$ and let $\mathcal{B} > 0$. The family of admissible functions $\eth_1[\Omega, \mathcal{B}]$, which includes the functions $\Psi: \mathbb{C}^4 \text{x } UD \to \mathbb{C}$ that satisfy the admissibility condition:

$$\Psi\left(1 + \mathcal{B}e^{i\theta}, 1 + \frac{(a + \zeta - n)\mathcal{B}e^{i\theta}}{a + \zeta}, 1 + \frac{\mathcal{L} + (a + \zeta)[(a + \zeta + 1) - 2n]\mathcal{B}e^{i\theta}}{(a + \zeta + 1)(a + \zeta)}, 1 + \frac{3\mathcal{L}(a + \zeta) - \mathcal{N} + (a + \zeta + 1)(a + \zeta)[(a + \zeta + 2) - 3n]\mathcal{B}e^{i\theta}}{(a + \zeta + 2)(a + \zeta + 1)(a + \zeta)}; z\right) \notin \Omega,$$

whenever $z \in UD$, $Re\{\mathcal{L}e^{-i\theta}\} \geq n(n-1)\mathcal{B}$ and $Re\{\mathcal{N}e^{-i\theta}\} \geq 0$ for every $0 \leq \theta \leq \pi$ and $n \geq i \geq 2$. the following result can be derived from Theorem 1 using the concept of the family of admissible functions.

Corollary 5: If $\Omega \subseteq \mathbb{C}$ and $\eth_1[\Omega, \mathcal{B}]$. If

$$\left| z\left(z^p \check{S}_{c,p}^{\zeta,a} f(z)\right)' \right| \leq n\mathcal{B},$$

then

$$\Psi\begin{pmatrix} z^p \check{S}_{c,p}^{\zeta,a} f(z), z^p \check{S}_{c,p}^{\zeta,a+1} f(z), \\ z^p \check{S}_{c,p}^{\zeta,a+2} f(z), z^p \check{S}_{c,p}^{\zeta,a+3} f(z); z \end{pmatrix} \in \Omega \qquad (z \in UD),$$

which leads to

$$z^p \check{S}_{c,p}^{\zeta,a} f(z) \prec 1 + \mathcal{B}e^{i\theta}.$$

Applications Involving $\check{S}_{c,p}^{\zeta,a}$

If it is assumed that $\Omega = \{\tau \in \mathbb{C}: |\tau - 1| < \mathcal{B}\}$ and $\eth_1[\Omega, \mathcal{B}]$ is simply indicated by $\eth_1[\mathcal{B}]$, then Corollary 5 is reduced to the next corollary.

Corollary 6: Let $\Psi \in \eth_1[\mathcal{B}]$ and suppose that

$$\left| z\left(z^p \check{S}_{c,p}^{\zeta,a} f(z)\right)' \right| \leq n\mathcal{B},$$

then

$$\left|\Psi\begin{pmatrix} z^p \check{S}_{c,p}^{\zeta,a} f(z), z^p \check{S}_{c,p}^{\zeta,a+1} f(z), \\ z^p \check{S}_{c,p}^{\zeta,a+2} f(z), z^p \check{S}_{c,p}^{\zeta,a+3} f(z); z \end{pmatrix} - 1\right| < \mathcal{B} \quad (z \in UD),$$

which implies

$$z^p \check{S}_{c,p}^{\zeta,a} f(z) \prec 1 + \mathcal{B}z.$$

Corollary 7: Suppose that

$$\left| z\left(z^p \check{S}_{c,p}^{\zeta,a} f(z)\right)' \right| \leq n\mathcal{B},$$

then

$$\left| z^p \check{S}_{c,p}^{\zeta,a+1} f(z) - 1 \right| < \mathcal{B} \qquad (z \in UD),$$

which lead to

$$z^p \check{S}_{c,p}^{\zeta,a} f(z) \prec 1 + \mathcal{B}z.$$

Proof:

If it is put $\Psi(a', b', c', d'; z) = b'$, Corollary 6 leads to the conclusion.

If it is taken $c = -4$ and $(a)_\zeta = \frac{1}{(3/2)_\zeta}$ in Corollary 7, then

Published Online First: March, 2024
https://doi.org/10.21123/bsj.2024.9619
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

$$\check{S}^{\zeta,a}_{-4,p} f(z) = f(z)$$

and $\check{S}^{\zeta,a+1}_{-4,p} f(z) = \frac{(a+\zeta-p)}{(a+\zeta)} f(z) - \frac{1}{(a+\zeta)} z f'(z)$

The following results will be produced:

Example 1: If $f \in \sum(p, \imath)$ satisfies the following conditions

$$\left| z\left(z^p \check{S}^{\zeta,a}_{c,p} f(z) \right)' \right| \leq n\mathcal{B}$$

and

$$\left| \frac{(a+\zeta-p)}{(a+\zeta)} z^p f(z) - \frac{1}{(a+\zeta)} z^{p+1} f'(z) - 1 \right| < \mathcal{B},$$

then

$$|z^p f(z) - 1| < \mathcal{B}.$$

In addition, if it is taken $c = -4$ and $(a)_\zeta = \frac{1}{(3/2)_\zeta}$ in Corollary 7 and $p = \imath = 1$ the following results will be obtained:

Example 2: If $f \in \sum(p, \imath)$ satisfies the following inequality

$$|z^2 f'(z) + z f(z)| \leq n\mathcal{B}$$

and

$$\left| \frac{(a+\zeta-1)}{(a+\zeta)} z f(z) - \frac{1}{(a+\zeta)} z^2 f'(z) - 1 \right| < \mathcal{B},$$

then

$$|z f(z) - 1| < \mathcal{B}.$$

Conclusion

The purpose of using the operator $\check{S}^{\zeta,a}_{c,p}$, is to provide some results for the $3^{rd} -$ order differential subordination for analytic function. Investigating pertinent classes of admissible function leads to the results. The findings presented in this paper offer fresh perspectives for further research and opportunities are provided for scholars to generalize the results to produce new results in the fields of meromorphic univalent and mermorphic multivalent function theory.

Authors' Declaration

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Tikrit.

Authors' Contribution Statement

This work was carried out in collaboration with all authors. A.S. did the diagnosis of the cases then collected the samples, and did the tests. S.J. wrote and edited the manuscript with revised ideas. A.S. and H.H. conducted the analysis of the data with revised ideas. All authors read and appr–oved the final manuscript.

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التبعية التفاضلية من الدرجة الثالثة لدالة ستروف المعممة المرتبطة بالدوال الميرومورفية (الدوال التحليلية باستثناء عدد منته من النقاط)

سهى جمعة حماد[1]، عبدالرحمن سلمان جمعة [2]، حسن حسين ابراهيم[3]

[1] قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة تكريت، تكريت، العراق.
[2] قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة الانبار، الرمادي، العراق.
[3] قسم الرياضيات، كلية علوم الحاسوب والرياضيات، جامعة تكريت، تكريت، العراق.

الخلاصة

سابقاً تناولت العديد من الاعمال دراسة التبعية التفاضلية من الدرجة الاولى وبعد فترة وجيزة تناولت دراسات اخرى التبعية التفاضلية من الدرجة الثانية في قرص الوحدة . مؤخراً تم تقديم التبعية التفاضلية من الدرجة الثالثة من قبل بوانسامي واخرون في عام 1992 و انتونيو ميلر عام 2011 . تبحث هذة الورقة في صنف اوسع بكثير من متراجحات التبعية التفاضلية من الدرجة الثالثة . عرف المؤلفون المعايير الخلصة بصنف من الاوبريترات المسموح بها وهذا يعني ضمان وجود التبعية التفاضلية من الدرجة الثالثة. الدوال الميرومورفية في D هي دوال تحليلية في المجال D بأستثناء الرواسب اذا كانت $D = \mathbb{C}$ فأن الدالة ميرومورفية وهي دوال يمكن تمثيلها كحاصل قسمة دالتين. دالة ستورف لها تطبيقات في قضايا الموجات السطحية و موجات الماء والديناميكا الهوائية غير المستقرة و الاتجاه البصري ونظرية عدم الاستقرار المقاوم ظهرت دالة ستروف مؤخرا في عدد من انظمة الجسيمات . فكرة التبعية التفاضلية في لغة $\mathbb{C}$ هي تعميم للمتباينات في لغة R, وقد بدأت في عام 1981 من خلال اعمال ميلر و موكانو و ريد. في هذا المقال يتم معاينة الفئات المناسبة من الدوال المقبولة ويتم انشاء خصائص التبعية التفاضلية من الدرجة الثالثة باستخدام المؤثر $\check{S}_{c,p}^{\zeta,a}$ للدوال متعددة التكافؤ التحليلية باستثناء عدد منته من النقاط(الاقطاب) المرتبطة بوظيفة ستروف المعممة . في هذة الدراسة هناك حاجة لعرض العديد من المفاهيم منها التبعية , الفوقية , المسيطر, افضل السائد, الالتواء (او منتج هادامارد) ,الدالة متعددة التكافؤ التحليلية باستثناء عدد منته من النقاط (الاقطاب) , دالة ستروف بالاضافة الى مفهوم المضروب المزاح(او رمز بوشهامر) و الدوال المسموح بها.

الكلمات المفتاحية: الدوال المسموح بها، الدالة التحليلية، الالتفاف، الدالة التحليلية باستثناء عدد منته من النقاط (الاقطاب)، دالة ستروف، التبعية التفاضلية من الدرجة الثالثة.