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Injective Eccentric Domination in Graphs

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Abstract

The concept of domination has inspired researchers which has contributed to a vast literature on domination. A subset D of V is said to be a dominating set, if every vertex not in D is adjacent to at least one vertex in D. The eccentricity $e(v)$ of v is the distance to a vertex farthest from v. Thus $e(v) = \max\{d(u, v): u \in V\}$. For a vertex v, each vertex at a distance $e(v)$ from v is an eccentric vertex. The eccentric set of a vertex v is defined as $E(v) = \{u \in V(G): d(u, v) = e(v)\}$. Let $S \subseteq V(G)$, then S is known as an eccentric point set of G if for every $v \in V - S$, S has at least one vertex u such that $u \in E(v)$. A dominating set S is called an eccentric dominating set if it is also an eccentric point set. In this article the concept of injective eccentric domination is introduced for simple, connected and undirected graphs. An eccentric dominating set S is called an injective eccentric dominating set if for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that $|\Gamma(v, u)| \geq 1$ where $\Gamma(v, u)$ is the set of vertices different from v and u, that are adjacent to both v and u. Theorems to determine the exact injective eccentric domination number for the basic class of graphs are stated and proved. Nordhaus-Gaddum results are proposed. The injective eccentric dominating set, injective eccentric domination number $\gamma_{ined}(G)$, upper injective eccentric dominating set and upper injective eccentric domination number $\Gamma_{ined}(G)$ for different standard graphs are tabulated.

Keywords: Common neighborhood, Domination, Eccentricity, Injective eccentric domination, Injective eccentric domination number.

Introduction

The concept of domination by Berge has inspired researchers which has led to many invariants of domination. A collection of wide varieties of domination can be studied in the books[1-4]. 'Eccentric domination in graphs' was introduced by Janakiraman et al[5] . A. Alwardi et al[6] presented injective domination of graphs. Different types of domination[7,8] were also developed. Complementary nil eccentric domination was introduced by Bhanumathi and Senthil[9], and equal eccentric domination and accurate eccentric domination were introduced by Mohamed Ismayil and Riyaz Ur Rehman [10,11]. Different types of eccentric domination were also developed[13-18]. Motivated by these research works the concept of injective eccentric domination is introduced for simple,

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connected, and undirected graphs. The concept of injective eccentric domination is introduced. Theorems related to injective eccentric domination are discussed. Propositions for any arbitrary graphs are stated. The injective eccentric dominating set, $\gamma_{ined}(G)$, upper injective eccentric dominating set and $\Gamma_{ined}(G)$ for different standard graphs are tabulated. The minimum degree and the maximum degree among the vertices of G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. Refer to the book 'Graph Theory' by F Harary[12] for undefined terminologies.

Injective Eccentric Domination in Graphs

In this section, injective eccentric dominating sets and their numbers are defined. Theorems related to injective eccentric domination of families of graphs such as complete, star, path, wheel, and cycle graphs are discussed. The injective eccentric dominating set, injective eccentric domination number $\gamma_{ined}(G)$, upper injective eccentric dominating set and upper injective eccentric domination number $\Gamma_{ined}(G)$ for different standard graphs are given in Table 1.

Definition 1: An eccentric dominating set (ED-set) S is called an injective eccentric dominating set (INED-set) if for every vertex of $v \in V - S$ there exists a vertex $u \in S$ such that $|\Gamma(v,u)| \geq 1$ where $\Gamma(v,u)$ is the set of vertices different from v and u, that are adjacent to both v and u.

Definition 2: An injective eccentric dominating set (INED-set) S is called a minimal INED-set if no proper subset of S is an INED-set.

Definition 3: The INED-number $\gamma_{ined}(G)$ of a graph, G is the minimum cardinality among the minimal INED-sets of G.

Definition 4: The upper INED-number $\Gamma_{ined}(G)$ of a graph, G is the maximum cardinality among the minimal INED-sets of G.

Example 1: Consider the graph G given in Fig 1, the graph G consists of 7 vertices and 7 edges.

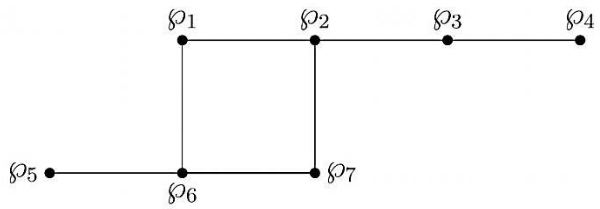

Figure 1. A graph G of order and size 7.

Here the dominating set is $\{\wp_3, \wp_6\}$.
The ED set is $\{\wp_2, \wp_4, \wp_5\}$.
The INED set is $\{\wp_1, \wp_2, \wp_4, \wp_5\}$, therefore $\gamma_{ined}(G) = 4$.
The upper INED set is $\{\wp_1, \wp_2, \wp_3, \wp_5, \wp_6, \wp_7\}$, therefore $\Gamma_{ined}(G) = 6$.

Observation 1: For any graph $G = (V, E)$, $|V(G)| = n$ and $|E(G)| = q$,

1. $\gamma(G) \leq \gamma_{ed}(G) \leq \gamma_{ined}(G) \leq \Gamma_{ined}(G)$.
2. $\gamma_{ined}(G) < n$ and $\Gamma_{ined}(G) < n$.
3. $\Gamma_{ined}(G) + 1 \leq n$.
4. If G is a star graph ($G = S_n$) then
   (a) $diam(S_n) = \gamma_{ined}(S_n)$,
   (b) $\gamma_{ed}(S_n) = \gamma_{ined}(S_n) = 2$.

Remark 1: Any graph with 2 vertices does not have an INED-set ie, $\gamma_{ined}(K_n) = 0$ where $|G| \leq 2$. Since $|\Gamma(\wp_i, \wp_j)| = 0 \ \forall \ \wp_i, \wp_j \in G$. In this paper, graphs of order greater than or equal to 3 are considered.

Theorem 1: For a complete graph K_n, where $n \geq 3$, $\gamma_{ined}(K_n) = 1$.

Proof: Let K_n be a complete graph, then $\forall \ \wp_i \in V(K_n)$, $\deg(\wp_i) = n - 1$. Therefore, every single vertex forms a dominating set, say $D = \{\wp_i\}$. Since $\deg(\wp_i) = n - 1$, $e(\wp_i) = 1$ and $E(\wp_i) = V(K_n) - \{\wp_i\}$, then $D = \{\wp_i\}$ forms an eccentric dominating set since $\forall \ \wp_j \in V(K_n) - D$, $E(\wp_j) = \wp_i$. Now, $\forall \ \wp_j \in V(K_n) - D$ there exists $\wp_i \in D$ such that the common neighbor $|\Gamma(\wp_i, \wp_j)| = n - 2 > 1$. Therefore, D becomes an INED-set and $\gamma_{ined}(K_n) = 1 \ \forall \ n \geq 3$.

Observation 2: For a complete graph K_n, $\delta = \Delta = (n-1)$ then $\gamma(K_n) = \gamma_{ed}(K_n) = \gamma_{ined}(K_n) = 1$.

Theorem 2: For star graph $S_n \ \forall \ n \geq 3$, $\gamma_{ined}(S_n) = 2$.

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Proof: Let $V(S_n) = \{\wp_1, \wp_2, \dots, \wp_c, \dots, \wp_n\}$. Let $\wp_1$ be the central vertex and all the other vertices are pendant vertices. $\deg(\wp_1) = n-1$ and $\deg(\wp_i) = 1$, $i = 2,3, \dots n$ where $\wp_i$ is any pendant vertex. $D = \{\wp_1\}$ is the only dominating set of cardinality 1. But $D = \{\wp_1\}$ is not an ED-set. $e(\wp_1) = 1$, $E(\wp_1) = V(S_n) - \{\wp_1\}$, $e(\wp_i) = 2$, $i = 2,3, \dots n$ and $E(\wp_i) = V(S_n) - \{\wp_1, \wp_j\}$, $i \neq j$. Since every pendant vertex is an eccentric vertex of each other. A set $D' = \{\wp_1, \wp_i\}$, $i \neq 1$ forms an ED-set, and for every $\wp_j \in V(S_n) = D'$ there exists $\wp_i \in D'$ such that $E(\wp_j) = \wp_i$ and $\wp_1$ dominates all other vertices. Then for every $\wp_j \in V(S_n) - D'$ there exists $\wp_i \in D'$ such that $|\Gamma(\wp_i, \wp_j)| = 1$. The set D' becomes INED-set and therefore $\gamma_{ined}(S_n) = 2 \ \forall\, n \geq 3$.

Theorem 3: For a path graph P_n where $n > 3$,

$$\gamma_{ined}(P_n) = \begin{cases} \left(\frac{n}{3}\right) + 2, & for\ n = 3k, \quad where\ k > 1 \\ \left(\frac{n-1}{3}\right) + 1, & for\ n = 3k+1, where\ k = 1,2, \dots \\ \left(\frac{n-2}{3}\right) + 2, & for\ n = 3k+2, where\ k = 1,2, \dots \end{cases}$$

Proof: Let P_n be the path graph with n vertices.

Case (i): For $n = 3k$ where $k > 1$, the path graphs are of the form $P_6, P_9, P_{12}, \dots, P_{3k}$. Let $D = \{\wp_2, \wp_5, \wp_8, \wp_{11}, \dots, \wp_{3k-1}\}$ be one of the minimal dominating sets of P_{3k} then $\gamma(G) = \left(\frac{n}{3}\right)$, but D is not an ED-set. The set $\{\wp_1, \wp_{3k}\}$ is an eccentric point set of G. Make the set D as $\{\wp_1, \wp_4, \wp_7, \dots, \wp_{3k-2}, \wp_{3k}\}$ is a minimum ED-set with $\left(\frac{n}{3}\right) + 1$ cardinality say D'. The set D' is not an injective dominating set because $|\Gamma(\wp_{3k-1}, \wp_i)| = \emptyset$, $\wp_i \in D$ and $\wp_{3k-1} \in V(P_{3k}) - D$. Now add the vertex $\wp_{3k-1}$ in D', then the set $D'' = D' \cup \{\wp_{3k-1}\}$ is an INED-set with $\left(\frac{n}{3}\right) + 2$ vertices. Hence, $\gamma_{ined}(P_{3k}) = \left(\frac{n}{3}\right) + 2$.

Case (ii): For $n = 3k+1$ where $k = 1,2, \dots$ the path graphs are of the form $P_4, P_7, P_{10}, \dots, P_{3k+1}$. Let $D = \{\wp_1, \wp_4, \wp_7, \dots, \wp_{3k+1}\}$ be one of the minimal dominating set of P_{3k+1}, then $\gamma(P_{3k+1}) = \left(\frac{n-1}{3}\right) + 1$. Since both the pendant vertices are in the set D therefore D is an ED-set. The same ED-set D forms an INED-set since for every $\wp_i \in V(P_{3k+1}) - D$ there exists $\wp_j \in D$ such that $d(\wp_i, \wp_j) = 2$, then $|\Gamma(\wp_i, \wp_j)| = 1$. Therefore $\gamma_{ined}(P_{3k+1}) = \left(\frac{n-1}{3}\right) + 1$.

Case (iii): For $n = 3k+2$, the path graphs are of the form $P_5, P_8, P_{11}, \dots P_{3k+2}$. Let $D = \{\wp_2, \wp_5, \wp_8, \dots \wp_{3k+2}\}$ be one of the minimal dominating set of P_{3k+2}, then $\gamma(P_{3k+2}) = \left(\frac{n-2}{3}\right) + 1$ but D is not an ED-set. Let $D' = D \cup \{\wp_1, \wp_{3k+2}\}$, this set is an ED-set, and for every $\wp_i \in V(P_{3k+2}) - D'$ there exists $\wp_j \in D'$ such that $|\Gamma(\wp_i, \wp_j)| = 1$ which satisfies the condition of an INED-set. Hence $\gamma_{ined}(P_{3k+2}) = \left(\frac{n-2}{3}\right) + 2$.

Lemma 1: For a path P_3, $\gamma_{ined}(P_3) = 2$.

Proof: For P_3, $V(P_3) = \{\wp_1, \wp_2, \wp_3\}$ then $|\Gamma(\wp_1, \wp_3)| = 1$. $\wp_2$ is the only common neighbor of $\wp_1$ and $\wp_3$. Therefore $D_1 = \{\wp_1, \wp_2\}$ and $D_2 = \{\wp_2, \wp_3\}$ forms an INED-set. Hence $\gamma_{ined}(P_3) = 2$.

Observation 3: For the path graphs $P_4, P_7, P_{10}, \dots P_{3k+1} = P_n$, the composition of an INED-set $D = \{\wp_1, \wp_a, \wp_b, \wp_c, \dots \wp_{3k+1} = \wp_n\}$ where $1 < a < b < c < \cdots < 3k+1 = n$ whose cardinality is $\left(\frac{n-1}{3}\right) + 1$ will be $d(\wp_1, \wp_a) = d(\wp_a, \wp_b) = d(\wp_b, \wp_c) = \cdots = d(\wp_{3k-2}, \wp_{3k+1}) = 3$.

Theorem 4: For the wheel graph W_n where $n \geq 4$,

$$\gamma_{ined}(W_n) = \begin{cases} 1, & if\ n = 4 \\ 2, & if\ n = 5,7 \\ 3, & if\ n = 6, n \geq 8 \end{cases}$$

Proof: Let W_n be the wheel graph with n vertices.

Case (i): For $n = 4$: Since the wheel W_4 is a complete graph K_4. From Theorem-1, $\gamma_{ined}(W_4) = 1$.

Case (ii): Subcase-1: For $n = 5$, let $V(W_5) = \{\wp_1, \wp_2, \wp_3, \wp_4, \wp_5\}$ where $\wp_1$ being the central vertex $\deg(\wp_1) = 4 = \Delta(W_4)$ and $\deg(\wp_i) = 3 = \delta(W_4) \ \forall\, \wp_i \in V(W_4) - \{\wp_1\}$. $\{\wp_1\}$ is the only dominating set but not eccentric dominating. Any two non-central vertices that are adjacent form the eccentric dominating set. The central vertex $\wp_1$ is the common neighbor of any two adjacent vertices that are non-central vertices. Therefore the two non-central vertices form an eccentric dominating and also an injective dominating set. Hence any two non-central adjacent vertices form an INED-set. Therefore $\gamma_{ined}(W_5) = 2$.

Subcase-2: For $n = 7$, let $V(W_7) = \{\wp_1, \wp_2, \wp_3, \wp_4, \wp_5, \wp_6, \wp_7\}$ where $\wp_1$ being the central vertex $\deg(\wp_1) = 6 = \Delta(W_7)$. Let $D = \{\wp_i, \wp_j\}$ be the diagonally opposite non-central vertex pair form the eccentric dominating set. Then W_7 contains exactly 3 pairs of such vertices. $|\Gamma(\wp_i, \wp_k)| = |\Gamma(\wp_j, \wp_k)| = 1 \ \forall \ \wp_k \in V(W_7) - D$, $(\wp_i, \wp_k), (\wp_j, \wp_k) \in E(W_7)$ and $|\Gamma(\wp_i, \wp_p)| = |\Gamma(\wp_j, \wp_p)| = 2 \ \forall \ \wp_p \in V(W_7) - D$ and $(\wp_i, \wp_p), (\wp_j, \wp_p) \notin E(W_7)$. Therefore $D = \{\wp_i, \wp_j\}$ forms the INED-set of W_7.

Case (iii): For W_6 and W_n when $n \geq 8$, there exists no ED-set whose cardinality is 2. Therefore the set $D = \{\wp_i, \wp_j, \wp_c\}$ forms a dominating set where $\wp_c$ is the central vertex and $\wp_i, \wp_j$ are adjacent non-central vertices. Then $|\Gamma(\wp_i, \wp_k)| = |\Gamma(\wp_j, \wp_k)| = 1$ if $\wp_k \in V - D$ is adjacent to $\wp_i$ or $\wp_j$. Hence, $|\Gamma(\wp_i, \wp_k)| = |\Gamma(\wp_j, \wp_k)| = |\Gamma(\wp_c, \wp_k)| = 2$ if $\wp_k$ is not adjacent to $\wp_i$ or $\wp_j$. Therefore $\gamma_{ined}(W_6) = \gamma_{ined}(W_n) = 3$ for all $n \geq 8$.

Theorem 5: For a cycle graph C_n where n is even

$$\gamma_{ined}(C_n) = \begin{cases} 4, & for\ n = 6 \\ \frac{n}{2}, & if\ n \neq 6 \end{cases}$$

Proof: Case (i): For a cycle graph C_6, every cycle graph with an even number of vertices is a self-centered graph ie, the eccentricity of any vertex is unique. Then for any vertex $\wp_i \in V(C_6)$ the diagonally opposite vertex will be the eccentric vertex of $\wp_i$. Let $D = \{\wp_p, \wp_q\}$ be a minimum dominating set with $\left(\frac{n}{3}\right) = \frac{6}{3} = 2$ vertices such that the distance between the vertices in the set D is 3 ie, $d(\wp_p, \wp_q) = 3$ but D is not an ED-set. Let D' be a set consisting of 3 vertices such that the distance between the vertices in the set D' is 2 ie, $D' = \{\wp_p, \wp_q, \wp_r\}$ such that $d(\wp_p, \wp_q) = d(\wp_q, \wp_r) = d(\wp_r, \wp_p) = 2$ thus the set D' is an ED-set but not an INED-set. Let $D'' = D' \cup \{\wp_s\}$ where $\wp_s \in V - D'$, then for every vertex $\wp_i \in V - D''$ there exists $\wp_p \in D''$ such that $|\Gamma(\wp_i, \wp_p)| = 1$. Hence a set D'' of cardinality 4 forms an INED-set. Thus $\gamma_{ined}(C_6) = 4$.

Case (ii): For cycle graphs C_n where n is even and $n \neq 6$. Let D be a minimum dominating set containing $|D| = \left(\frac{n}{2}\right)$ vertices, here D must not contain two vertices that are diagonally opposite to each other. Since the diagonally opposite vertices are the eccentric vertices to each other. Take a set D such that the distance between the vertices $\wp_i, \wp_j \in D$ is either 1 or 2 or 3 ie, $d(\wp_i, \wp_j) = 1$ or 2 or 3 forms an ED-set since for every vertex $\wp_p \in V(C_n) - D$ there exists $E(\wp_p) \in D$. D also forms an INED-set as $|\Gamma(\wp_p, \wp_i)| = 1$ for every $\wp_p \in V(C_n) - D, \wp_i \in D$. Therefore $\gamma_{ined}(C_n) = \left(\frac{n}{2}\right)$.

Theorem 6: For a cycle graph C_n where n is odd

$$\gamma_{ined}(C_n) = \begin{cases} \left(\frac{n}{3}\right), & where\ n = 3k, k\ is\ odd \\ \left(\frac{n+1}{3}\right) + 1, & where\ n = 3k + 2, k\ is\ odd \\ \left(\frac{n-1}{3}\right) + 2, & where\ n = 3k + 1, k\ is\ even \end{cases}$$

Proof: For cycle graph C_n where n is odd,

Case (i): For cycle graph C_{3k}, $k = 1,3,5, \ldots$ then C_{3k} will be of the form $C_3, C_9, C_{15}, \ldots$ Every vertex $\wp_i \in V(C_{3k})$ has two vertices adjacent to it, $\Delta(C_n) = \delta(C_n) = \deg(\wp_i) = 2 \ \forall \ \wp_i \in V(C_{3k})$. Every vertex $\wp_i$ can dominate itself and the vertices adjacent to it, therefore $\left(\frac{n}{3}\right)$ vertices are enough to dominate the cycle graph C_{3k}. Then the set $D = \{\wp_i, \wp_j, \wp_k, \ldots \wp_r, \wp_s\}$ such that $d(\wp_i, \wp_j) = d(\wp_j, \wp_k) = \cdots = d(\wp_r, \wp_s) = 3$ forms a dominating set of cardinality $\left(\frac{n}{3}\right)$. Here every vertex has two eccentric vertices. Therefore for every vertex $\wp_p \in V - D$ there exists $\wp_i \in D$ such that $E(\wp_p) = \wp_i$. Then D becomes an eccentric

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dominating set whose cardinality is $\left(\frac{n}{3}\right)$. Further, D also forms an INED-set since for every $\wp_p \in V(C_{3k}) - D$ there exists $\wp_i \in D$ such that $\left|\Gamma\left(\wp_i, \wp_p\right)\right| = 1$. Hence $\left(\frac{n}{3}\right)$ vertices form the INED-set of C_{3k}. Therefore $\gamma_{ined}(C_{3k}) = \left(\frac{n}{3}\right)$.

Case (ii): For cycle graph C_{3k+2} where k is odd, then the cycle graphs are of the form $C_5, C_{11}, C_{17}, \ldots$ Let $D = \{\wp_p, \wp_q, \wp_r, \ldots\}$ be a minimum dominating set with cardinality $\left(\frac{n+1}{3}\right)$ but D is not an ED-set. Let $D' = D \cup \{\wp_s\}$, $\wp_s \in V - D$ such that D' contains 2 pairs of vertices where $d\left(\wp_p, \wp_q\right) = d(\wp_r, \wp_s) = 1$, then $|D'| = \left(\frac{n+1}{3}\right) + 1$ then for every vertex $\wp_i \in V - D'$ there exists $\wp_p \in D'$ such that $E(\wp_i) \in D'$ and $\left|\Gamma\left(\wp_i, \wp_p\right)\right| = 1$. Therefore $\gamma_{ined}(C_{3k+2}) = \left(\frac{n+1}{3}\right) + 1$.

Case (iii): For cycle graph C_{3k+1} where k is even, the cycle graphs are of the form $C_7, C_{13}, \ldots C_{3k+1}$. Every vertex $\wp_i \in V(C_{3k+1})$ has two eccentric vertices. Let $D = \{\wp_i, \wp_j, \ldots\}$ be a minimum dominating set with cardinality $\left(\frac{n-1}{3}\right) + 1$ but not an INED-set since there exists at least one vertex $\wp_p \in V - D$ which does not have an eccentric vertex $\wp_i \in D$ or $\left|\Gamma\left(\wp_i, \wp_p\right)\right| \ngeq 1$. Let $D' = D \cup \{\wp_k\}$ then $D' = \{\wp_i, \wp_j, \wp_k, \ldots\}$ whose cardinality is $\left(\frac{n-1}{3}\right) + 2$ forms an INED-set since for every $\wp_p \in V - D'$, $E(\wp_p)$ lies in D' and $\left|\Gamma\left(\wp_p, \wp_i\right)\right| = 1$, where $\wp_i \in D'$ and $\wp_p \in V - D'$. Hence $\gamma_{ined}(C_{3k+1}) = \left(\frac{n-1}{3}\right) + 2$.

Theorem 7: For any graph G, $\frac{n}{1+\Delta(G)} \leq \gamma_{ined}(G)$.

Proof: Let D be the INED-set of G. $|D| = \gamma_{ined}(G)$ where $V(G) = n$ and $\frac{n}{1+\Delta(G)} = \frac{|V(G)|}{1+\Delta(G)}$. Since $\frac{|V(G)|}{1+\Delta(G)}$ either $V(G) = 1 + \Delta(G)$ or $V(G) > 1 + \Delta(G)$.

Case(i): If $V(G) = 1 + \Delta(G)$. Then $\frac{|V(G)|}{1+\Delta(G)} = 1$. Therefore $\frac{n}{1+\Delta(G)} \leq \gamma_{ined}$. This case is true and holds for a complete graph K_n.

Case(ii): If $V(G) > 1 + \Delta(G)$. Suppose $\Delta(G) = 1$ then $\Delta(G) = \delta(G)$ which is not possible, by $|\Gamma(u, v)| \geq 1$. Therefore $|V(G)| \geq 3$. Then if $\Delta(G) = 2$, $\frac{|V(G)|}{1+\Delta(G)} \leq 2$ for $3 \leq n \leq 6$. Similarly, if the value of $\Delta(G) > 2$. Then $\frac{|V(G)|}{1+\Delta(G)} = \frac{n}{1+\Delta(G)} \leq \gamma_{ined}$. Since $\frac{|V(G)|}{1+\Delta(G)}$ decreases as $\Delta(G)$ increases for any value of $V(G)$. Therefore $\frac{n}{1+\Delta(G)} \leq \gamma_{ined}(G)$.

Theorem 8: If a graph contains one or more pendant vertices then every minimum INED-set contains at least one pendant vertex.

Proof: The pendant vertex v has a degree one and $\delta(G) = \deg(v) = 1$. Since v is adjacent to only one vertex it is not directly connected to other vertices which increases the distance between v and its non-adjacent vertices. Other vertices can find the shortest path between them through the immediate neighbours which shortens the path between them. Therefore the pendant vertex will be one among those vertices that are farthest from every vertex. The pendant vertex is included in the periphery. The pendant vertex v forms the eccentric vertex of most of the vertices. Since an INED-set is also an eccentric dominating set and for many non-pendant vertices u_i, $E(u_i) = v$. Therefore an INED-set contains v, since for some $u_i \in V - S$, $\exists$ a pendant vertex $v \in S$ such that $E(u_i) = v$ and $|\Gamma(u_i, v)| \geq 1$.

Proposition 1: For any graph G,

(i) $\gamma_{ined}(G) + 2 \leq n$, (Except for P_3 graph),
(ii) If $rad(G) > 2$, $\gamma_{ined}(G) \leq n - \Delta$,
(iii) If $rad(G) = 2$ and $diam(G) = 3$ then $\gamma_{ined}(G) \leq \Delta$,
(iv) If $diam(G) = 2$ then $\gamma_{ined}(G) = \gamma_{ed}(G)$,
(v) $\gamma_{ined}(G) \leq \left\lceil\frac{n+1}{2}\right\rceil$,
(vi) $\gamma_{ined}(G) \leq n - \delta + 1$,
(vii) $\gamma_{ined}(G) \leq \left\lceil\frac{n+3-\delta}{2}\right\rceil$,
(viii) $\gamma_{ined}(G) \geq \left\lceil\frac{diam+1}{3}\right\rceil$,
(ix) $\gamma_{ined}(G) \leq \left\lceil\frac{2n}{3}\right\rceil$,
(x) $\gamma_{ined}(G) \leq \left\lceil\frac{n+2-(\delta-1)\frac{\Delta}{\delta}}{2}\right\rceil$,
(xi) $\gamma_{ined}(G) \leq \left\lceil n + 1 - \sqrt{2q}\right\rceil$,

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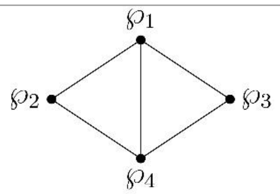
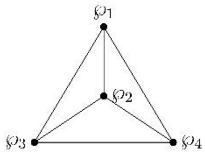
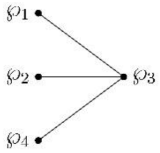
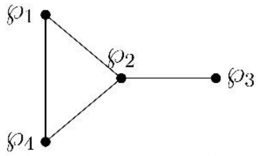
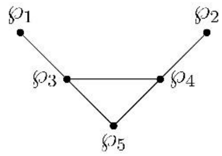
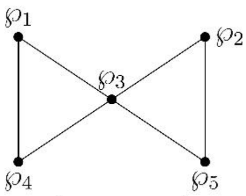
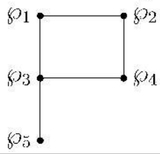
(xii) With $n \geq 3$ then $1 \leq \gamma_{ined}(G) \leq (n-1)$.

Proposition 2: For any wheel graph W_n, $\gamma_{ined}(W_n) \leq \left\lceil \frac{n}{2} \right\rceil$.

Theorem 8: For complete graph $\delta = \Delta$ and $\gamma_{ined}(K_n) = \Gamma_{ined}(K_n)$ then $n = \delta + \gamma_{ined}(K_n)$.

Proof: For any complete graph, a vertex $u \in V(K_n)$ is adjacent to every other vertex in the graph ie, $degree(u) = n-1$, where $|V(K_n)| = n$. Since every vertex is adjacent to every other vertex of the graph, therefore the degree of every vertex is the same ie, $\delta = (n-1) = \Delta$. From the Theorem 1, $\gamma_{ined}(K_n) = 1$. Hence $n = \delta + \gamma_{ined}(K_n)$.

Table 1. The injective eccentric dominating set, $\gamma_{ined}(G)$, upper injective eccentric dominating set and $\Gamma_{ined}(G)$ of standard graphs are tabulated.

Graph	Figure	Minimum INED set	$\gamma_{ined}(G)$	Upper INED set	$\Gamma_{ined}(G)$
Diamond		$\{\wp_1,\wp_2\}$, $\{\wp_1,\wp_3\}$, $\{\wp_2,\wp_3\}$, $\{\wp_2,\wp_4\}$, $\{\wp_3,\wp_4\}$.	2	$\{\wp_1,\wp_2\}$, $\{\wp_1,\wp_3\}$, $\{\wp_2,\wp_3\}$, $\{\wp_2,\wp_4\}$, $\{\wp_3,\wp_4\}$.	2
Tetrahedral		$\{\wp_1\}$, $\{\wp_2\}$, $\{\wp_3\}$, $\{\wp_4\}$.	1	$\{\wp_1\}$, $\{\wp_2\}$, $\{\wp_3\}$, $\{\wp_4\}$.	1
Claw		$\{\wp_1,\wp_3\}$, $\{\wp_2,\wp_3\}$, $\{\wp_3,\wp_4\}$.	2	$\{\wp_1,\wp_3\}$, $\{\wp_2,\wp_3\}$, $\{\wp_3,\wp_4\}$.	2
Paw		$\{\wp_1,\wp_3\}$, $\{\wp_2,\wp_3\}$, $\{\wp_3,\wp_4\}$.	2	$\{\wp_1,\wp_2,\wp_4\}$.	3
Bull		$\{\wp_1,\wp_2,\wp_3\}$, $\{\wp_1,\wp_2,\wp_4\}$, $\{\wp_1,\wp_2,\wp_5\}$, $\{\wp_1,\wp_3,\wp_4\}$, $\{\wp_2,\wp_3,\wp_4\}$.	3	$\{\wp_1,\wp_2,\wp_3\}$, $\{\wp_1,\wp_2,\wp_4\}$, $\{\wp_1,\wp_2,\wp_5\}$, $\{\wp_1,\wp_3,\wp_4\}$, $\{\wp_2,\wp_3,\wp_4\}$.	3
Butterfly		$\{\wp_1,\wp_2\}$, $\{\wp_1,\wp_5\}$, $\{\wp_2,\wp_4\}$, $\{\wp_4,\wp_5\}$.	2	$\{\wp_1,\wp_3,\wp_4\}$, $\{\wp_2,\wp_3,\wp_4\}$, $\{\wp_2,\wp_3,\wp_5\}$.	3
Banner		$\{\wp_2,\wp_5\}$.	2	$\{\wp_1,\wp_2,\wp_3\}$, $\{\wp_1,\wp_3,\wp_5\}$, $\{\wp_2,\wp_3,\wp_4\}$, $\{\wp_3,\wp_4,\wp_5\}$.	3

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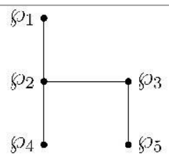
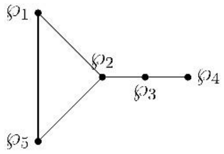
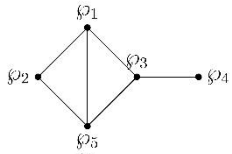
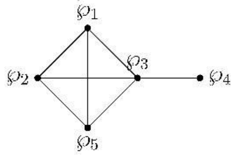
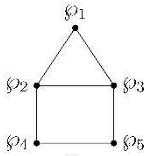
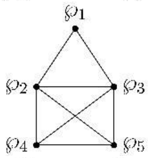
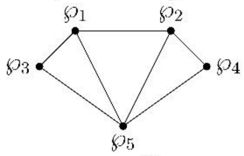
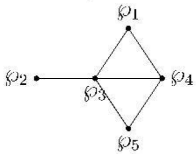
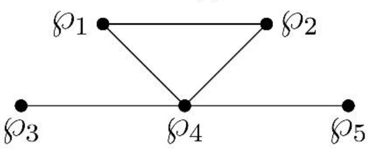
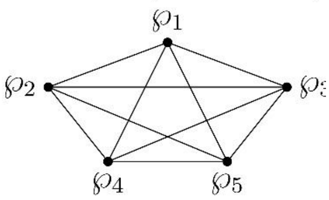
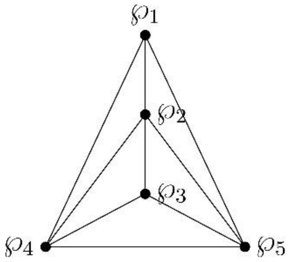
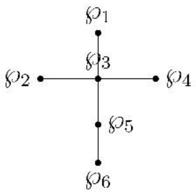
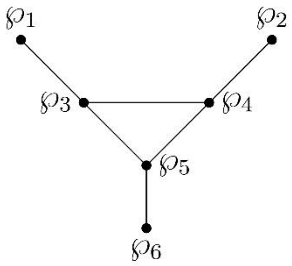
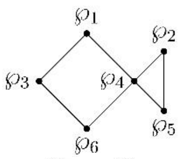
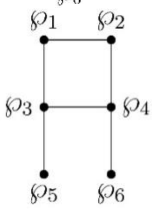
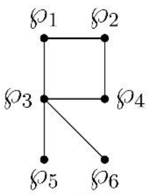
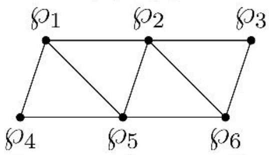
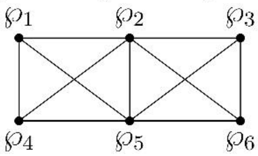
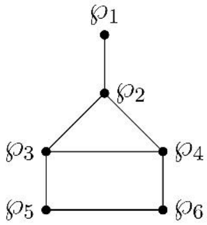
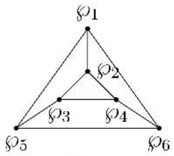
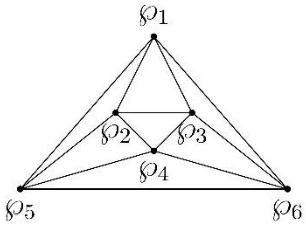
Fork		$\{\wp_1,\wp_2,\wp_5\}$, $\{\wp_1,\wp_4,\wp_5\}$, $\{\wp_2,\wp_3,\wp_5\}$, $\{\wp_2,\wp_4,\wp_5\}$.	3	$\{\wp_1,\wp_2,\wp_3,\wp_4\}$.	4
(3,2)Tadpole		$\{\wp_1,\wp_4\}$, $\{\wp_4,\wp_5\}$.	2	$\{\wp_1,\wp_2,\wp_3,\wp_5\}$.	4
Kite		$\{\wp_2,\wp_4\}$.	2	$\{\wp_1,\wp_2,\wp_3,\wp_5\}$.	4
(4,1)-Lollipop		$\{\wp_1,\wp_4\}$, $\{\wp_2,\wp_4\}$, $\{\wp_3,\wp_4\}$, $\{\wp_4,\wp_5\}$.	2	$\{\wp_1,\wp_2,\wp_3,\wp_5\}$.	4
House		$\{\wp_2,\wp_4\}$, $\{\wp_3,\wp_5\}$.	2	$\{\wp_1,\wp_2,\wp_3\}$, $\{\wp_1,\wp_4,\wp_5\}$.	3
House X		$\{\wp_1,\wp_2\}$, $\{\wp_1,\wp_3\}$, $\{\wp_1,\wp_4\}$, $\{\wp_1,\wp_5\}$.	2	$\{\wp_2,\wp_3,\wp_4,\wp_5\}$.	4
Gem		$\{\wp_1,\wp_2\}$, $\{\wp_3,\wp_4\}$.	2	$\{\wp_1,\wp_3,\wp_5\}$, $\{\wp_2,\wp_4,\wp_5\}$.	3
Dart		$\{\wp_2,\wp_3\}$, $\{\wp_2,\wp_4\}$.	2	$\{\wp_1,\wp_3,\wp_4,\wp_5\}$.	4
Cricket		$\{\wp_3,\wp_4\}$, $\{\wp_4,\wp_5\}$.	2	$\{\wp_1,\wp_2,\wp_3,\wp_5\}$.	4
Pentatope		$\{\wp_1\}$, $\{\wp_2\}$, $\{\wp_3\}$, $\{\wp_4\}$, $\{\wp_5\}$.	1	$\{\wp_1\}$, $\{\wp_2\}$, $\{\wp_3\}$, $\{\wp_4\}$, $\{\wp_5\}$.	1

Johnson solid skeleton 12		$\{\wp_1,\wp_2\}$, $\{\wp_1,\wp_3\}$, $\{\wp_1,\wp_4\}$, $\{\wp_1,\wp_5\}$, $\{\wp_2,\wp_3\}$, $\{\wp_3,\wp_4\}$, $\{\wp_3,\wp_5\}$.	2	$\{\wp_1,\wp_2\}$, $\{\wp_1,\wp_3\}$, $\{\wp_1,\wp_4\}$, $\{\wp_1,\wp_5\}$, $\{\wp_2,\wp_3\}$, $\{\wp_3,\wp_4\}$, $\{\wp_3,\wp_5\}$.	2
Cross		$\{\wp_1,\wp_3,\wp_6\}$, $\{\wp_2,\wp_3,\wp_6\}$, $\{\wp_3,\wp_4,\wp_6\}$, $\{\wp_3,\wp_5,\wp_6\}$.	3	$\{\wp_1,\wp_2,\wp_3,\wp_4,\wp_5\}$.	5
Net		$\{\wp_1,\wp_2,\wp_6\}$.	3	$\{\wp_1,\wp_2,\wp_3,\wp_5\}$, $\{\wp_1,\wp_2,\wp_4,\wp_5\}$, $\{\wp_1,\wp_3,\wp_4,\wp_5\}$, $\{\wp_1,\wp_3,\wp_4,\wp_6\}$, $\{\wp_1,\wp_4,\wp_5,\wp_6\}$, $\{\wp_2,\wp_3,\wp_4,\wp_5\}$, $\{\wp_2,\wp_3,\wp_4,\wp_6\}$, $\{\wp_2,\wp_3,\wp_5,\wp_6\}$, $\{\wp_3,\wp_4,\wp_5,\wp_6\}$.	4
Fish		$\{\wp_2,\wp_3\}$, $\{\wp_3,\wp_5\}$.	2	$\{\wp_1,\wp_2,\wp_4,\wp_5,\wp_6\}$.	5
A		$\{\wp_1,\wp_5,\wp_6\}$, $\{\wp_2,\wp_5,\wp_6\}$.	3	$\{\wp_1,\wp_2,\wp_3,\wp_4\}$, $\{\wp_1,\wp_2,\wp_3,\wp_6\}$, $\{\wp_1,\wp_2,\wp_4,\wp_5\}$, $\{\wp_1,\wp_3,\wp_4,\wp_5\}$, $\{\wp_2,\wp_3,\wp_4,\wp_6\}$, $\{\wp_3,\wp_4,\wp_5,\wp_6\}$.	4
R		$\{\wp_1,\wp_2,\wp_3\}$, $\{\wp_2,\wp_3,\wp_4\}$, $\{\wp_2,\wp_3,\wp_5\}$, $\{\wp_2,\wp_3,\wp_6\}$, $\{\wp_2,\wp_5,\wp_6\}$.	3	$\{\wp_1,\wp_3,\wp_5,\wp_6\}$, $\{\wp_3,\wp_4,\wp_5,\wp_6\}$.	4
4-polynomial		$\{\wp_3,\wp_4\}$.	2	$\{\wp_1,\wp_2,\wp_4,\wp_5\}$, $\{\wp_2,\wp_3,\wp_5,\wp_6\}$,	4
(2,3)-King		$\{\wp_1,\wp_3\}$, $\{\wp_1,\wp_6\}$, $\{\wp_3,\wp_4\}$, $\{\wp_4,\wp_6\}$.	2	$\{\wp_1,\wp_2,\wp_4\}$, $\{\wp_1,\wp_4,\wp_5\}$, $\{\wp_2,\wp_3,\wp_6\}$, $\{\wp_3,\wp_5,\wp_6\}$.	3

Antenna		$\{\wp_1,\wp_2,\wp_5\}$, $\{\wp_1,\wp_2,\wp_6\}$, $\{\wp_1,\wp_3,\wp_5\}$, $\{\wp_1,\wp_4,\wp_6\}$, $\{\wp_1,\wp_5,\wp_6\}$.	3	$\{\wp_1,\wp_2,\wp_3,\wp_4\}$, $\{\wp_2,\wp_3,\wp_5,\wp_6\}$, $\{\wp_2,\wp_4,\wp_5,\wp_6\}$.	4
3-prism		$\{\wp_1,\wp_2\}$, $\{\wp_3,\wp_5\}$, $\{\wp_4,\wp_6\}$.	2	$\{\wp_1,\wp_5,\wp_6\}$, $\{\wp_2,\wp_3,\wp_4\}$.	3
Octahedral		$\{\wp_1,\wp_2,\wp_3\}$, $\{\wp_1,\wp_2,\wp_5\}$, $\{\wp_1,\wp_3,\wp_6\}$, $\{\wp_1,\wp_5,\wp_6\}$, $\{\wp_2,\wp_3,\wp_4\}$, $\{\wp_2,\wp_4,\wp_5\}$, $\{\wp_3,\wp_4,\wp_6\}$, $\{\wp_4,\wp_5,\wp_6\}$.	3	$\{\wp_1,\wp_2,\wp_3\}$, $\{\wp_1,\wp_2,\wp_5\}$, $\{\wp_1,\wp_3,\wp_6\}$, $\{\wp_1,\wp_5,\wp_6\}$, $\{\wp_2,\wp_3,\wp_4\}$, $\{\wp_2,\wp_4,\wp_5\}$, $\{\wp_3,\wp_4,\wp_6\}$, $\{\wp_4,\wp_5,\wp_6\}$.	3

Conclusion

In this paper, the concept of injective eccentric domination of a graph is introduced. The injective eccentric domination of some families of graphs is calculated. Interesting Nordhaus-Gaddum results are proposed. The injective eccentric dominating set, injective eccentric domination number $\gamma_{ined}(G)$, upper injective eccentric dominating set and upper injective eccentric domination number $\Gamma_{ined}(G)$ for different standard graphs are tabulated.

Authors' Declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Jamal Mohamed College (Affiliated with Bharathidasan University).

Authors' Contribution Statement

This work was carried out in collaboration between all authors. RURA conceived the idea of injective eccentric domination and developed the theory under the supervision of AMI. The statement of contribution is prepared and approved by both authors.

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حقن الهيمنة غريب الأطوار في الرسوم البيانية

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الخلاصة

لقد ألهم مفهوم الهيمنة الباحثين الذين ساهموا في تأليف مؤلفات واسعة حول الهيمنة. يقال أن المجموعة الفرعية D من V هي مجموعة مهيمنة، إذا كان كل قمة ليست في D مجاورة لقمة واحدة على الأقل في D. الانحراف المركزي $e(v)$ لـ v هو المسافة إلى الرأس الأبعد من v. وبالتالي فان $e(v) = \max\{d(u,v): u \in V\}$. بالنسبة للقمة v، كل قمة على مسافة $e(v)$ من v هي قمة غريبة الأطوار. يتم تعريف المجموعة اللامركزية للقمة v على أنه E(v)={u∈V(G):d(u,v)=e(v)} على فرض $S \subseteq V(G)$ ، إذن S تُعرف بمجموعة نقاط غريب الأطوار من G إذا كان لكل $v \in V - S$ ، S لديه قمة واحدة على الأقل u حيث $u \in E(v)$. تسمى المجموعة المهيمنة S بالمجموعة المهيمنة اللامركزية إذا كانت أيضًا مجموعة نقطية غريبة الأطوار. في هذه المقالة يتم تقديم مفهوم الهيمنة اللامركزية عن طريق الحقن للرسوم البيانية البسيطة والمتصلة وغير الموجهة. تُسمى المجموعة المسيطرة اللامركزية S بالمجموعة المهيمنة اللامركزية إذا كان لكل رأس $v \in V - S$ رأس $u \in S$ بحيث تكون Γ(v,u)|≥1| حيث Γ(v,u) هي مجموعة الرؤوس المختلفة عن v و u، المجاورتين لكل من v و u. تم ذكر وإثبات النظريات لتحديد رقم التحكم اللامركزي الدقيق للفئة الأساسية من الرسوم البيانية. كما تم اقتراح نتائج Nordhaus-Gaddum. يتم جدولة مجموعة السيطرة اللامركزية عن طريق الحقن، ورقم الهيمنة اللامركزية عن طريق الحقن $\gamma_{ined}(G)$، ومجموعة الهيمنة اللامركزية عن طريق الحقن العليا ورقم الهيمنة اللامركزية عن طريق الحقن $\Gamma_{ined}(G)$ للرسوم البيانية القياسية المختلفة .

الكلمات المفتاحية: الجوار المشترك، الهيمنة، الانحراف، الهيمنة اللامركزية بالحقن، رقم الهيمنة اللامركزية بالحقن.